Name: Solutions Class #:

3. PRACTICE FUNCTION NOTATION

Let $f(x) = x^2 - 3x + 4$.

3A. Find the value of $f(2)$.
3B. Find the value of $f(2+h)$.
3C. Find the average rate of change between points $(2, f(2))$ and $(2+h, f(2+h))$
3D. For any constant a, find the average rate of change over interval $(a, f(a))$ and $(a+h, f(a+h))$

Let $f(x) = x^2 - 3x + 4$.

3A. To find the value of $f(2)$ we note that input $x = 2$ and then find

$$f(2) = f(x)\Big|_{x=2}$$

$$= (x^2 - 3x + 4)\Big|_{x=2}$$

$$= (2)^2 - 3\cdot(2) + 4$$

$$= 4 - 6 + 4$$

$$= 2 \Rightarrow \boxed{f(2) = 2} \checkmark$$

3B. Now let's find the value of $f(2+h)$.

we notice in this notation

$$\underbrace{f(x)}_{\substack{\text{input value}\\ \text{is inside the}\\ \text{parenthesis}}} = f(\underbrace{2+h}_{\substack{\text{input value}\\ \text{is inside the}\\ \text{parenthesis}}})$$

the input value is given by $x = 2+h$.

In other words, everywhere we see an x

we substitute the "value" of $2+h$. Thus

$$f(2+h) = f(x)\Big|_{x=2+h}$$

$$= (x^2 - 3x + 4)\Big|_{x=2+h}$$

3B continued ...

$$\Rightarrow \quad f(2+h) = (2+h)^2 - 3(2+h) + 4$$

Side note:

$$(2+h)^2 = (2+h)\cdot(2+h)$$
$$= 2\cdot(2+h) + h\cdot(2+h)$$
$$= 4 + 2h + 2h + h^2$$
$$= 4 + 4h + h^2$$

$$= 4 + 4h + h^2 - 6 - 3h + 4$$

$$= (4+4-6) + (4h - 3h) + h^2$$

combine like terms

$$= 2 + h + h^2$$

$$\Rightarrow \quad \boxed{f(2+h) = h^2 + h + 2}$$

3c. Recall from Lesson 6 of Math 48A,

to find the average rate of change between the points $(2, f(2))$ and $(2+h, f(2+h))$,

we find the slope of the secant line through those two points.

Side Note:	Math memories make money To solve problems, we have to remember stuff from class

- ☐ The average rate of change is the slope of a secant line through points $(a, f(a))$ and $(b, f(b))$
- ☐ A secant line is a line passing through two points on the graph of a function
- ☐ To calculate the average rate of change, we find

$$m = \frac{f(b) - f(a)}{b - a} = \frac{\text{change in } y}{\text{change in } x}$$

3C continued...

We see we can find our desired rate of change between the points $(2, f(2))$ and $(2+h, f(2+h))$ using the formula:

$$m = \frac{f(2+h) - f(2)}{2+h-2}$$

$$= \frac{h^2+h+2-2}{h}$$

$$= \frac{h^2+h}{h}$$

$$= \boxed{\frac{h(h+1)}{h}} \leftarrow \text{unsimplified}$$

$$= \frac{h}{h} \cdot (h+1) = \boxed{h+1 \quad \text{if } h \neq 0} \leftarrow \text{simplified}$$

3D. Let a be any constant. For $f(x) = x^2 - 3x + 4$ we'll find the average rate of change between the points $(a, f(a))$ and $(a+h, f(a+h))$.

We'll begin by finding $f(a)$ and $f(a+h)$:

$$f(a) = f(x)\Big|_{x=a}$$

$$= \left(x^2 - 3x + 4\right)\Big|_{x=a}$$

$$= a^2 - 3a + 4$$

To find $f(a+h)$ note:

$$f(a+h) = f(x)\Big|_{x=a+h}$$

$$= (x^2 - 3x + 4)\Big|_{x=a+h}$$

$$= (a+h)^2 - 3(a+h) + 4$$

Side Note:

$$(a+h)^2 = (a+h)\cdot(a+h)$$

$$= a(a+h) + h(a+h)$$

$$= a^2 + ah + ah + h^2$$

$$= a^2 + 2ah + h^2$$

$$= a^2 + 2ah + h^2 - 3a - 3h + 4$$

$$= (a^2 - 3a + 4) + h^2 - 3h + 2ah$$

3D, continued...

To find our desired average rate of change, we write

$$m = \frac{f(a+h) - f(a)}{a+h - a}$$

$$= \frac{a^2 - 3a + 4 + h^2 - 3h + 2ah - (a^2 - 3a + 4)}{h}$$

$$= \frac{a^2 - 3a + 4 + h^2 + 2ah - 3h - a^2 + 3a - 4}{h}$$

$$= \frac{(\cancel{a^2 - a^2})^{0} + (\cancel{3a - 3a})^{0} + (\cancel{4 - 4})^{0} + h^2 + 2ah - 3h}{h}$$

$$= \frac{h^2 + 2ah - 3h}{h}$$

3D, continued ...

$$\Rightarrow \underbrace{m}_{\text{average rate of change}} = \frac{h^2 + 2ah - 3h}{h}$$

$$= \frac{h \cdot (h + 2a - 3)}{h}$$

$$= \underbrace{\frac{\cancel{h}}{h}}_{1 \text{ if } h \neq 0} \cdot (h + 2a - 3)$$

Remember that $\frac{h}{h} = 1$

as long as the denominator

$h \neq 0$

$$\Rightarrow \boxed{m = h + 2a - 3} \checkmark$$

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4. SOLVE QUADRATIC EQUATIONS

Consider the following quadratic equation:

$$x^2 + 6x = 16$$

4A. Solve this equation by using the standard form and factoring.

Math memories make money

Recall that the standard form of a quadratic equation is given as

$$ax^2 + bx + c = 0$$

To get this equation in standard form, we get the right-hand side to be zero

$$x^2 + 6x = 16$$
$$-16 \quad -16$$

$$\Rightarrow \quad x^2 + 6x - 16 = 0$$

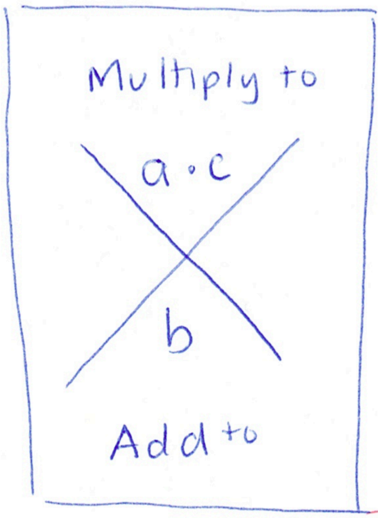
$$ax^2 + bx + c$$

$$\Rightarrow \quad x^2 + 6x + -16 = 0$$

$a = 1$, $b = 6$, $c = -16$

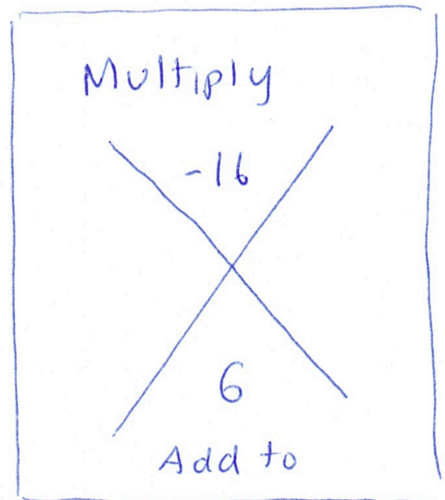
$\boxed{\text{4A, continued...}}$

Now we factor using the AC-method. Let's create our diamond in the form



Recall: we have

$a = 1,\ b = 6,\ c = -16$



Now we want to find two numbers that

multiply to : -16

add to : $+6$

4A continued ...

To do that, we notice

$$-16 = \underbrace{-1 \cdot 16}_{}$$

these factors don't work →

Check additions
$-1 + 16 = 15 \neq 6$
$1 + -16 = -15 \neq 6$

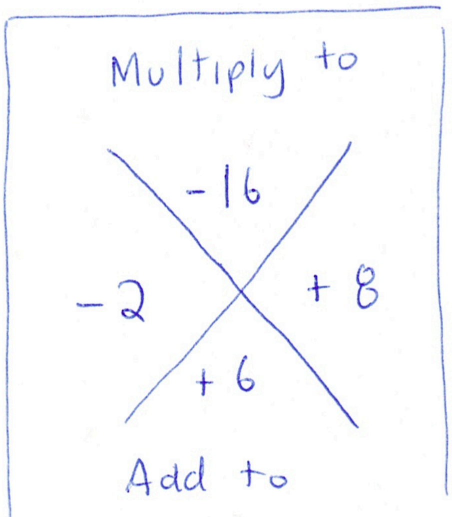
OR

$$-16 = \underbrace{-2 \cdot 8}_{}$$

these factors work :)

Check additions
$2 + -8 = -6 \neq 6$
$-2 + 8 = 6 = 6$

We fill out our AC diamond



$\Longrightarrow$

We see from our AC method that

$$6 = -2 + 8$$

$$\Rightarrow 6 \cdot x = (-2 + 8)x$$

$$\Rightarrow \underbrace{6x}_{} = \underbrace{-2x + 8x}_{}$$

middle term of our original quadratic

split into two seperate terms

4A continued ...

Now we split our middle term into the two terms we found using the AC diamond

$$x^2 + 6x - 16 = 0$$

split the middle term

$$\Rightarrow \underbrace{x^2 - 2x} + \underbrace{8x - 16} = 0$$

factor by grouping

$$\Rightarrow x(x-2) + 8(x-2) = 0$$

$$\Rightarrow (x+8)\cdot(x-2) = 0$$

$$\Rightarrow x + 8 = 0 \quad \text{or} \quad x - 2 = 0$$

$$\quad -8 \quad -8 \qquad\qquad +2 \quad +2$$

$$\Rightarrow \boxed{x = -8} \quad \text{or} \quad \boxed{x = 2}$$

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4B. Solve the equation

$$x^2 + 6x = 16$$

using the method of completing the square.

We recall our complete the square method from Lesson 15. Our goal is to get a perfect square trinomial on the left-hand side.

Math memories make money

Recall that the complete the square method is based on the relationship:

if $(x + d)^2 = x^2 + bx + c$

then $d = \frac{b}{2}$ and $c = d^2$

Using our complete the square technique we see

$$x^2 + 6x + 9 = (x + 3)^2$$

$b = 6$ $\quad c = d^2 = 3^2$ $\quad d = \frac{b}{2} = \frac{6}{2} = 3$

1:13pm – 1:17pm

Problem 4B, continued...

$$\Rightarrow \quad x^2 + 6x \underbrace{+\ 9}_{} = 16 + 9$$

transform the left-hand side into a perfect-square trinomial

$$\Rightarrow \quad \underbrace{(x+3)^2}_{} = 25$$

factor left-hand side as a perfect square in form $(x+d)^2$

$$\Rightarrow \quad \underbrace{\sqrt[2]{(x+3)^2}}_{} = \sqrt[2]{25}$$

take inverse operation of the second power

1:17pm - 1:20pm

Problem 4B, continued...

Recall:

$$\sqrt[2]{x^2} = |x|$$

$$\Rightarrow \quad |x + 3| = 5$$

$$\Rightarrow \quad x + 3 = -5 \quad \text{or} \quad x + 3 = +5$$

$$-3 \quad -3 \qquad\qquad -3 \quad -3$$

$$\Rightarrow \quad \boxed{x = -8} \quad \text{or} \quad \boxed{x = 2}$$

Notice these are the same solutions

we found in problem 4A.

1:21pm - 1:22pm

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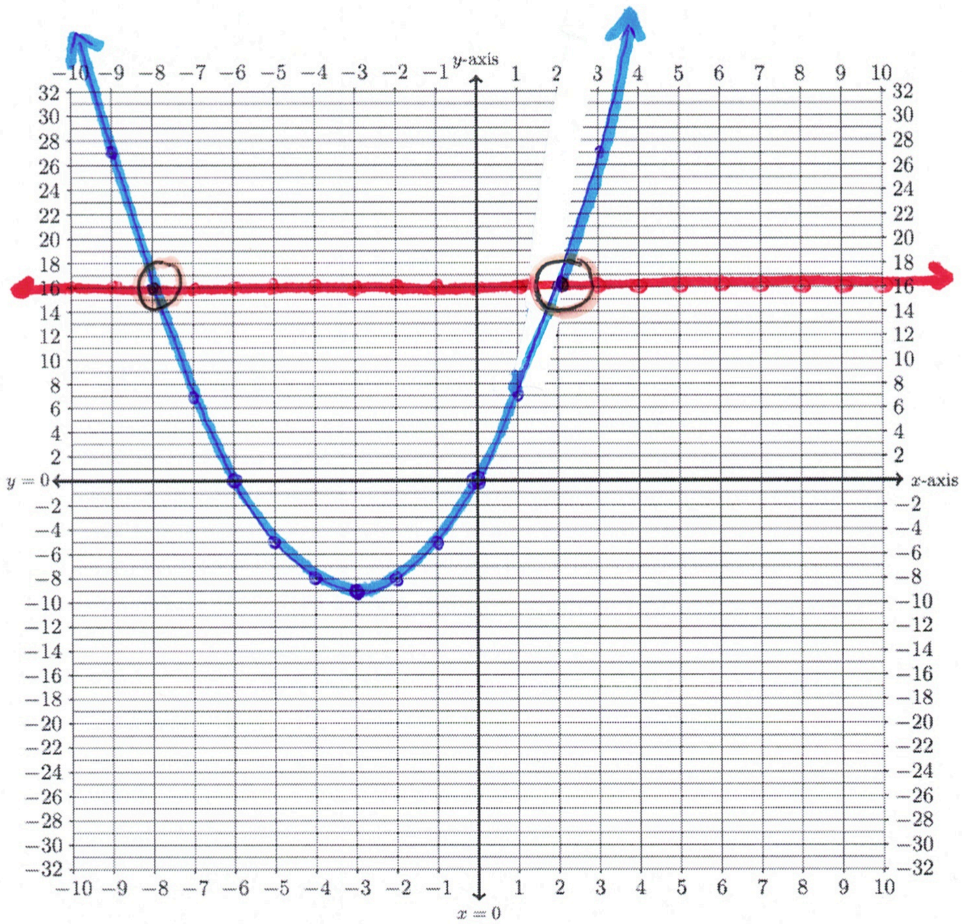
4C. Use our graphical method to solve the equation $x^2 + 6x = 16$

Input	LHS	RHS
x	$x^2 + 6x$	16
-10	40	16
-9	27	16
-8	16	16
-7	7	16
-6	0	16
-5	-5	16
-4	-8	16
-3	-9	16
-2	-8	16
-1	-5	16
0	0	16
1	7	16
2	16	16
3	27	16
4	40	16
5	55	16
6	72	16
7	91	16
8	112	16
9	135	16
10	160	16

left point of intersection

Right point of intersection



Points of Intersection between blue curve and red curve

Left point of Intersection
(-8, 16)

Right point of intersection
(2, 16)

$\Rightarrow$ the solutions to our original equation are the x-values (1st coordinates) of the points of intersection

$\Rightarrow$ $\boxed{x = -8}$ or $\boxed{x = 2}$

Notice these are same solutions as problems 4AB.

1:23 pm – 1:34 pm

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5. GRAPHING PIECEWISE FUNCTIONS

Consider the piecewise function below:

$$f(x) = \begin{cases} 16 - 2x & \text{if} \quad x < -1 \\ x^2 - 4 & \text{if} \quad -1 \le x < 3 \\ 6 & \text{if} \quad 3 \le x \end{cases}$$

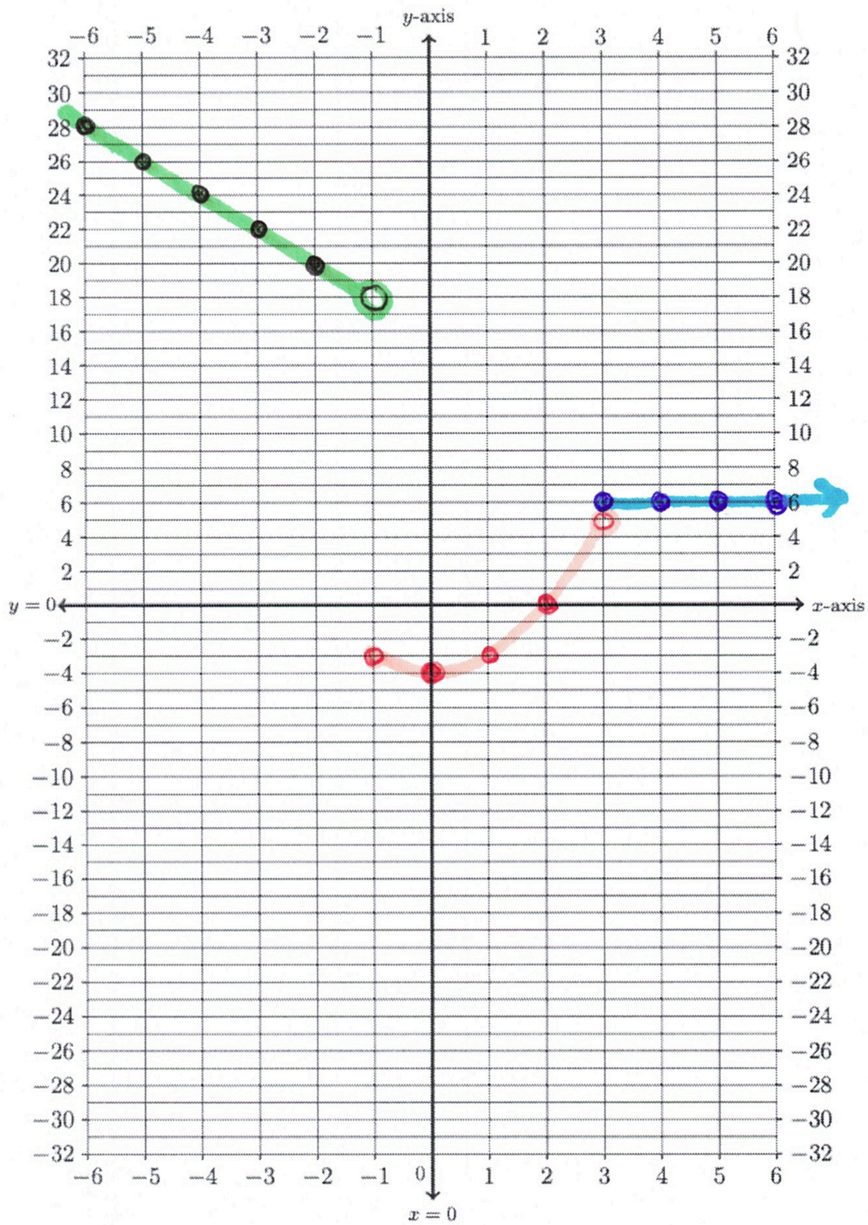
← piece # 1: $x < -1$

← piece # 2: $-1 \le x < 3$

← piece # 3: $3 \le x$

Using this description, fill out the table of values below and then graph the function

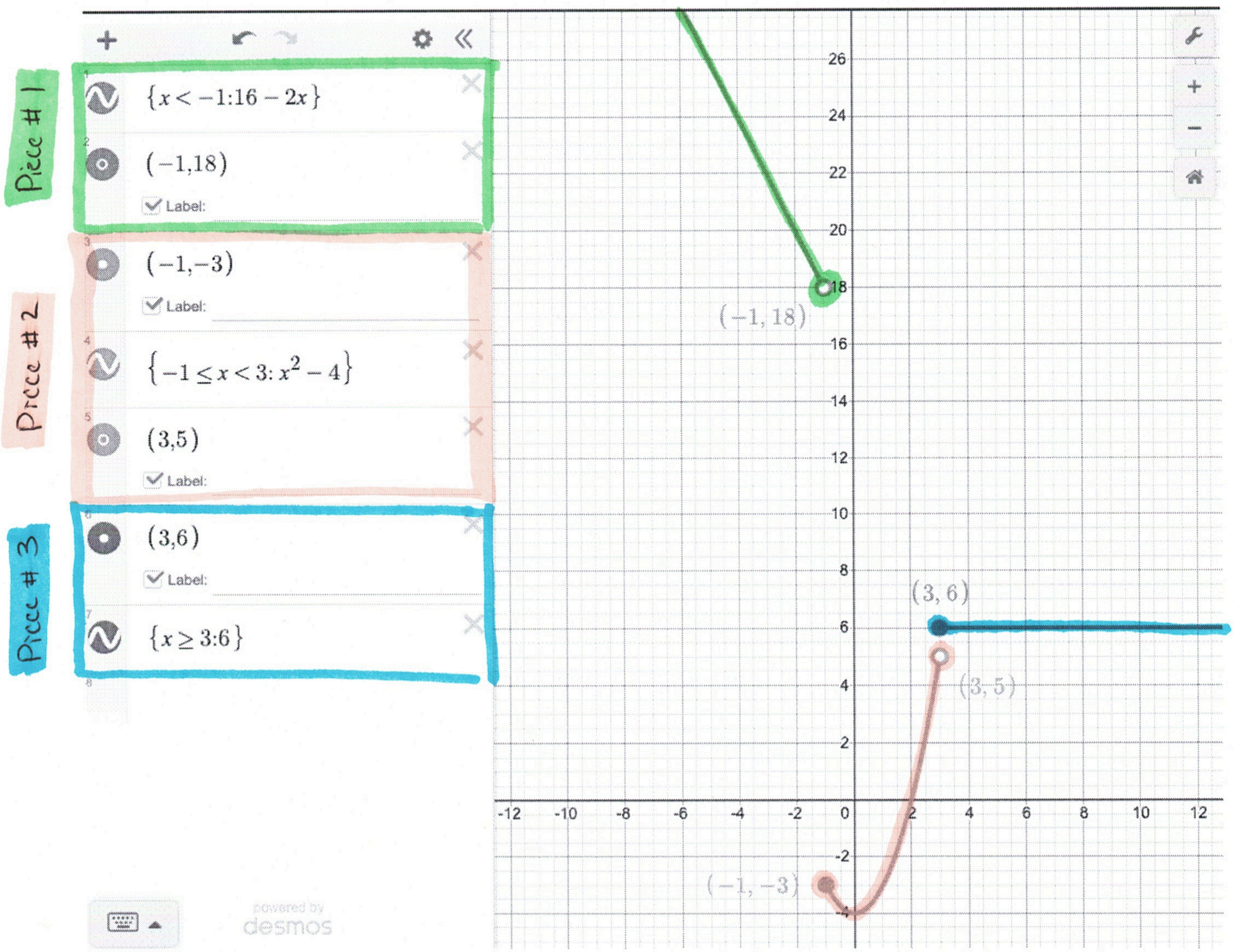
	Input	Output
	x	$f(x)$
Piece # 1 Input x-values strictly less than −1	−6	28
	−5	26
	−4	24
	−3	22
	−2	20
Piece # 2 $-1 \le x < 3$	−1	−3
	0	−4
	1	−3
	2	0
Piece # 3 $x \ge 3$	3	6
	4	6
	5	6
	6	6



On the next page, we view the Desmos.com version of the same graph.

1:35pm – 1:43pm

Problem 5: Graphing Piecewise Function



1:44pm - 1:52pm

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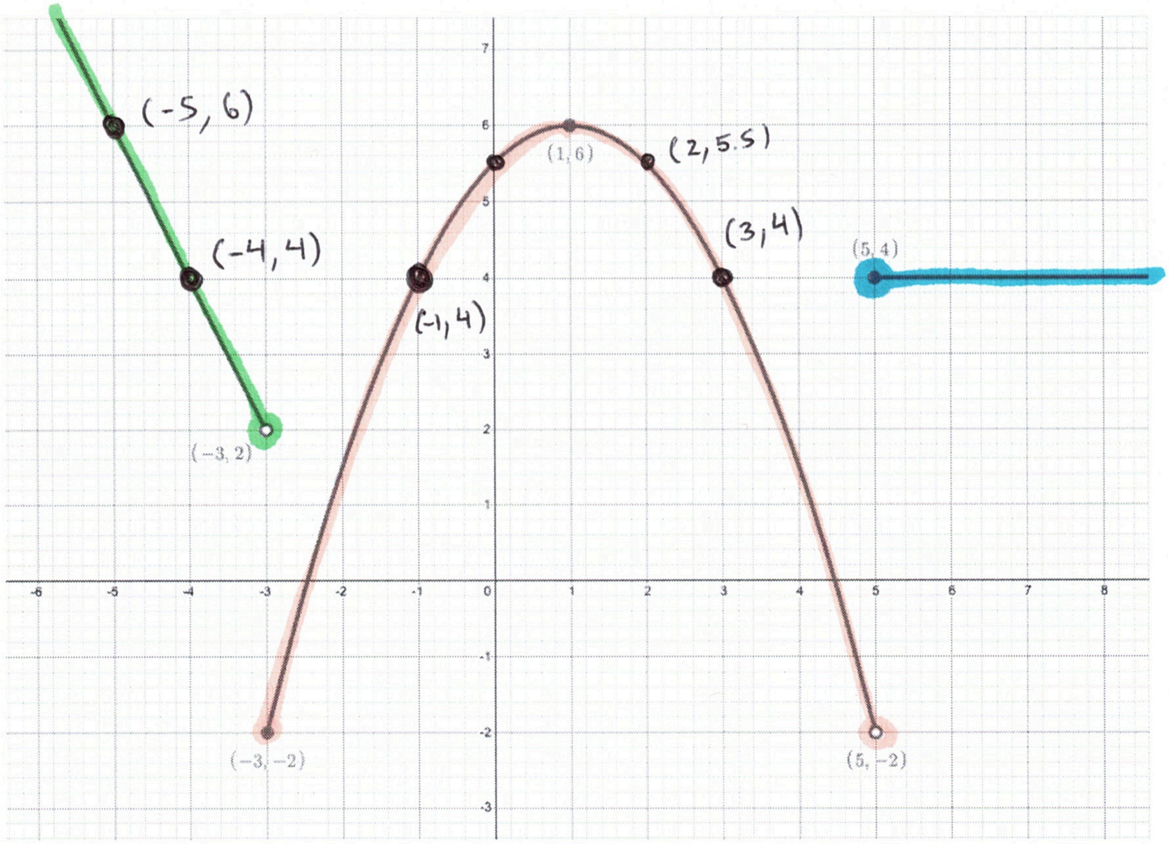
6. ANALYZE THE GRAPH OF A PIECEWISE FUNCTIONS

The following is a graph of a piecewise defined function $g(x)$.



Find the formula (rule) for each part of the function and the x-values for which it applies. Explain your reasoning.

$$g(x) = \begin{cases} -2x - 4 & \text{if } x < -3 \\ 6 - \frac{1}{2}(x-1)^2 & \text{if } -3 \leq x < 5 \\ 4 & \text{if } x \geq 5 \end{cases}$$

← piece #1
← piece #2
← piece #3

1:52 pm – 2:08 pm

Piece # 1

We notice that piece # 1 is a line through the points $(-4, 4)$ and $(-5, 6)$.

We can find the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 4}{-5 - (-4)} = \frac{2}{-1} = -2$$

Our point-slope form is given by:

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 4 = -2(x - -4)$$

$$\Rightarrow y - 4 = -2(x + 4)$$

$$\Rightarrow y = 4 + -2x - 8 = \boxed{-2x - 4}$$

Piece #2

We notice that piece #2 is a parabola with vertex at $(h, k) = (1, 6)$.

We recall from Lesson 16 that the vertex form of our quadratic function is given as

$$a(x-h)^2 + k$$

In our case we know $h = 1$, $k = 6$

The last question we have is what is our value of a. To answer that question we note point $(-3, -2)$ is on parabola

$$\Rightarrow \quad a(x-h)^2 + k = f(x)$$

$$\Rightarrow \quad a(x-1)^2 + 6 = f(x)$$

At point $(-3, -2)$ we see $x = -3$ and $f(x) = -2$

$$\Rightarrow \quad a(-3-1)^2 + 6 = -2$$

$$\Rightarrow \quad a(-4)^2 + 6 = -2$$

$$\Rightarrow \quad a \cdot 16 + 6 = -2$$

$$-6 \qquad -6$$

$$\Rightarrow \quad 16a = -8$$

$$\Rightarrow \quad a = \frac{-8}{16} = -\frac{1}{2} \Rightarrow$$

Piece #2

$$-\frac{1}{2}(x-1)^2 + 6$$

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7. PRACTICE FACTORING USING THE AC-METHOD

Use the AC-Method to factor each of the polynomials given below. For a reminder about using the AC Method to factor a quadratic polynomial, check out the Factoring Review Sheet.

7A. $12x^2 + 5x - 2$
7B. $9x^4 + 6x^3 - 3x^2$

To factor using the AC Method, we will use the standard form of our quadratic expression

$$ax^2 + bx + c$$

Problem 7A

Let's find the values of a,b,c for

$$12x^2 + 5x - 2$$

$$= 12x^2 + 5x + -2$$

$$ax^2 + bx + c$$

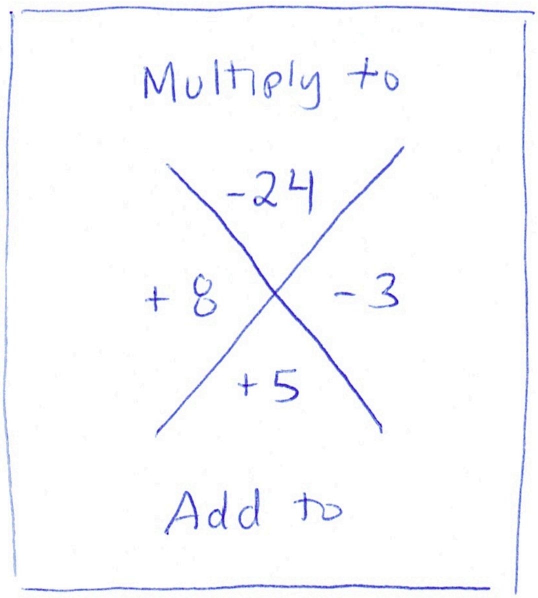
We see for this quadratc expression, we have

$$a = 12, \quad b = 5, \quad c = -2$$

2:09pm - 2:12pm

Problem 7A

Just like we did in problem 4 on this exam, we will use the AC method to factor our quadratic expression:



Multiply to: -24

Add to: $+5 = +8 - 3$

$$-24 = -1 \cdot 24 \leftarrow \text{NO}$$
$$= -2 \cdot 12 \leftarrow \text{NO}$$
$$= -3 \cdot 8 \leftarrow \text{Yes}$$

Now we split the middle term:

$$+5 = +8 - 3$$

$$\Rightarrow \quad 5x = (8-3)x = 8x - 3x$$

$$\Rightarrow \quad 12x^2 + 5x - 2 = 12x^2 + 8x - 3x - 2$$

2:12pm – 2:17pm

Problem 7A, continued

Now we factor by grouping

$$\underbrace{12x^2 + 8x}_{\text{group 1}} \underbrace{- 3x - 2}_{\text{group 2}}$$

$$= 4x(3x+2) - 1 \cdot (3x+2)$$

$$= (4x-1) \cdot (3x+2)$$

$$\Rightarrow \boxed{12x^2 + 5x - 2 = (4x-1) \cdot (3x+2)}$$

2:18pm – 2:19pm

Problem 7B

Lets find the values of a, b, c for

$$9x^4 + 6x^3 - 3x^2$$

$$ax^2 + bx + c$$

We notice a problem: the two expressions are not in the same form. So let's begin by looking for the greatest common factor

$$9x^4 + 6x^3 - 3x^2 = 3x^2 (3x^2 + 2x - 1)$$

Now we can focus on

$$3x^2 + 2x + -1$$

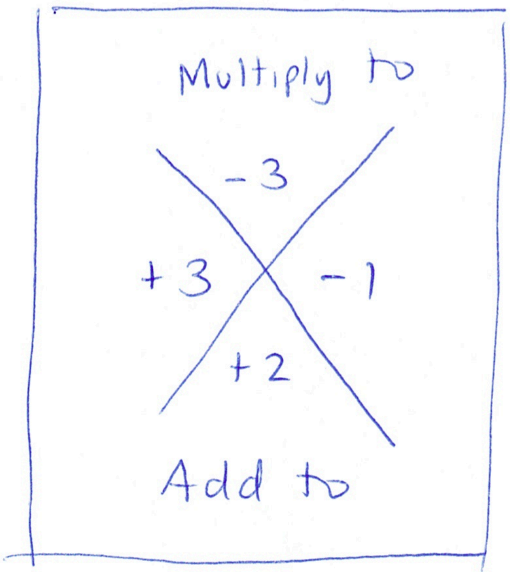
$$ax^2 + bx + c$$

$a = 3$, $b = 2$, $c = -1$

2:20pm - 2:22pm

Problem 7B, continued...

Let's repeat the AC method:



Add to: $+2$

Multiply to: $-3 = -1 \cdot 3$

Now let's split the middle term

$$+2 = 3 - 1$$

$$\Rightarrow \quad 2x = (3-1)x$$

$$\Rightarrow \quad 2x = 3x - x$$

$$\Rightarrow \quad 3x^2 + 2x - 1 = 3x^2 + 3x - x - 1$$

2:23pm – 2:24pm

Now we can factor by grouping

$$3x^2 + 2x - 1 = \underbrace{3x^2 + 3x}_{\text{group 1}} \underbrace{- x - 1}_{\text{group 2}}$$

$$= \overbrace{3x(x+1)}^{} \overbrace{- 1(x+1)}^{}$$

$$= (3x - 1) \cdot (x + 1)$$

We go back to original expression to find

$$9x^4 + 6x^3 - 3x^2 = \boxed{3x^2 (3x - 1)(x + 1)}$$

2:24pm - 2:25pm