Free Response

1. (16 points) Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a two-variable function with explicit representation $z = f(x, y)$. Let $A\big(a, b, f(a, b)\big)$ be a point on the surface $z = f(x, y)$. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the domain of function f.

   A. Using the 5 steps process to constructing a derivative that we discussed in our Lesson 11 videos, derive the limit definition of the directional derivative.

Solution: Recall the 5 step process for constructing derivative included each of the following:

I. Graph a curve C

In order to create the curve on which we will plot our tangent line, we begin with the graph of the surface defined by the explicit equation $z = f(x, y)$. To create the curve C along the surface, we restrict our input points in the domain to move along the line

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t \cdot \mathbf{u}, \\ &= \langle a, b \rangle + t \langle u_1, u_2 \rangle, \\ &= \langle a + tu_1, b + tu_2 \rangle, \\ &= \langle x(t), y(t) \rangle,\end{aligned}$$

where $x(h) = a + tu_1$ and $y(h) = b + tu_2$. This is equivalent to intersecting the surface $z = f(x, y)$ with a plane through the point $A\big(a, b, f(a, b)\big)$ with normal vector $\mathbf{n} = \langle -u_2, u_1, 0 \rangle$. This results in a single-variable function given by

$$g(t) = f(x(t), y(t)) = f\big(a + tu_1, b + tu_2\big).$$

Below, we visualize this curve and the points from step 2 of this process.

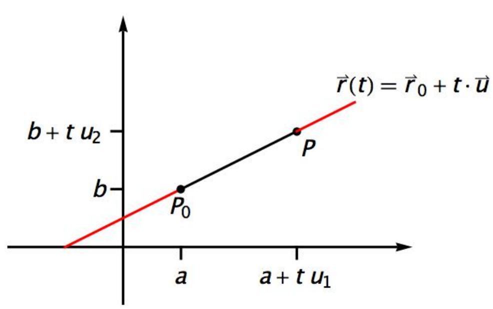


Figure 1A: Restricted domain inputs

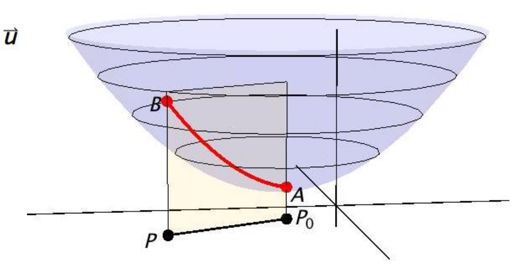


Figure 1B: Resulting curve on surface

Solution:

II. Find two points on the curve and draw a secant line between these two points.

We now find two points on the curve C. Since we will be finding the derivative at the point $A\big(a, b, f(a, b)\big)$, we start by noticing that the output value on the surface at this point is given by

$$g(0) = f(x(0), y(0)) = f(a, b).$$

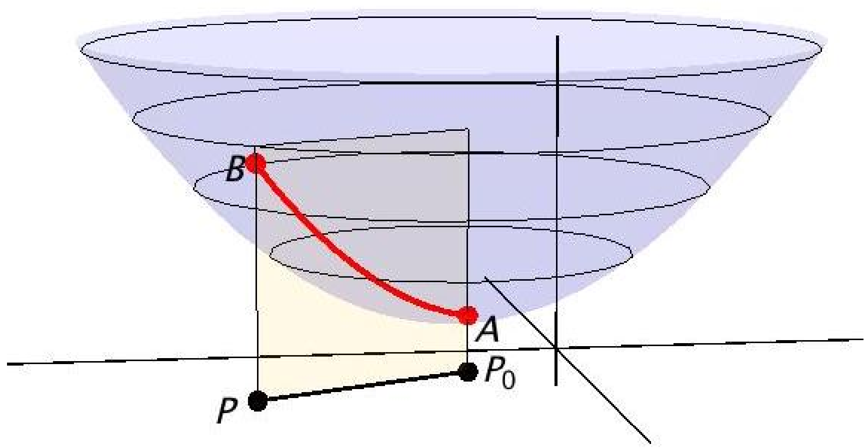
If we assume that $h \in \mathbb{R}$ with $h \neq 0$, we can get that output value of another point on the curve C by evaluating

$$g(h) = f(x(h), y(h)) = f\big(a + hu_1, b + hu_2\big)$$

This yields two points A and B on the curve C with coordinates are given by

$$A\big(a, b, f(a, b)\big) \qquad \text{and} \qquad B\big(a + hu_1, b + hu_2, f(a + hu_1, b + hu_2)\big)$$

As discussed before, we see the two points on the surface in the visual below:



Solution:

III. Measure the slope of the secant line.

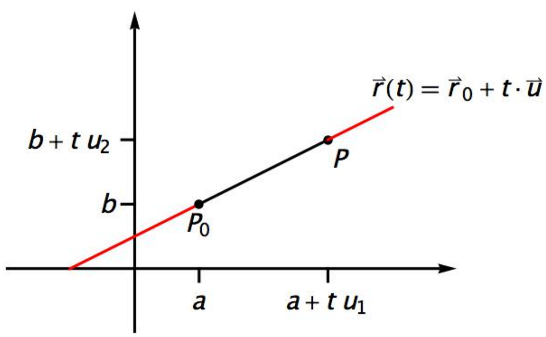
To measure the slope m_{AB} of the secant line through the points A and B, recall that we say that the slope

$$m_{AB} = \frac{\text{change in output}}{\text{signed `distance' traveled in input}}$$

We can calculate the change in output values on the surface to be given by

$$\text{change in output} = g(h) - g(0) = f\big(a + hu_1, b + hu_2\big) - f(a, b).$$

On the other hand, the signed 'distance' traveled in the input requires some deeper thought. To this end, consider the diagram below:



When moving from point P_0 to point P in the domain, we notice that the scalar h encodes both the magnitude and orientation of this movement. In other words, we see that the nonnegative distance traveled when moving from point P_0 to point P is given by the magnitude:

$$\left\|\overrightarrow{P_0P}\right\|_2 = \|h \cdot \langle u_1, u_2\rangle\|_2 = |h| \cdot \|\mathbf{u}\|_2 = |h|.$$

The fact that the length of this vector is the value of the scalar h directly results from our assumption that $\mathbf{u}$ is a unit vector. To get the signed 'distance' traveled, we remember that in producing the point P, we only required that $h \neq 0$. This corresponds to two scenarios: a positive scalar $h > 0$ or a negative scalar $h > 0$. In each case, the signed 'distance' will just be the value of h. This results in a slope of the secant line through the points A and B given by

$$m_{AB} = \frac{g(h) - g(0)}{h} = \frac{f\big(a + hu_1, b + hu_2\big) - f(a, b)}{h}$$

IV. Transform the secant line into a tangent line using a limit.

V. Construct the "derivative" as the slope of a tangent line.

We recall that we can force point B toward point A by forcing point P to point P_0 in the domain. In particular, we can measure the slope of the tangent line between these points as the following limit:

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f\big(a + hu_1, b + hu_2\big) - f(a, b)}{h}$$

B. Use the multivariable chain rule with two intermediate variables and one independent variables to derive the dot product formula for the directional derivative.

Solution: By construction, we see that the limit definition of the directional derivative in part A above is given as

$$D_{\mathbf{u}} f(a,b) = \lim_{h\to 0} \frac{f\big(a+hu_1, b+hu_2\big) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

Using ordinary derivative notation, we see this is equivalent to taking the ordinary derivative of the single-variable function

$$g'(0) = \frac{d}{dt}\,\Big[g(t)\Big]\Big|_{t=0}$$

Using the multivariable chain rule, we know

$$\begin{aligned}
\frac{d}{dt}\Big[g(t)\Big]\Big|_{t=0} &= \frac{d}{dt}\Big[f\big(x(t), y(t)\big)\Big]\Big|_{t=0} \\
&= \left[\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right]\Big|_{t=0} \\
&= f_x(a,b)\cdot x'(0) + f_y(a,b)\cdot y'(0) \\
&= f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2 \\
&= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle \\
&= \nabla f(a,b)\cdot \mathbf{u}
\end{aligned}$$

This gives us an alternative method to calculate the directional derivative without requiring limits.

C. Use the cosine formula for the dot product to explain which unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ gives the direction of steepest ascent on the surface. Please explain your reasoning.

Solution: By combining part B above with the cosine formula for the dot product we see

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \|\nabla f(a,b)\|_2 \cdot \|\mathbf{u}\|_2 \cdot \cos(\theta) = \|\nabla f(a,b)\|_2 \cdot \cos(\theta)$$

where θ is the angle between the vectors $\nabla f(a,b)$ and $\mathbf{u}$. We see this derivative has maximum value when $\theta = 0$. In other words, the slope of this tangent line is maximum when we move in the direction of the gradient vector.

2. (12 points) Let $f(x, y) = x^2 + y^2 - 4x$.

A. Find a vector-valued equaton for the tangent line to the level curve $L_1(f) = \{(x, y) : f(x, y) = 1\}$ at the point $(1, 2)$.

Solution: If $D = \mathrm{Dom}(f)$, then we notice that the level curve $L_1(f) \subseteq D \subseteq \mathbb{R}^2$. To find the vector-valued equation of the tangent line to $L_1(f)$ given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v}$$

where $\mathbf{r}_0 \in \mathbb{R}^2$ is a point on the line and $\mathbf{v} \in \mathbb{R}^2$ represents the direction of the line. By the problem statement, we know that $\mathbf{r}_0 = \langle 1, 2\rangle$. To find the "direction" of this line, we will use implicit differentiation:

$$\begin{aligned} \frac{d}{dx}\Big[x^2 + y^2 - 4x\Big] = \frac{d}{dx}\Big[1\Big] \qquad &\Longrightarrow \qquad 2x + 2y \cdot \frac{dy}{dx} - 4 = 0 \\ &\Longrightarrow \qquad \frac{dy}{dx} = \frac{2-x}{y} \\ &\Longrightarrow \qquad \frac{dy}{dx}\Big|_{(1,2)} = \frac{1}{2} \\ &\Longrightarrow \qquad \mathbf{v} = \langle 2, 1\rangle \end{aligned}$$

Using this calculation, we find

$$\mathbf{r}(t) = \langle 1, 2\rangle + t \cdot \langle 2, 1\rangle = \langle 1 + 2t\, , \, 2 + t\rangle$$

(Problem 2 continued on next page)

B. Show that the gradient $\nabla f(1,2)$ is orthogonal to the direction of the line you found in part A above.

Solution: Let us calculate the gradient at a general point:

$$\nabla f(x,y) = \langle f_x(x,y)\,,\ f_y(x,y)\rangle$$

$$= \langle 2x - 4\,,\ 2y\rangle$$

We can evaluate the gradient at the given input point:

$$\nabla f(1,2) = \langle -2\,,\ 4\rangle$$

The dot product between the gradient and the direction vector for the tangent line:

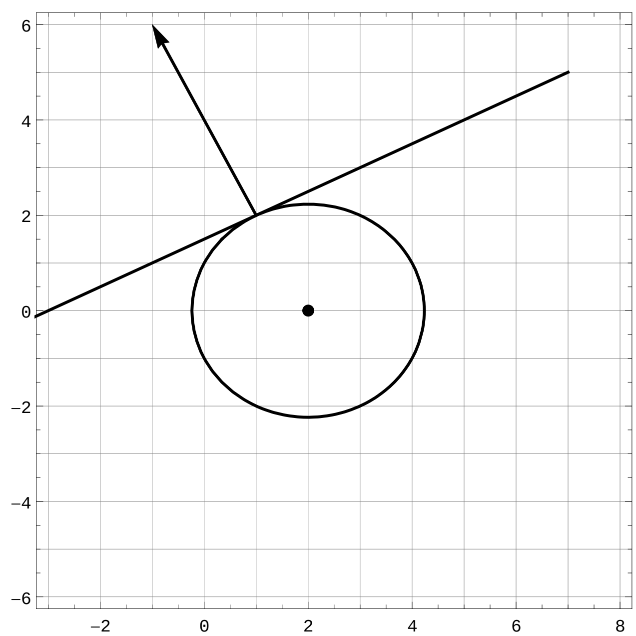
$$\nabla f(1,2)\ \cdot\ \mathbf{v} = \langle -2\,,\ 4\rangle\ \cdot\ \langle 2\,,\ 1\rangle$$

$$= -2\cdot 2 + 4\cdot 1$$

$$= 0$$

We see that the gradient is orthogonal to the vector $\mathbf{v}$. This is what was to be shown.

C. Sketch the level curve, the tangent line, and the gradient vector from parts A. and B. on the axis below.



3. (12 points) Using the second partial derivative test, find the minimum distance from the point $P(1,-2,4)$ to the plane $3x+2y+6z=5$. Please explain your answer and specifically identify the steps you took to arrive at your final answer. NOTE: To earn full credit, you must use the second partial derivative test (NOT projections or any other method).

Solution: Let $Q(x,y,z)$ be any point on the plane. Since Q is on the plane, we know by definition that

$$z = \frac{5-3x-2y}{6}.$$

We define a two-variable objective function

$$f(x,y) = \left\|\overrightarrow{PQ}\right\|_2^2 = (x-1)^2+(y+2)^2+\left(\frac{5-3x-2y}{6}-4\right)^2$$

$$= (x-1)^2+(y+2)^2+\left(\frac{19+3x+2y}{6}\right)^2$$

We know that an extreme value of this function will occur only where $\nabla f(x,y) = \mathbf{0}$. To this end, let us consider

$$\nabla f(x,y) = \left\langle \frac{15x+2y+7}{6}, \frac{3x+20y+55}{9} \right\rangle = \langle 0,0\rangle$$

Solving this linear system of equations, we see

$$x = -\frac{5}{49}, \qquad y = -\frac{134}{49}$$

Then, substituting this into the function $f(x,y)$ and taking the square root, we see that the minimum distance from the point to the plane is

$$\frac{18}{7}$$

4. (10 points) Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

Solution: Using the method of Lagrange Multipliers, we set up the following system of 3 equations in 3 unknowns:

Equation 1:	$2x = \lambda 2x$
Equation 2:	$4y = \lambda 2y$
Equation 2:	$x^2 + y^2 = 1$

Starting with equation 1, we see that

$$\lambda 2x - 2x = 2x \cdot (\lambda - 1) = 0$$

This corresponds to either $x = 0$ or $\lambda = 1$.

Case I: $x = 0$

By equation 3, we see this corresponds to two points on the constraint curve given by

$$(0, 1) \qquad (0, -1)$$

Case II: $\lambda = 1$

By equation 2, we see this corresponds to two points on the constraint curve given by

$$(1, 0) \qquad (-1, -0)$$

With this, we see that the first two points are maximum values and the second two are maximum values on this constraint curve.

Challenge Problem

5. (Optional, Extra Credit, Challenge Problem) Suppose that the general form of a tangent quadratic approximation to a function $f(x, y)$ at point (α, β) is given by

$$ax^2 + bxy + cy^2.$$

Using this information, explain each of the four conclusions of the second partial derivative test based on the behavior of the quadratic approximation. Make explicit connections to the scalar values of a, b, c and the geometric interpretations of these values based on the behavior of the corresponding quadratic surfaces.

