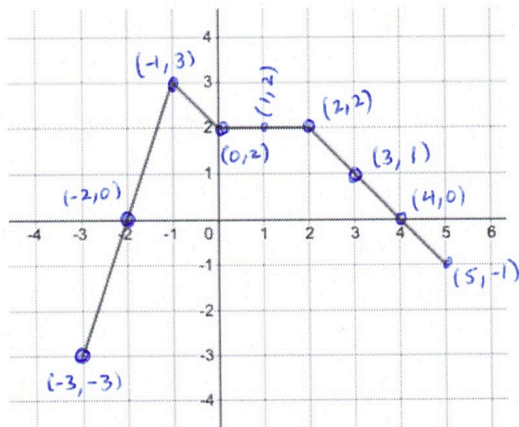


9. TRANSFORMATIONS OF GRAPHS (from Lessons 11 - 12)

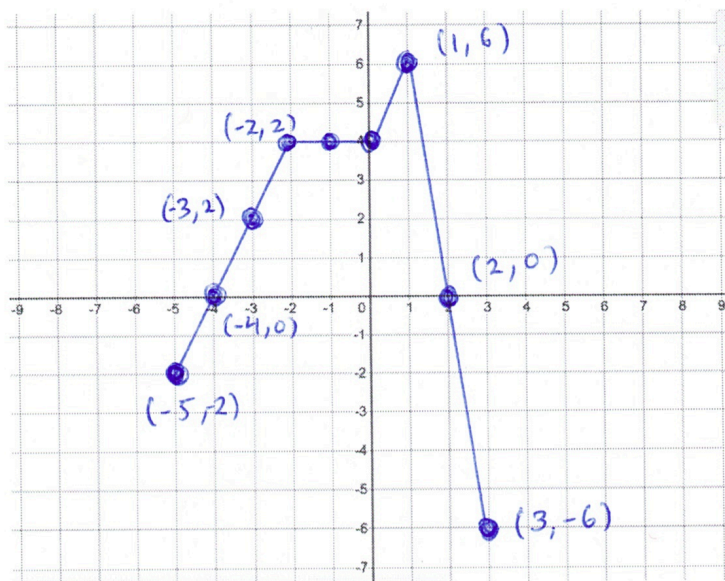
Below is a graph of a function $f(x)$.

Let's begin by analyzing the behavior of the graph of $f(x)$. To do so, let's identify some points on the graph.



Now let's create a table of values and a piecewise description of this function. See next pages for this work.

9A. Sketch a graph of $g(x) = 2f(-x)$.



9B. Explain, in words, the effect of each transformation in the new function $g(x) = 2f(-x)$

Notice that for the ordered pair

$$(x, g(x)) = (x, 2 \cdot f(-x))$$

$$= (-x, 2 \cdot f(x))$$

transform
Input into negative values

multiply output values by 2

Problem 9, continued ...

Let's take a look at the input-output relationship for the function $f(x)$. To this end, consider

Input-values x	output-values $f(x)$
-3	-3
-2	0
-1	3
0	2
1	2
2	2
3	1
4	0
5	-1

Now let's generate this same information using Desmos.com and make a testable conjecture for the piecewise description of $f(x)$

4:37am - 4:46am

Problem 9, continued...

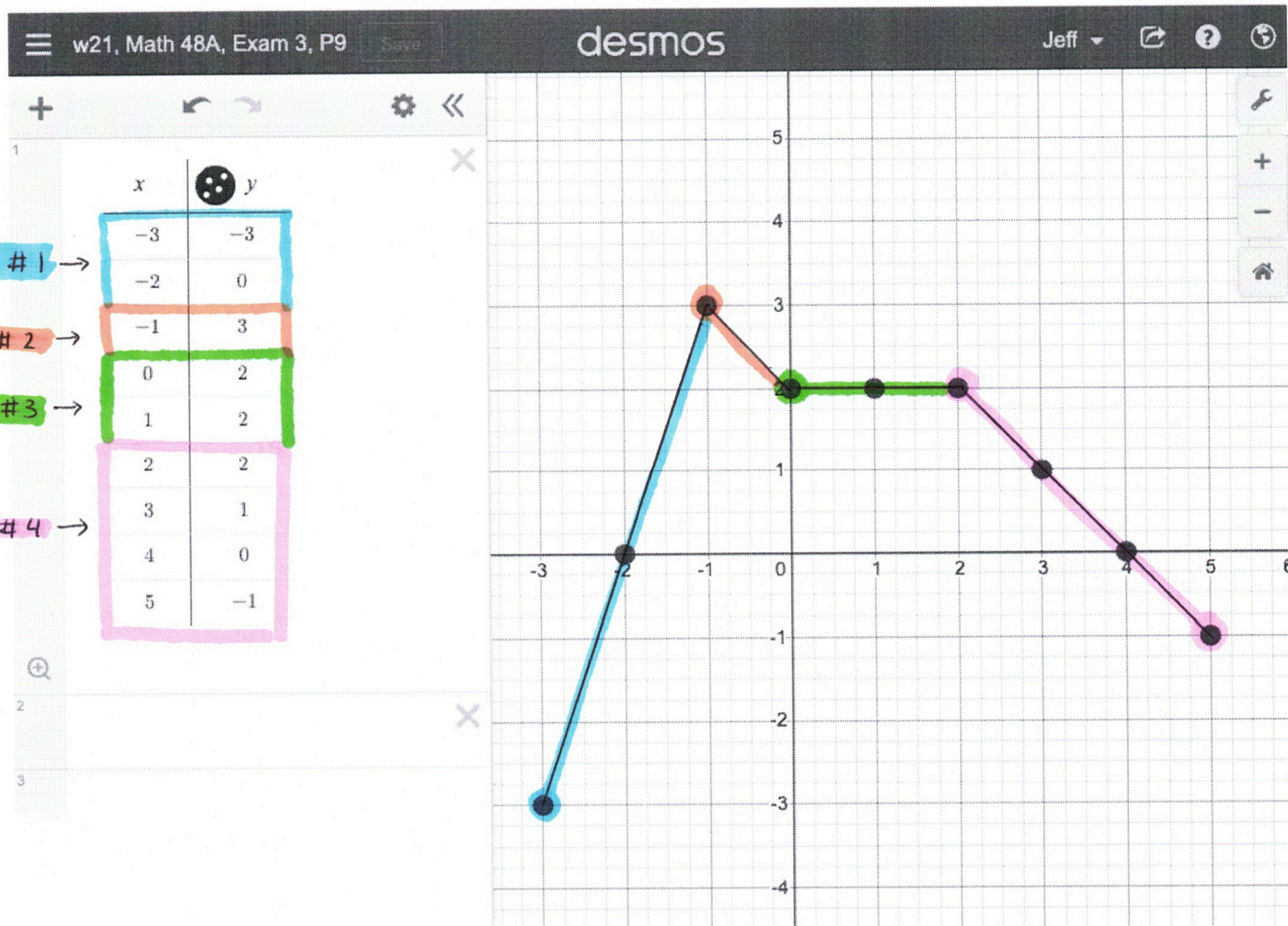
Below we see an image of $f(x)$ that highlights 4 distinct pieces of this function:

Piece # 1: $-3 \leq x < -1$

Piece # 2: $-1 \leq x < 0$

Piece # 3: $0 \leq x < 2$

Piece # 4: $2 \leq x \leq 5$



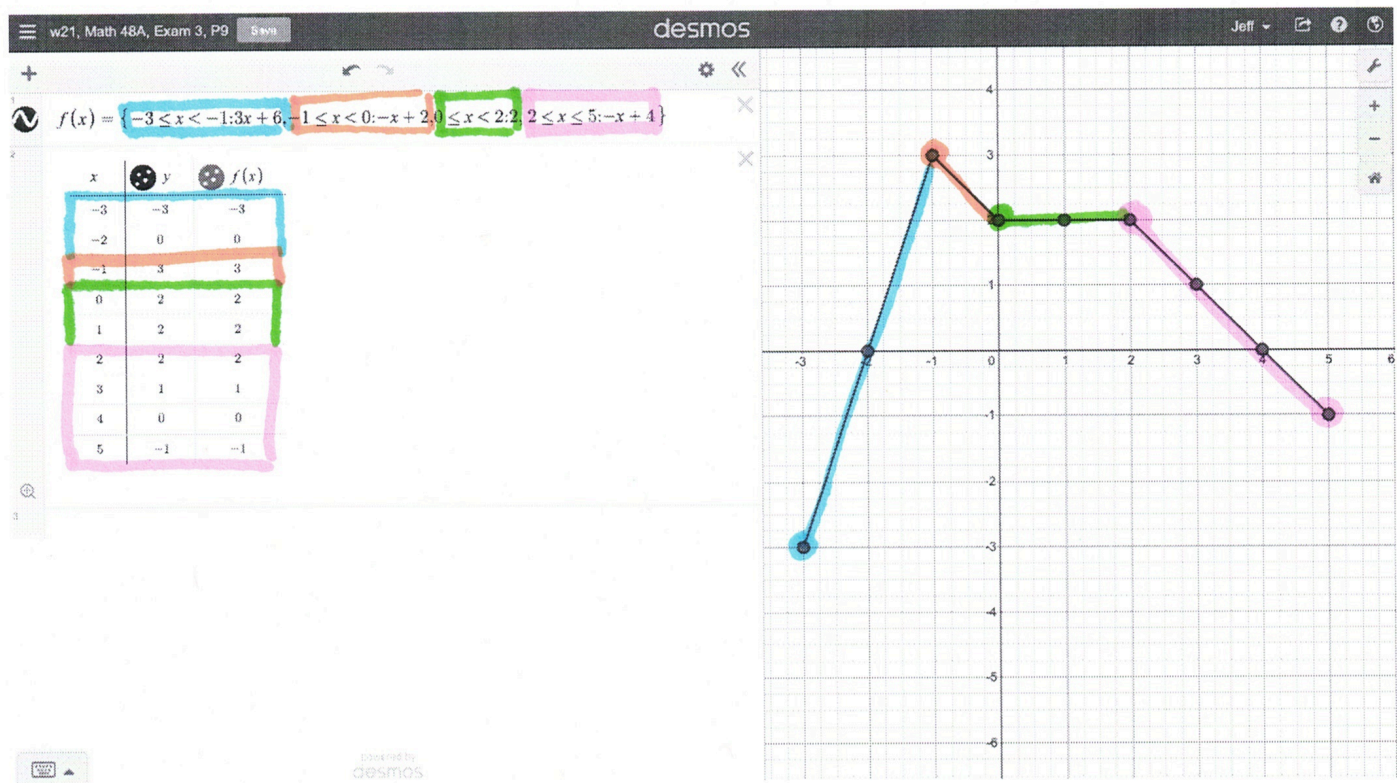
Then, we can write the piecewise definition for this function

$$f(x) = \begin{cases} 3x + 6 & \text{if } -3 \leq x < -1 \leftarrow \text{piece 1} \\ -x + 2 & \text{if } -1 \leq x < 0 \leftarrow \text{piece 2} \\ 2 & \text{if } 0 \leq x < 2 \leftarrow \text{piece 3} \\ -x + 4 & \text{if } 2 \leq x \leq 5 \leftarrow \text{piece 4} \end{cases}$$

4:46am - 4:54am

Problem 9, continued...

Now we confirm using Desmos.com that our guess for $f(x)$ was valid (see screen shot below):



Now that we've come up with an equivalent description for the outputs of $f(x)$, we have a mechanism to make guesses (conjectures) about the shape of the graphs for problems 9a - 9d and check our work quickly.

4:55am - 5:02am

Problem 9 continued ...

Let's continue with the two shifted functions from this problem given by

$$g(x) = 2 \cdot f(-x)$$

Vertical stretch by factor of 2

Horizontal reflection about y-axis

$$h(x) = 3 - f(x+2)$$

vertical shift upward by three units

horizontal shift towards left (negative direction) by 2 units

vertical reflection about x-axis

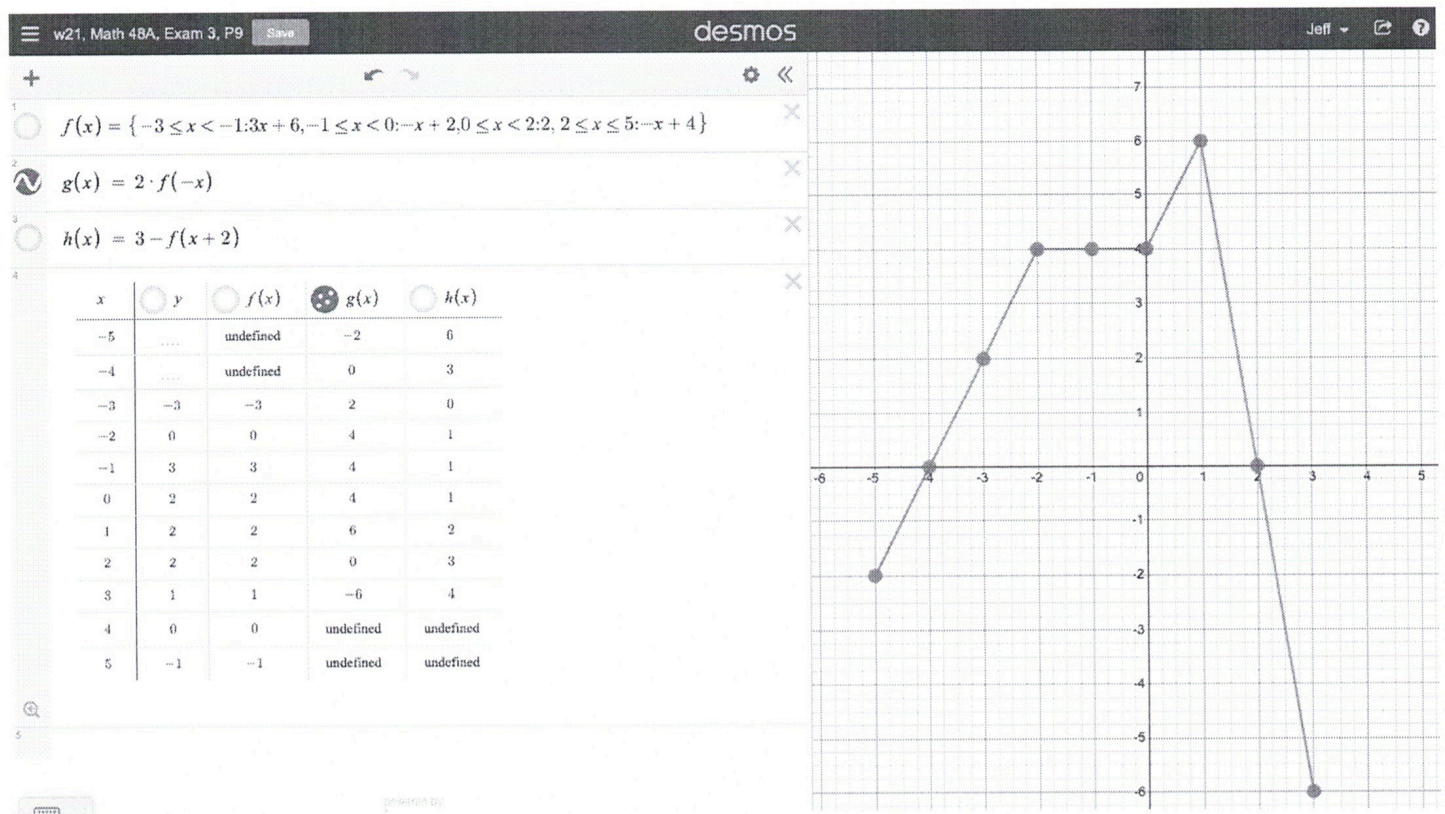
We can use Desmos.com to confirm these descriptions and draw our desired graphs.

Problem 9a, continued...

Below we see the graph of $g(x) = 2 \cdot f(-x)$

combined with the associated table of values.

Notice we've hidden all other data that is not related to $g(x)$.

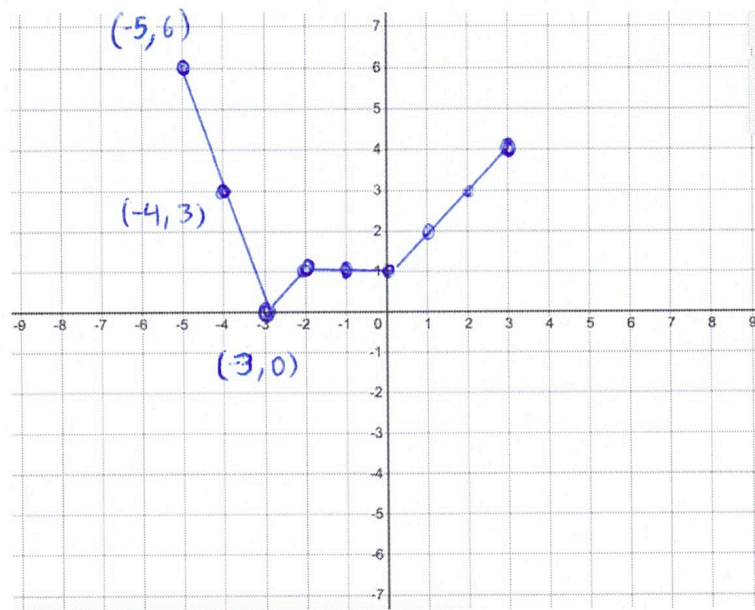


We see $g(x) = 2 \cdot f(-x)$ has two transformations

1. Horizontal reflection about y-axis $f(x) \rightarrow f(-x)$

2. Vertical scaling by factor of 2 $f(-x) \rightarrow 2 \cdot f(-x)$

9C. Sketch a graph of $h(x) = 3 - f(x + 2)$.



9D. Explain, in words, the effect of each transformation in the new function $h(x) = 3 - f(x + 2)$

Notice that for the ordered pair

$$(x, h(x)) = (x, 3 - f(x+2))$$

$$= (\underbrace{x-2}, 3 + \underbrace{-f(x)})$$

inputs
shift to the
left by
two units

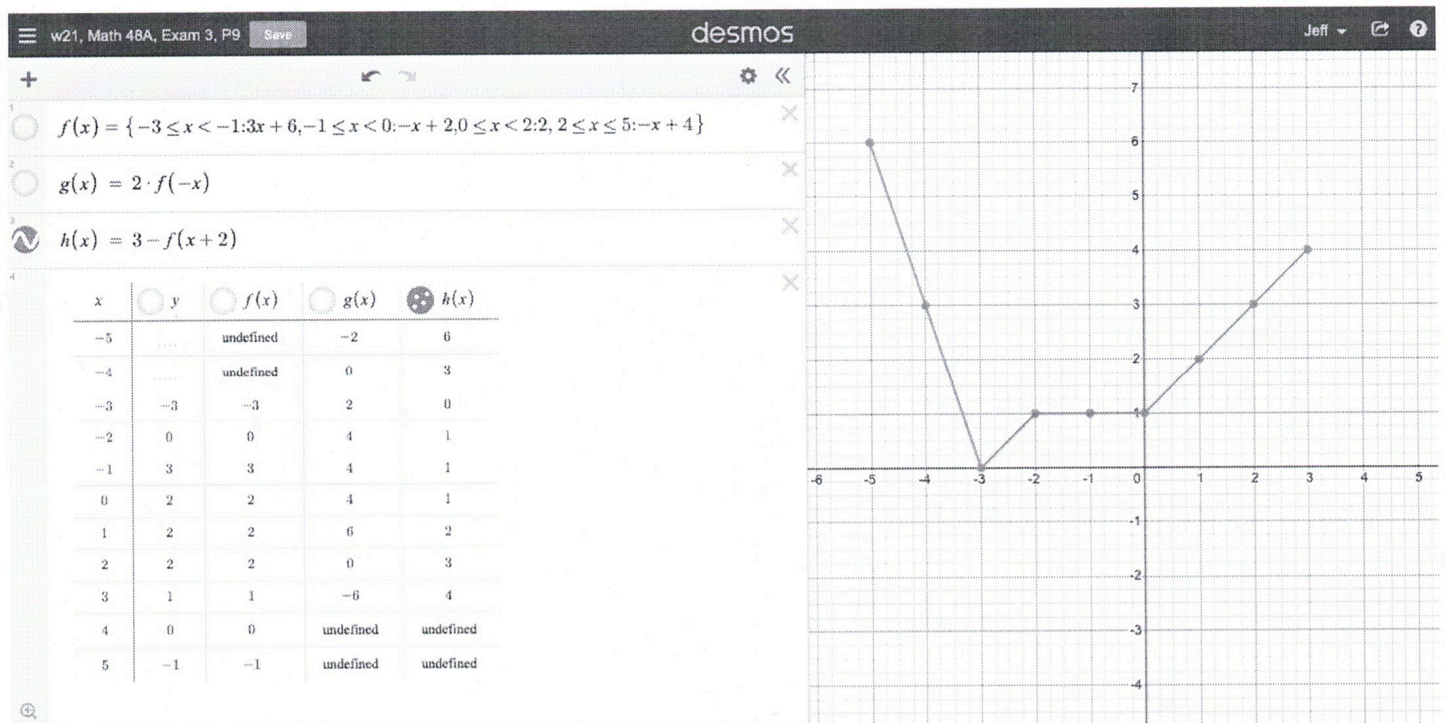
outputs flip signs
and then shift upward
by 3 units.

For more on this, see next page.

Problem 9c, continued...

Below we see the graph of $h(x) = 3 - f(x+2)$ combined with associated table.

We've hidden all other data that is not related to $h(x)$.



Notice that $h(x) = 3 - f(x+2)$ has three translations:

1. Horizontal shift to the left by 2 units: $f(x) \rightarrow f(x+2)$
2. Vertical reflection about x-axis: $f(x+2) \rightarrow -f(x+2)$
3. Vertical shift upward by 3 units: $-f(x+2) \rightarrow 3 - f(x+2)$

5:29am - 5:31am

10. FIND COMPOSITION OF FUNCTIONS (Lesson 13)

Let $f(x) = x^2 - 3$ and $h(x) = 2x + 1$. Find each of the following:

10A. $(h \circ f)(3)$

10B. $f(h(x))$

Problem 10 A

Recall that $h \circ f(3) = h(f(x)) \Big|_{x=3}$

To start, let's consider

$$h(f(x)) = h(y) \quad \text{where } y = f(x) = x^2 - 3$$

$$= 2y + 1 \Big|_{y=f(x)}$$

$$= 2y + 1 \Big|_{y=x^2-3}$$

$$= 2(x^2 - 3) + 1$$

$$= 2x^2 - 6 + 1$$

$$= 2x^2 - 5$$

Problem 10A, continued...

$$\begin{aligned}\Rightarrow h \circ f(3) &= h(f(x)) \Big|_{x=3} \\ &= 2x^2 - 5 \Big|_{x=3} \\ &= 2(3)^2 - 5 \\ &= 2 \cdot 9 - 5 \\ &= 18 - 5 \\ &= 13\end{aligned}$$

$$\Rightarrow \boxed{h \circ f(3) = 13} \quad \checkmark$$

Check: $f(3) = 3^2 - 3 = 9 - 3 = 6$

$$h(6) = 2 \cdot 6 + 1 = 12 + 1 = 13 \quad \checkmark$$

12:30pm - 12:34pm

Problem 10B

We begin by noticing:

$$f(h(x)) = f(y) \quad \text{where } y = h(x) = 2x+1$$

$$= y^2 - 3 \quad \Big| \quad y = 2x+1$$

$$= (2x+1)^2 - 3$$

Sidenote:

$$\begin{aligned} (2x+1)^2 &= (2x+1) \cdot (2x+1) \\ &= 2x(2x+1) + 1(2x+1) \\ &= 4x^2 + 2x + 2x + 1 \\ &= 4x^2 + 4x + 1 \end{aligned}$$

$$= 4x^2 + 4x + 1 - 3$$

$$\Rightarrow f(h(x)) = 4x^2 + 4x - 2 = (2x+1)^2 - 3$$

11. SOLVE EQUATIONS ALGEBRAICALLY

Solve each of the following equations. Show your work.

11A. $2 + 3|x - 2| = 11$

11B. $1 + \sqrt{3x - 5} = x$

Problem 11A

Lets solve the absolute-value equation

$$2 + 3|x - 2| = 11$$

Using the techniques from Lessons 8 - 9:

$$\begin{array}{r} 2 + 3 \cdot |x - 2| = 11 \\ -2 \qquad \qquad \qquad -2 \end{array}$$

$$\Rightarrow \frac{3 \cdot |x - 2|}{3} = \frac{9}{3}$$

$$\Rightarrow |x - 2| = 3$$

We can break this equation into two parts:

$$\Rightarrow \begin{array}{r} x - 2 = -3 \\ +2 \quad +2 \end{array}$$

or

$$\begin{array}{r} x - 2 = +3 \\ +2 \quad +2 \end{array}$$

$$\Rightarrow \boxed{x = -1} \checkmark$$

or

$$\boxed{x = 5} \checkmark$$

$$\boxed{\text{Check:}} \quad x = -1$$

$$\text{Left-hand side: } 2 + 3|x-2| = 2 + 3|-1-2|$$

$$= 2 + 3 \cdot |-3|$$

$$= 2 + 3 \cdot 3$$

$$= 2 + 9$$

$$= 11 \leftarrow \text{right-hand side}$$

$$\boxed{\text{Check:}} \quad x = 5$$

$$\text{left-hand side: } 2 + 3|x-2| = 2 + 3|5-2|$$

$$= 2 + 3 \cdot |3|$$

$$= 2 + 3 \cdot 3$$

$$= 2 + 9 = 11 \leftarrow \text{right-hand side}$$

Problem 11A, continued ...

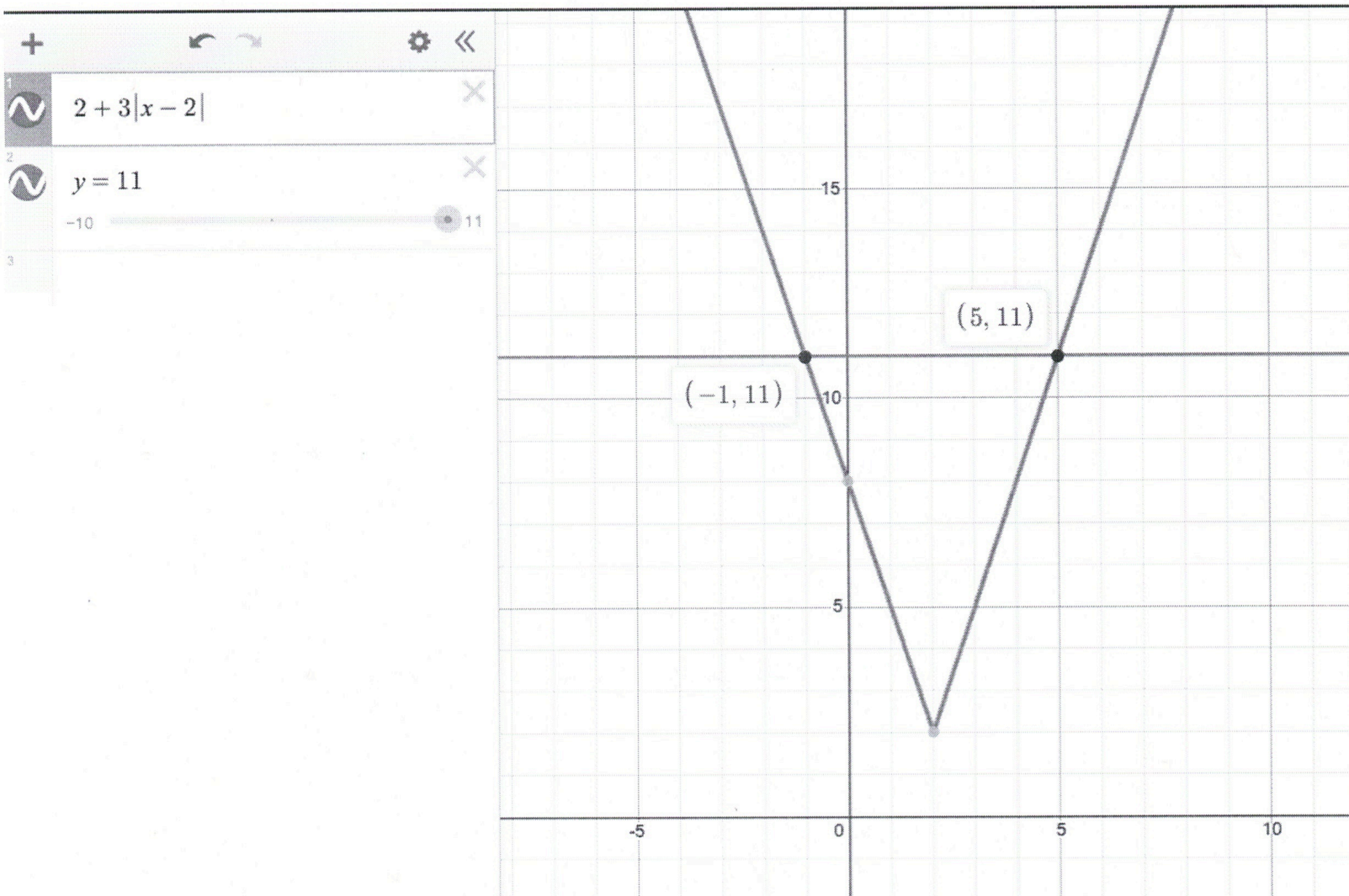
Let's check our work using Desmos.com.

We see two points of intersection

$$(-1, 11)$$

and

$$(5, 11)$$



⇒ we have two solutions at

$$x = -1$$

and

$$x = 5$$

Problem 11B

Let's solve the radical equation

$$1 + \sqrt{3x-5} = x$$

using techniques from Lesson 10:

$$\begin{array}{r} 1 + \sqrt[2]{3x-5} = x \\ -1 \qquad \qquad \qquad -1 \end{array}$$

$$\Rightarrow \sqrt[2]{3x-5} = x-1$$

$$\Rightarrow \left(\sqrt[2]{3x-5}\right)^2 = (x-1)^2$$

$$\Rightarrow 3x-5 = (x-1) \cdot (x-1)$$

Problem 11B, continued ...

$$\Rightarrow 3x - 5 = x \cdot (x-1) - 1 \cdot (x-1)$$

$$\Rightarrow 3x - 5 = x^2 - x - x + 1$$

$$\Rightarrow \begin{array}{r} 3x - 5 \\ -3x + 5 \end{array} = \begin{array}{r} x^2 - 2x + 1 \\ -3x + 5 \end{array}$$

$$\Rightarrow 0 = x^2 - 5x + 6$$

$$a=1, b=-5, c=6$$

Multiply to

$$\begin{array}{ccc} & 6 & \\ -2 & \times & -3 \\ & -5 & \end{array}$$

add to

$$\Rightarrow 0 = (x - 2)(x - 3)$$

$$\Rightarrow x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

5:31pm - 5:33pm

Problem 11B, continued...

$$\Rightarrow \boxed{x = 2} \quad \text{or} \quad \boxed{x = 3}$$

Check: $x = 2$ ✓

$$\begin{aligned} \text{Left-hand side: } 1 + \sqrt{3x-5} &= 1 + \sqrt{3 \cdot 2 - 5} \\ &= 1 + \sqrt{6-5} \\ &= 1 + \sqrt{1} \\ &= 1 + 1 = 2 \leftarrow \text{right-hand side} \end{aligned}$$

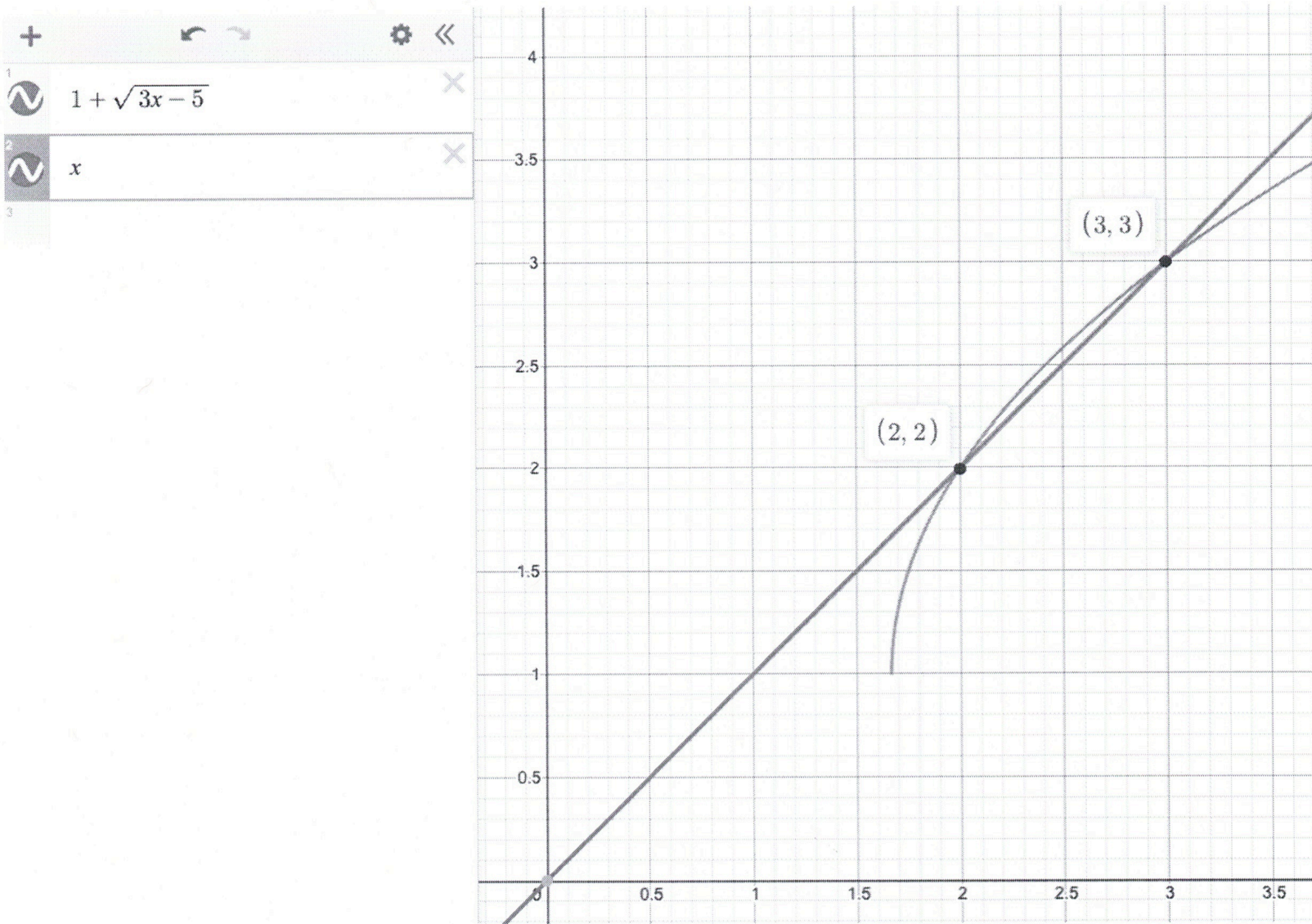
Check: $x = 3$ ✓

$$\begin{aligned} \text{Left-hand side: } 1 + \sqrt{3x-5} &= 1 + \sqrt{3 \cdot 3 - 5} \\ &= 1 + \sqrt{9-5} \\ &= 1 + \sqrt{4} \\ &= 1 + 2 \\ &= 3 = x \leftarrow \text{right-hand side} \end{aligned}$$

5:34pm - 5:37pm

Problem 11 B, continued...

We can also check our work graphically using Desmos.com. We see two points of intersection at $(2, 2)$ and $(3, 3)$



\Rightarrow two solutions at

$x = 2$ and $x = 3$.

12. FIND THE INVERSE FUNCTION (Lesson 14)

Use algebra to find the inverse of the following functions.

12A. $f(x) = \frac{4x-2}{3x+1}$

12B. $g(x) = \frac{6-x}{5}$

To find the desired inverse functions, we switch the x - and y -values and then isolate y in terms of x .

Problem 12A

We start with original problem

$$y = \frac{4x-2}{3x+1}$$

and then switch the x - and y -variables

$$x = \frac{4y-2}{3y+1}$$

$$\Rightarrow x(3y+1) = 4y-2$$

Problem 12A, continued...

$$\Rightarrow \begin{array}{r} 3xy + \boxed{x} = 4y - 2 \\ -3xy \qquad +2 \qquad +2 - 3xy \end{array}$$

$$\Rightarrow \boxed{x+2} = 4y - 3xy$$

$$\Rightarrow \frac{\boxed{x+2}}{4-3x} = \frac{y(4-3x)}{(4-3x)}$$

$$\Rightarrow y = \frac{x+2}{4-3x} = f^{-1}(x)$$

To check our work, we will check to see if

$$f \circ f^{-1}(x) = x = f^{-1} \circ f(x)$$

Problem 12A, continued...

$$f \circ f^{-1}(x) = f(f^{-1}(x))$$

$$= f(y) \quad \text{where} \quad y = f^{-1}(x) = \frac{x+2}{4-3x}$$

$$= \frac{4y-2}{3y+1} \quad \left| \quad y = \frac{x+2}{4-3x} \right.$$

$$= \frac{\left[\frac{4 \cdot [x+2]}{4-3x} - 2 \right]}{\left[\frac{3 \cdot [x+2]}{4-3x} + 1 \right]}$$

$$= \frac{\left[\frac{4x+8}{4-3x} - \frac{2[4-3x]}{[4-3x]} \right]}{\left[\frac{3x+6}{4-3x} + \frac{[4-3x]}{[4-3x]} \right]}$$

Problem 12A, continued...

$$\Rightarrow f \circ f^{-1}(x) = \left[\frac{4x+8-8+6x}{4-3x} \right] \div \left[\frac{3x+6+4-3x}{4-3x} \right]$$

$$= \frac{10x}{\cancel{4-3x}} \cdot \frac{\cancel{4-3x}}{10}$$

$$= \frac{10x}{10}$$

$$= x$$

$$\Rightarrow f \circ f^{-1}(x) = x$$

$$\Rightarrow \boxed{f^{-1}(x) = \frac{x+2}{4-3x}} \quad \checkmark$$

12:50pm - 12:56pm