Name: Solutions Class #:

Math 48B, Quiz 2, Lessons 3 – 6: Polynomial Division and Zeros of a Polynomial

In your first draft solutions to this quiz, I encourage you to take extra space and make your work very easy to read. I might encourage you to write one solution per page. I want to focus your mind here on two goals. First, this is designed to help build understanding of the material. Second, as you write your solutions, think about creating a document that you can look back on and understand years into the future. In this way, your solutions can become a so-called second brain where you store math knowledge for future reference. For more about ideas on how to format your solutions, please take a look at Jeff's Conquering College Study Skills Activity 5.

1. REVIEW : REMAINDERS AND LONG DIVISION

1A. For each of the following expressions in column 1 of the table below, write the solutions to the division problem in column 2. Then, write this solution as a multiplication problem in column 3. The first row is done for you.

Column 1: Expression	Column 2: Division Problem	Column 3 Multiplication Problem
$\frac{31}{5}$	$\frac{31}{5} = 6 + \frac{1}{5}$	$6 \cdot 5 + 1 = 31$
$\frac{31}{7}$	$\frac{31}{7} = 4 + \frac{3}{7}$	$4 \cdot 7 + 3 = 31$
$\frac{45}{4}$	$\frac{45}{4} = 11 \text{ R } 1$	$11 \cdot 4 + 1 = 45$
$\frac{45}{6}$	$\frac{45}{6} = 7\frac{3}{6}$	$7 \cdot 6 + 3 = 45$

1B. For each entry in the table above from Problem 1A, develop a verbal (using both abuelita and nerdy language), visual, and symbolic representation of the work you used to solve each problem. I encourage you to use about one page per problem. For an example of how to develop verbal, visual, and symbolic representations, take a look at our Lesson 3 solutions or the LiveStream recording from that day's class.

2. REVIEW : LONG DIVISION

Describe in words, using both abuelita (intuitive) and nerdy (formal) language, the connection between the following two formulas:

Division Formula

$$\frac{N}{D} = q + \frac{r}{D} = q \text{ R } r$$

Multiplication Formula

$$D \cdot q + r = N$$

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3. REVIEW : LONG DIVISION

Solve each of the following problems using long division. Use all three of the remainder notations (see page 10 of the Lesson 3 handout) to state your results. Also, do your best to use sentences to explain what you are doing.

3A. $\frac{761}{3}$

3B. $\frac{373}{9}$

4. POLYNOMIAL DIVISION

Use polynomial long division to solve each of the following problems. Then, write the solution of each problem in two different forms, given by the equations below. For a reminder on this, see page 6 of the Lesson 5 Handout.

Division Equation

$$\frac{N(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}$$

Multiplication Equation

$$N(x) = D(x) \cdot q(x) + r(x)$$

4A. $\frac{x^3-4x^2-19x-14}{(x-7)}$

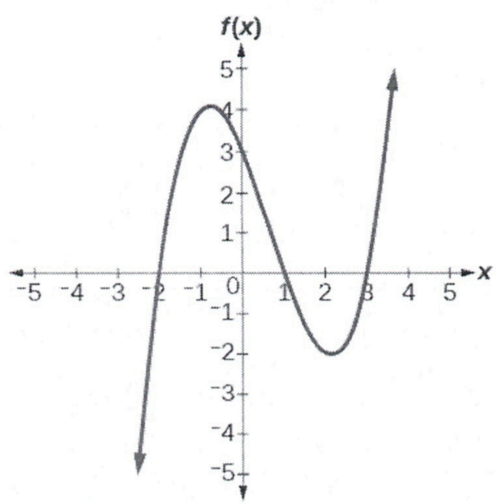
4B. $\frac{x^3-4x^2-19x-14}{(x-7)}$

5. ZEROS OF A POLYNOMIAL

Construct a polynomial $P(x)$ that has four zeros at $-1, 0, 2$, and $\frac{1}{2}$. Create both forms of this polynomial: the complete factorization form and also the standard form. When looking at the standard form, make sure that the degree three term x^3 has a coefficient of $a_3 = 3$. Using Desmos.com, create a graph to confirm that your polynomial has the desired zero points.

6. FIND THE FORMULA FOR A POLYNOMIAL

Use the graph below to write a formula for a polynomial function of least degree whose graph looks like the curve given in this figure. Use Desmos.com to confirm your conjecture. In other words, graph the formula you create to make sure your work aligns with the graph below. Then, discuss how this problem is related to the polynomial division problems given above.



Math 48B, Quiz 2, Problem 1A

Division Problem

Let's consider our division problem

Numerator

remainder

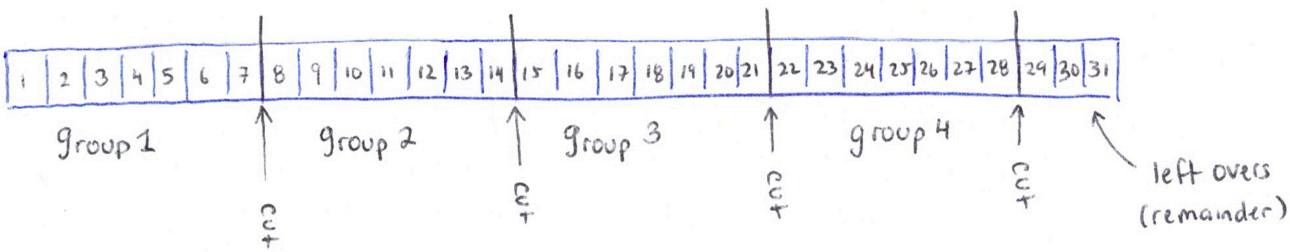
$$\frac{\boxed{31}}{\boxed{7}} = \boxed{4} + \frac{\boxed{3}}{7}$$

← this answer is written in addition notation

Denominator

quotient

Here, we are taking 31 units, represented by the 31 boxes drawn below, and breaking those boxes into equal-sized groups where we want each group to include 7 individual boxes.



Notice that when we break 31 boxes (units) into groups of size 7, we get 4 such groups containing a total of 28 boxes total. We also have three left-over boxes that don't form a complete group of size 7.

Problem 1A, continued ...

We can re-write the division problem, the act of cutting 31 units into groups of size 7 and then counting the number of groups that result. Specifically, we can think of the multiplicative analog:

$$\frac{31}{7} = 4 \text{ R } 3 \quad \Rightarrow \quad 31 = 4 \cdot 7 + 3$$

Group 1 →	1	2	3	4	5	6	7
Group 2 →	1	2	3	4	5	6	7
Group 3 →	1	2	3	4	5	6	7
Group 4 →	1	2	3	4	5	6	7
Extras →	1	2	3				

In this multiplication problem, we ask: How many total units do we have if we take 4 seperate groups, each with seven units inside and combine those together with an extra set of 3 units left over.

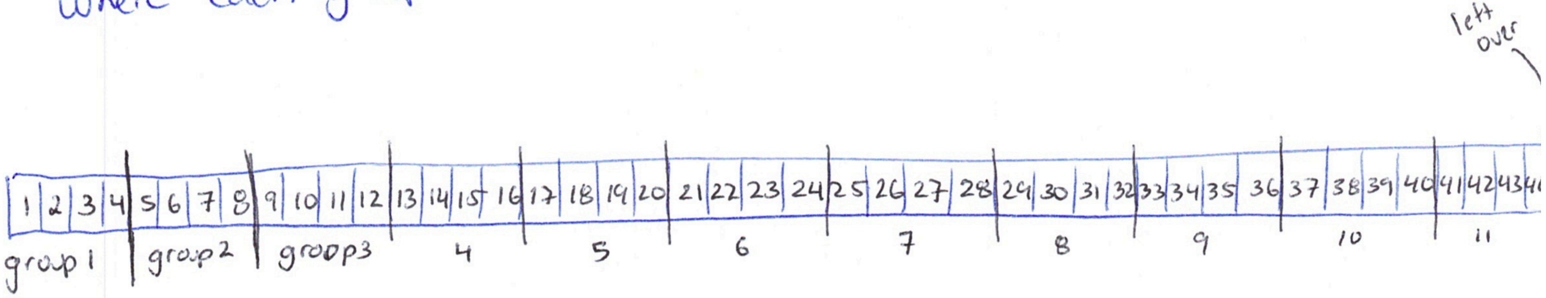
Problem 1A, continued...

Numerator   remainder

$$\frac{\boxed{45}}{\boxed{4}} = \boxed{11} + \frac{\boxed{1}}{4}$$

Denominator   quotient

Here we take 45 units and break into equal-sized groups where each group has 4 boxes



We've got a total of 11 groups, each having 4 boxes with one remaining box left over. In other words

$$\frac{45}{4} = 11 \text{ R } 1 \quad \Rightarrow \quad 45 = 11 \cdot 4 + 1$$

Problem 1A, continued ---

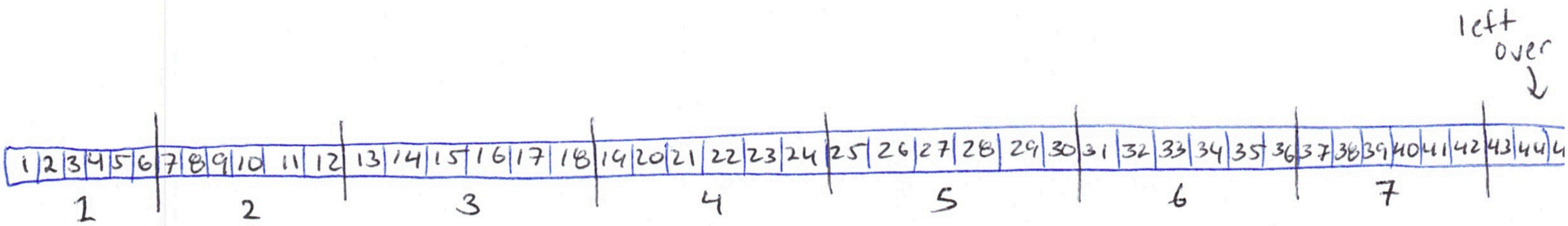
we keep going with our final division problem

Numerator Quotient remainder

$$\frac{\boxed{45}}{\boxed{6}} = \boxed{7} + \frac{\boxed{3}}{6}$$

Denominator

Here we take 45 units and cut into equal-sized groups each having 6 units inside. Then we ask how many such groups exist



We have a total of 7 groups each with 6 boxes and three left over units. In other words

$$\frac{45}{6} = 7 \text{ R } 3 \implies 45 = 7 \cdot 6 + 3$$

Problem 2

Let's consider the division formula:

remainder

numerator

$$\frac{N}{D} = q + \frac{r}{D}$$

Denominator

quotient

Intuitive language:

Take N units and cut into equal-sized groups each having D units. There will be a total of exactly q such groups with an additional r units left-over where $0 \leq r < D$.

Problem 2, continued --

Let's consider the multiplication formula

$$\underset{\text{numerator}}{N} = \overset{\text{quotient}}{q} \cdot \underset{\text{denominator}}{D} + \underset{\text{remainder}}{r}$$

Intuitive language:

Problem 3A: Long Division Review

Let's run the long division algorithm on our ratio

numerator

$$\frac{\boxed{761}}{\boxed{3}}$$

denominator

When we do so, we write the following

long division symbol

$$\boxed{3}\,\overline{\big)\,\boxed{7\ 6\ 1}}$$

numerator goes here

denominator goes here

we begin our long division algorithm by focusing on the most-significant digit of our denominator

$$3\,\overline{\big)\,\boxed{7}\ 6\ 1}$$

we start by looking at left-most digit

Problem 3A, continued...

Then we ask ourselves: how many copies of the denominator can be found inside this first digit of our numerator:

$$\begin{array}{r} \boxed{} \\ \boxed{3}\overline{)\boxed{7}\;6\;1} \end{array}$$

we write the answer to this probem here

How many copies of denominator 3 can be found inside left-most digit 7

In our case, we see that we have 2 copies of the number 3 inside 7 with one extra left over

1	2	3	4	5	6	7
1	2	3	1	2	3	1

left-over (remainder)

We write our answer in the box above the long division sign

$$\begin{array}{r} \boxed{2} \\ 3\overline{)\boxed{7}\;6\;1} \end{array}$$

there are two groups of size 3 in the number 7

Problem 3A, continued ...

Next, we track the remainder of the division problem from step 1

$$\begin{array}{rl} & 2 \\ \cline{2-2} 3 \,\big) & 7\ 6\ 1 \\ & -\boxed{6} \\ \cline{2-2} & \boxed{1} \end{array}$$

when we count our 2 groups of size 3, we get 6 units

here we have a remainder

The next step in our algorithm is to bring down the second-most significant digit in our number next to the remainder we found from our first digit

$$\begin{array}{rl} & 2 \\ \cline{2-2} 3 \,\big) & 7\ \boxed{6}\ 1 \\ & -6 \\ \cline{2-2} & 1\ 6 \end{array}$$

bring down this digit next to previous remainder

Problem 3A, continued ...

We now look at the combined remaider and ask ourselves: how many copies of the denominator can be found in this number

$$\begin{array}{r|l} & \ \ 2\ \boxed{} \\ \hline \boxed{3} & 1\ 7\ 6\ 1 \\ & -6 \\ \cline{2-2} & \ \ \boxed{1\ 6} \end{array}$$

We write our answer to this problem up here

How many copies of denominator 3 can be found inside this combined remainder

For this step, we see that we have 5 copies of the number 3 inside combined remainder 16 with one extra unit left over:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1

left-over (remainder)

Problem 3A, continued...

We write our answer in box above long division sign, as seen below

$$\begin{array}{rccc} & 2 & \boxed{5} & \\ \cline{2-4} 3 \big) & 7 & 6 & 1 \\ - & 6 & & \\ \cline{1-3} & \boxed{1} & \boxed{6} & \end{array}$$

there are five groups of size 3 inside the number 16

Next we track the remainder of the division problem from our last step

$$\begin{array}{rccc} & 2 & 5 & \\ \cline{2-4} 3 \big) & 7 & 6 & 1 \\ - & 6 & & \\ \cline{2-3} & 1 & 6 & \\ - & 1 & 5 & \\ \cline{2-3} & & \boxed{1} & \end{array}$$

when we count five groups of size three, we get 15 units

here we have our remainder

Problem 34, continued --

We continue by bringing down the next-most significant digit in our number and put this digit right next to our previous remainder

$$
\begin{array}{r|l}
 & \;\;2\;\;5 \\
\hline
3 & \;\;7\;\;6\;\;\boxed{1} \\
 & -6 \\
\hline
 & \;\;1\;\;6 \\
 & -1\;\;5 \\
\hline
 & \;\;\;\;\;\;1\;\;1
\end{array}
$$

bring down this digit next to previous remainder

We now look at this next "combined remainder" and ask how many copies of the denominator can be found inside this number

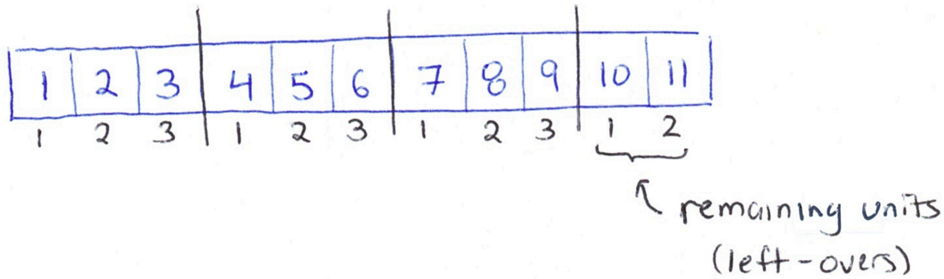
$$
\begin{array}{r|l}
 & \;\;2\;\;5\;\;\boxed{} \\
\hline
\boxed{3} & \;\;7\;\;6\;\;1 \\
 & -6 \\
\hline
 & \;\;1\;\;6 \\
 & -1\;\;5 \\
\hline
 & \;\;\;\;\;\;\boxed{1\;\;1}
\end{array}
$$

← we write our answer to this problem here

How many copies of denominator 3 can be found inside the combined remainder

Problem 3A, continued →

For this case, we have 3 copies of denominator 3 inside the combined remainder 11 with two extra units left over



We write our answer in the box above the long division sign, as seen below

$$\begin{array}{r} 25\boxed{3} \\ 3\,\overline{)\,7\,6\,1} \\ -6 \\ \hline 16 \\ -15 \\ \hline \boxed{11} \end{array}$$

there are three groups of size three inside the number 11

Problem 3A, continued --

Finally, we track the remainder of our division problem from our last step

$$
\begin{array}{r}
253 \\
3\overline{)761} \\
\underline{-6} \\
16 \\
\underline{-15} \\
11 \\
\underline{-9} \\
2
\end{array}
$$

← when we count three groups of size three, we get nine units

here we have our final remainder

We see we can write

$$\frac{761}{3} = \underbrace{253 + \frac{2}{3}}_{\text{addition notation}} = \underbrace{253\tfrac{2}{3}}_{\text{mixed number notation}} = \underbrace{253\,R\,2}_{\text{remainder notation}}$$

Long - Division Algorithm in Fraction Form

Problem 3A

$$\frac{761}{3} = \frac{\boxed{7}00}{3} + \frac{61}{3}$$

$$= \frac{600+100}{3} + \frac{61}{3}$$

$$= 200 + \frac{100}{3} + \frac{61}{3}$$

$$= 200 + \frac{100}{3} + \frac{60}{3} + \frac{1}{3}$$

$$= 200 + \frac{160}{3} + \frac{1}{3}$$

$$= 200 + \frac{150}{3} + \frac{10}{3} + \frac{1}{3}$$

$$= 200 + 50 + \frac{11}{3}$$

15

$$= 200 + 50 + \frac{9}{3} + \frac{2}{3} = \boxed{200+50+3+\frac{2}{3}}$$

✓

Problem 3B: Long Division Review

Consider the following long division problem

$$
\begin{array}{r|rrr}
 & 0 & 4 & 1 \\
\hline
9 & 3 & 7 & 3 \\
 & -0 & & \\
\cline{2-2}
 & 3 & 7 & \\
 & -3 & 6 & \\
\cline{2-3}
 & & 1 & 3 \\
 & & & 9 \\
\cline{3-4}
 & & & 4
\end{array}
$$

We can now write

$$\frac{373}{9} = \underbrace{41 + \frac{4}{9}}_{\text{addition notation}} = \underbrace{41\,{}^{4}\!/_{9}}_{\text{mixed-number notation}} = \underbrace{41\,R\,4}_{\text{remainder notation}}$$

Problem 4A: Polynomial Long Division

Let's execute our polynomial long division algorithm from Lessons 3 – 4. Consider

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x - 7 & x^3 - 4x^2 - 19x - 14 \\ & -x^3 + 7x^2 \\ \hline & 0 + 3x^2 \\ & \quad -3x^2 + 21x \\ \hline & \quad 0 + 2x - 14 \\ & \quad\quad -2x + 14 \\ \hline & \quad\quad 0 + \boxed{0} \end{array}$$

remainder

Notice we want to put in form

$$\frac{N}{D} = q + \frac{r}{D}$$

Problem 4A

$$\frac{N(x)}{D(x)} = \frac{x^3 - 4x^2 - 19x - 14}{x - 7} = \underbrace{x^2 + 3x + 2}_{Q(x)} + \frac{\overset{r(x)}{0}}{x-7}$$

$$\Rightarrow \quad x^3 - 4x^2 - 19x - 14 = (x-7)(x^2+3x+2)$$

$$\Rightarrow \quad x^3 - 4x^2 - 19x - 14 = (x-7)(x+1)\cdot(x-2)$$

Check: $(x-7)\cdot(x+1)\cdot(x+2)$

$$= (x^2 + x - 7x - 7)\cdot(x+2)$$

$$= (x^2 - 6x - 7)(x+2)$$

$$= x(x^2 - 6x - 7) + 2(x^2 - 6x - 7)$$

$$= x^3 - 6x^2 - 7x + 2x^2 - 12x - 14$$

$$= x^3 - 4x^2 - 19x - 14 \checkmark$$

Problem 4B)

Use polynomial long division to find

$$\frac{9x^3 - x + 2}{3x - 1}$$

Solution: The first thing we do is to write our denominator in standard form

$$
\begin{array}{r|l}
 & 3x^2 + x \\
\hline
3x - 1 & 9x^3 + 0x^2 - x + 2 \\
 & -\;9x^3 + 3x^2 \\
\cline{2-2}
 & \quad 3x^2 - x \\
 & \quad -\;3x^2 + x \\
\cline{2-2}
 & \qquad 0 + \boxed{2}
\end{array}
$$

Remainder

Claim: $$\frac{9x^3 - x + 2}{3x - 1} = \underbrace{(3x^2 + x)}_{q(x)} + \frac{\overset{r(x)}{2}}{3x-1}$$

$$\Rightarrow \quad 9x^3 - x + 2 = (3x - 1)(3x^2 + x) + 2$$

Check: Let's check our work

$$(3x-1)(3x^2+x)+2 = 3x(3x^2+x) - 1(3x^2+x) + 2$$

$$= 9x^3 + 3x^2 - 3x^2 - x + 2$$

$$= 9x^3 - x + 2 \checkmark$$

Thus, we see that

$$\frac{N(x)}{D(x)} = \frac{9x^3 - x + 2}{3x - 1} = (3x^2 + x) + \frac{2}{3x-1}$$ Division equation

$$\Leftrightarrow \quad N(x) = 9x^3 - x + 2 = (3x-1)(3x^2+x) + 2$$

$$= (3x-1)(3x+1)\cdot x + 2$$

$$= x\cdot(3x+1)(3x-1) + 2$$

Multiplication Equation

Problem 5: Zeros of a Polynomial

Let's start with a Polynomial $f(x)$ which has zeros given by

$$x=-1, \quad x=0, \quad x=2, \text{ and } x=\frac{1}{2}$$

We can create the complete factorization form of this polynomial by getting the right-hand sides of each equality to zero with

$$\underbrace{x+1}_{\text{factor 1}}=0, \quad \underbrace{x}_{\text{factor 2}}=0, \quad \underbrace{x-2}_{\text{factor 3}}=0, \quad \underbrace{2x-1}_{\text{factor 4}}=0$$

If we multiply each of these four factors together, we create the desired polynomial.

Problem 5, continued ...

$$f(x) = \underbrace{x \cdot (x+1)}_{\downarrow} \cdot \underbrace{(x-2)\cdot(2x-1)}_{\downarrow}$$

$$= (x^2 + x) \cdot (2x^2 - x - 4x + 2)$$

$$= x^2(2x^2 - 5x + 2) + x(2x^2 - 5x + 2)$$

$$= 2x^4 - 5x^3 + 2x^2 + 2x^3 - 5x^2 + 2x$$

$$= 2x^4 - 3x^3 - 3x^2 + 2x$$

Then, we claim that polynomial

$$f(x) = 2x^4 - 3x^3 - 3x^2 + 2x$$

$b_4 = 2$, $b_3 = -3$, $b_2 = -3$, $b_1 = 2$, $b_0 = 0$

has zeros at $-1, 0, 2,$ and $\frac{1}{2}$.

Problem 5, continued...

Recall, we want to construct a polynomial $P(x)$

with standard form

$$P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 x^0$$

where coefficient $a_3 = +3$. To do this, let

$$P(x) = -1 \cdot f(x)$$

$$\Rightarrow P(x) = -1 \cdot (2x^4 - 3x^3 - 3x^2 + 2x)$$

$$\Rightarrow \boxed{P(x) = -2x^4 + 3x^3 + 3x^2 - 2x}$$

$$\Rightarrow \boxed{P(x) = -1 \cdot x \cdot (x+1) \cdot (x-2) \cdot (2x-1)}$$

We can verify the behavior of this polynomial graphically using Desmos.com.

Problem 5, continued ...

Below is the graph of our desired polynomial

$$P(x) = -2x^4 + 3x^3 + 3x^2 - 2x$$ ← standard form

$$= -x \cdot (x+1) \cdot (x-2) \cdot (2x-1)$$ ← complete factorization form

Notice this polynomial satisfies all our desired conditions

☑ Has zeros at $-1, 0, 2,$ and $\frac{1}{2}$

☑ The value of $a_3 = +3$

Problem 6 : Find the formula for a Polynomial

We see from the Problem statement that the polynomial graph has three zeros at points

(-2, 0) (1, 0) and (3, 0)

$\Rightarrow \quad x = -2 \quad , \quad x = 1 \quad , \quad x = 3$

$\Rightarrow \quad \underbrace{x+2}_{\text{factor 1}} = 0 \quad , \quad \underbrace{x-1}_{\text{factor 2}} = 0 \quad , \quad \underbrace{x-3}_{\text{factor 3}} = 0$

We can create a first-guess function by multiplyng these zero factors together:

$$\underbrace{g(x)}_{\substack{\uparrow \\ \text{first guess:} \\ \text{(in factored form)}}} = (x+2)\cdot(x-1)(x-3)$$

$$= (x^2 - x + 2x - 2)(x-3)$$

$$= (x^2 + x - 2)(x-3)$$

$$= x(x^2 + x - 2) - 3(x^2 + x - 2)$$

Problem 6, continued ...

$$\Rightarrow \quad g(x) = x^3 + x^2 - 2x - 3x^2 - 3x + 6$$

$$= x^3 - 2x^2 - 5x + 6$$

Notice that the y-intercept of our first guess happens where $x = 0$ at point $(0, 6)$ since

$$g(0) = 6$$

But, our desired graph has y-intercept at point

$$(0, 3)$$

We can create our function $f(x)$ by multiplying $g(x)$ by scalar a with

$$f(x) = a \cdot g(x) \quad \Rightarrow \quad 3 = a \cdot 6$$

$$\Rightarrow \quad a = \frac{1}{2}$$

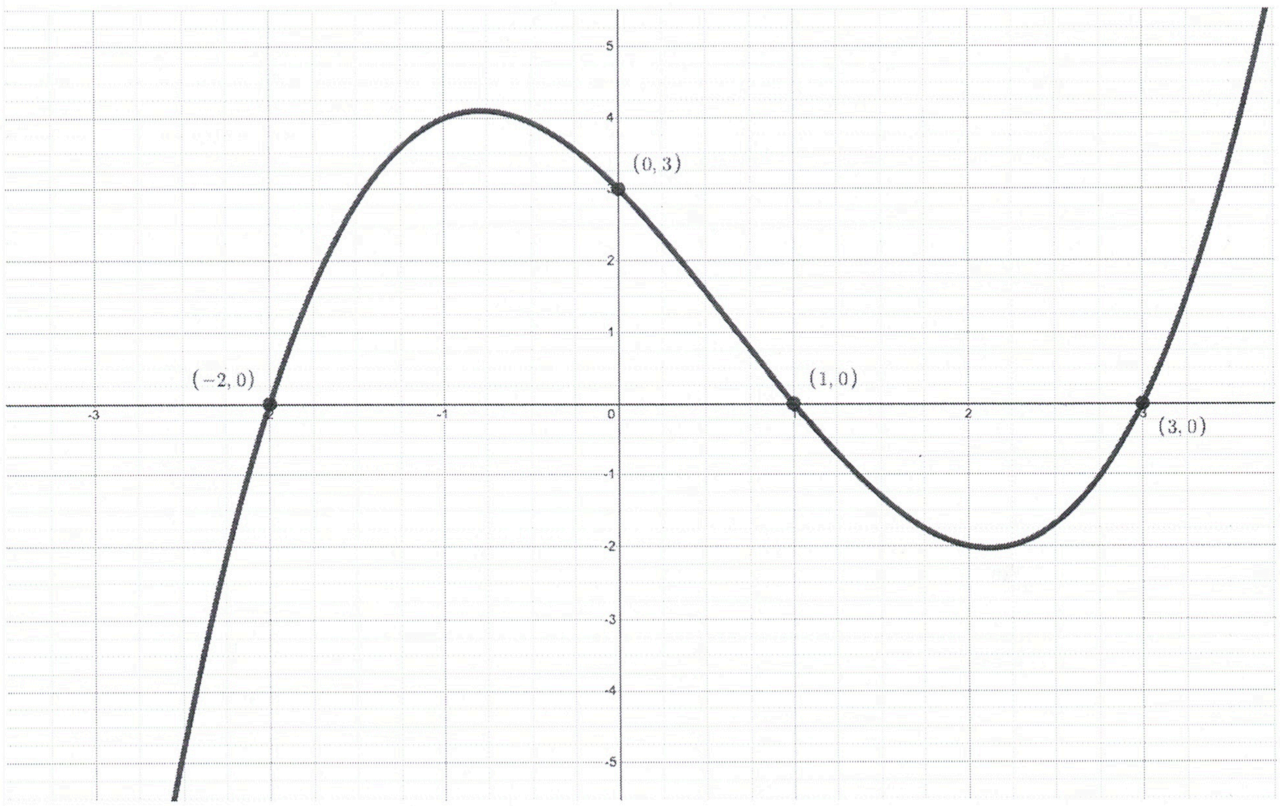
Problem 6, continued ...

We can write

$$f(x) = \frac{1}{2}(x+2)\cdot(x-1)\cdot(x-3)$$

$$\Rightarrow f(x) = \frac{1}{2}x^3 - x^2 - \frac{5}{2}x + 3$$

The graph this function below.



We see that this graph matches the desired figure from the problem statement.