Video: Lesson 12.3 4:36pm - 5:17pm

1. (10 points) Suppose a surface in $\mathbb{R}^3$ is defined using equation

$$F(x, y, z) = 0$$

where the function $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable for all input values in the domain D. Suppose point $P_0(x_0, y_0, z_0)$ is on this surface so that $F(x_0, y_0, z_0) = 0$. Derive the equation for the tangent plane to this surface at the point P_0. In your derivation, you can assume that the derivative of the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ without having to show this fact. For top scores, your answer should look different from Jeff's work in many ways and should demonstrate your mastery of this concept. Please do your best to provide evidence that your concept image includes multiple categories of knowledge including verbal, graphical, and symbolic representations of the ideas used in this derivation.

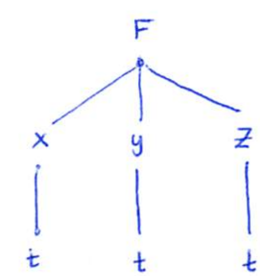
Let $S = \{(x,y,z) : F(x,y,z) = 0\}$ be the surface created by using the equation above (see next page).

Suppose $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve on the surface. that goes through the point $P_0(x_0, y_0, z_0)$ at $t = t_0$.

Define the single-variable composite function

$$g(t) = F(\vec{r}(t)) = F(x(t), y(t), z(t)) = 0$$

Then, we know

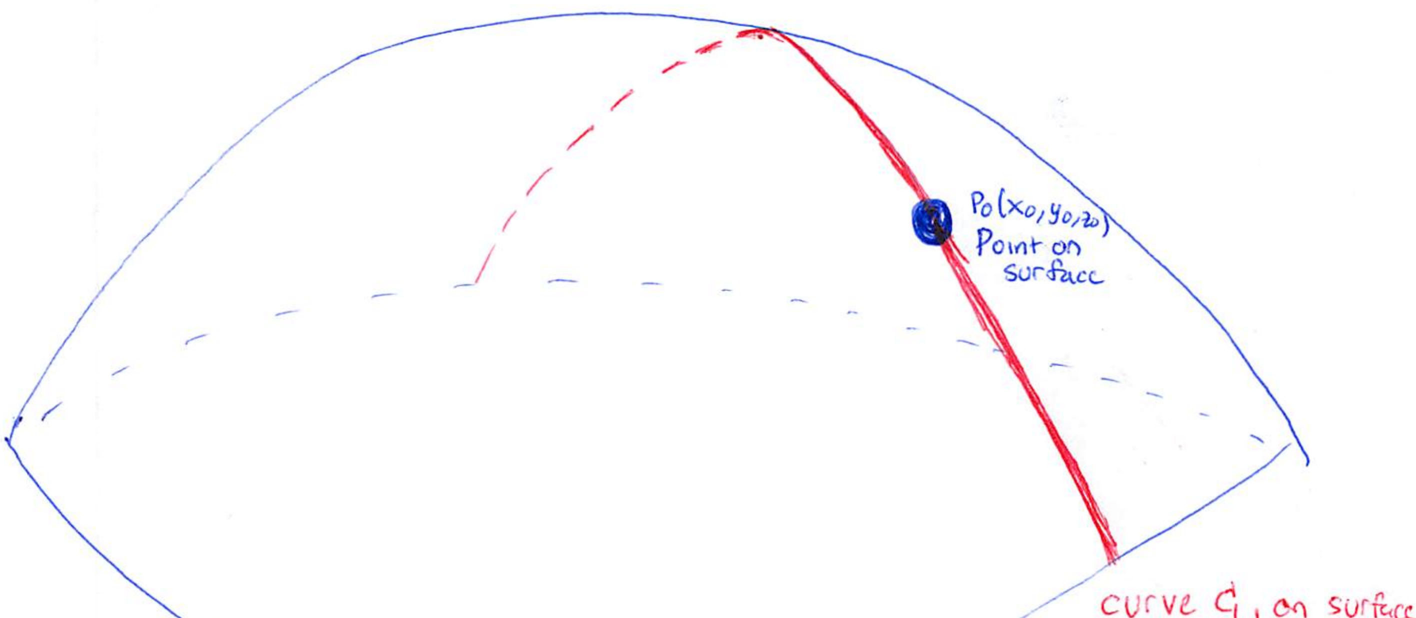


$$g'(t_0) = \frac{d}{dt}\left[g(t)\right]\Big|_{t=t_0}$$

$$= \frac{d}{dt}\left[F(x(t), y(t), z(t))\right]\Big|_{t=t_0} = \frac{d}{dt}[0]$$

$$\Rightarrow \left[\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt}\right]\Bigg|_{t=t_0} = 0$$

Let's visualize our surface S below. Notice our point $P_0(x_0, y_0, z_0)$ is on the surface and the parameterized curve $C_1 = \{ \vec{r}_1(t) : F(\vec{r}_1(t)) = 0 \}$ is also on surface



$C_1 = \{ (x(t), y(t), z(t)) : F(\vec{r}(t)) = 0 \}$

where $\vec{r}_1{}'(t_0) = \langle x_0, y_0, z_0 \rangle$

$$\Rightarrow \quad \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \Bigg|_{t=0} = 0$$

$$\Rightarrow \quad \vec{\nabla} F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$$

$$\Rightarrow \quad \vec{\nabla} F(x_0, y_0, z_0) \perp \vec{r}\,'(t_0) = 0$$

But since $\vec{r}(t)$ was any curve on the surface and since the collection of tangent line to these curves forms the tangent plane, we know the normal vector to the tangent plane of our surface at point $P_0(x_0, y_0, z_0)$ is in the direction of the gradient

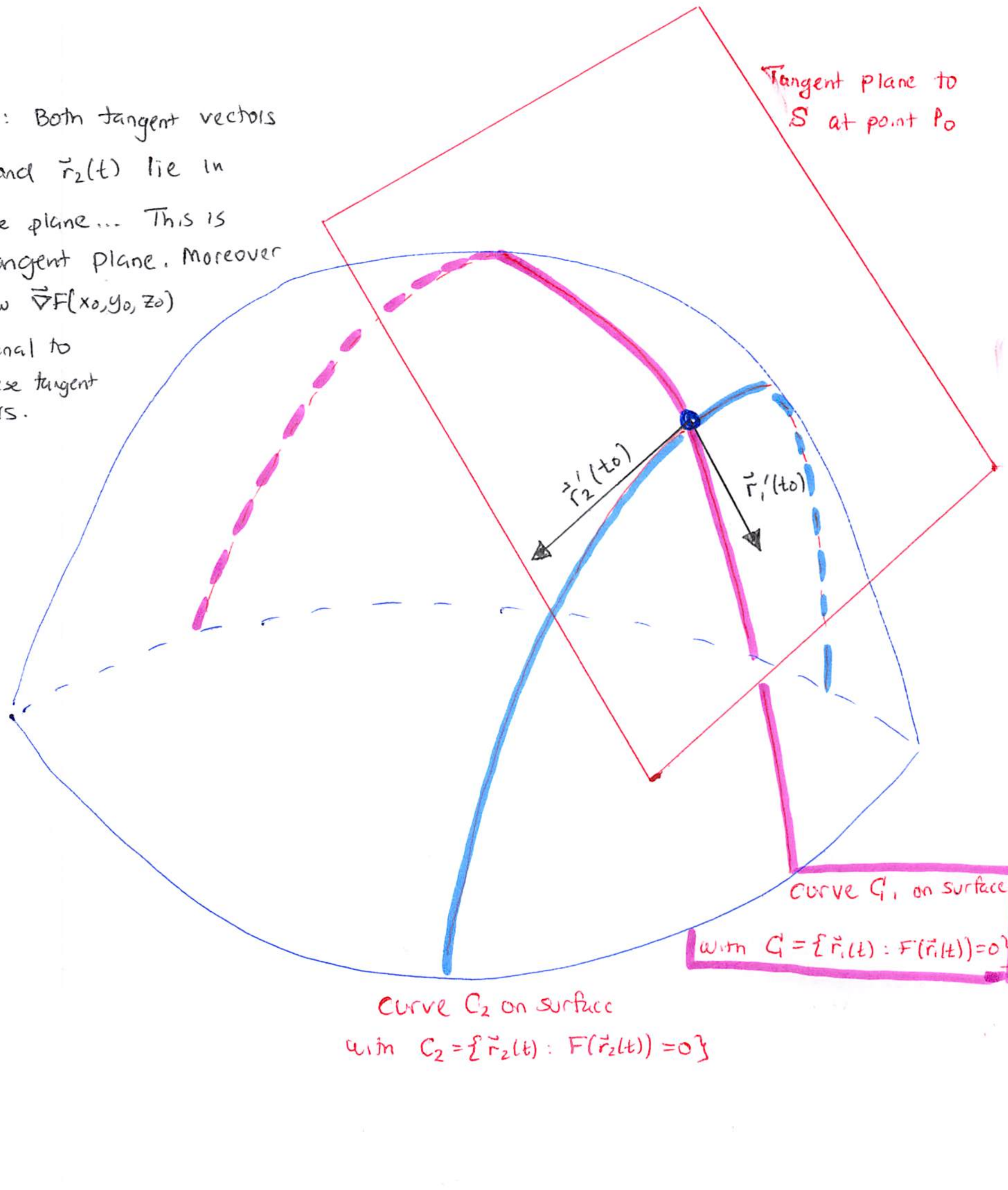
$$\vec{n} = \vec{\nabla} F(\vec{r}(t_0)) = \vec{\nabla} F(x_0, y_0, z_0)$$

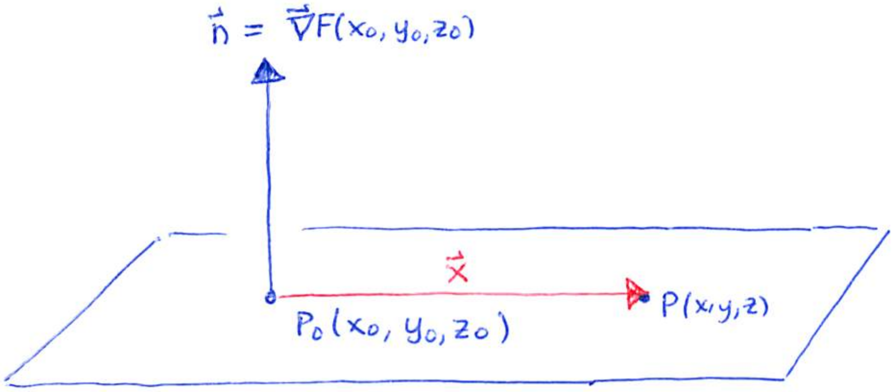
Then, we have both pieces of information that we need.

For a visual representation, see next page.

Now let's visualize the tangent plane to our surface at the point $P_0(x_0, y_0, z_0)$

□ Notice: Both tangent vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$ lie in the same plane... This is the tangent plane. Moreover we know $\vec{\nabla}F(x_0, y_0, z_0)$ is orthogonal to both these tangent vectors.





Where $P(x,y,z)$ is any variable point on the plane,

the vector $\vec{x} = \overrightarrow{P_0P}$, and $\vec{n} = \vec{\nabla}F(x_0, y_0, z_0)$. Then, we have

$$\vec{n} \perp \vec{x} \Rightarrow \vec{n} \cdot \vec{x} = 0$$

$$\Rightarrow \vec{\nabla}F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\Rightarrow \langle F_x, F_y, F_z \rangle \Big|_{P_0} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

This results in our equation for the tangent plane:

$$F_x(x_0, y_0, z_0) \cdot (x - x_0) + F_y(x_0, y_0, z_0) \cdot (y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

2. (6 points) Find the point(s) on the surface

$$x^2 - 2x + 2y^2 + z^2 - 2z = 2$$

with a horizontal tangent plane.

Using our work from problem 1, we want to get RHS equal to zero

$$x^2 - 2x + 2y^2 + z^2 - 2z - 2 = 0$$

Let $F(x,y,z) = x^2 - 2x + 2y^2 + z^2 - 2z + 2$. We know the normal vector to the tangent plane is

$$\vec{\nabla} F(x,y,z) = \langle 2x-2,\ 4y,\ 2z-2 \rangle$$

In order for the tangent plane to be horizontal, we need

$$\vec{\nabla} F(x,y,z) \parallel \langle 0,0,1 \rangle$$

$$\Rightarrow \langle 2x-2,\ 4y,\ 2z-2 \rangle = \alpha \langle 0,0,1 \rangle$$

$$\Rightarrow 2x-2 = 0 \quad \text{and} \quad 4y = 0 \quad \text{and} \quad 2z-2 = \alpha \cdot 1$$

$$\Rightarrow x = 1 \quad \text{and} \quad y = 0$$

$\Rightarrow \quad 1-2+0+z^2-2z=2$

$\Rightarrow \quad -1+0+z^2-2z=2$

$\Rightarrow \quad z^2-2z-3=0$

$\Rightarrow \quad (z-3)\cdot(z+1)=0$

$\Rightarrow \quad z=3 \quad$ or $\quad z=-1$

$\Rightarrow$ we have two points on surface w/ a horizontal tangent plane at

$$\boxed{\begin{array}{ll} \text{point 1:} & (1,0,3) \\ \text{point 2:} & (1,0,-1) \end{array}}$$

For problems 3 - 4, let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a two-variable function with explicit representation $z = f(x,y)$. Let $A(a,b,f(a,b))$ be a point on the surface $z = f(x,y)$. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the domain of function f. Suppose you have derived the limit definition for the directional derivative of f in the direction of $\mathbf{u}$ at the point (a,b), given by

$$D_{\mathbf{u}}f(a,b) = \lim_{h\to 0} \frac{f\big(a + hu_1, b + hu_2\big) - f(a,b)}{h}$$

3. (6 points) Show how to construct a composite function $g(t)$ that enables us to use the multivariable chain rule with two intermediate variables and one independent variables to derive the dot product formula for the directional derivative.

Solution: By construction, we see that the limit definition of the directional derivative in part A above is given as

$$D_{\mathbf{u}}f(a,b) = \lim_{h\to 0} \frac{f\big(a + hu_1, b + hu_2\big) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

Using ordinary derivative notation, we see this is equivalent to taking the ordinary derivative of the single-variable function

$$g'(0) = \frac{d}{dt}\left[g(t)\right]\Big|_{t=0}$$

Using the multivariable chain rule, we know

$$\frac{d}{dt}\left[g(t)\right]\Big|_{t=0} = \frac{d}{dt}\Big[f\big(x(t), y(t)\big)\Big]\Big|_{t=0}$$

$$= \left[\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right]\Big|_{t=0}$$

$$= f_x(a,b)\cdot x'(0) + f_y(a,b)\cdot y'(0)$$

$$= f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$

$$= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$

$$= \nabla f(a,b)\cdot \mathbf{u}$$

This gives us an alternative method to calculate the directional derivative without requiring limits.

4. (8 points) Explain which unit vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ gives the directions of steepest ascent, no change, and steepest descent on the surface. Please provide evidence that your concept images associated with these directions incorporate multiple categories of knowledge including verbal, graphical, and symbolic representations of these ideas. To earn top scores, your solution should combine the work you did in

problem 3 with the cosine formula for the dot product. Also, please make specific connections to between your explanations of each direction and your knowledge of the extreme values of the cosine function.

Solution: By combining our solution in problem 3 above with the cosine formula for the dot product we see

$$\begin{aligned} D_{\mathbf{u}} f(a,b) &= \nabla f(a,b) \cdot \mathbf{u} \\ &= \|\nabla f(a,b)\|_2 \cdot \|\mathbf{u}\|_2 \cdot \cos(\theta) \\ &= \|\nabla f(a,b)\|_2 \cdot \cos(\theta) \end{aligned}$$

where θ is the angle between the vectors $\nabla f(a,b)$ and $\mathbf{u}$. We have three cases to consider.

Case 1: $\theta = 0 \implies \cos(\theta) = 1$

We know that the maximum value of the cosine curve is 1 and this occurs when $\theta = 0$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}} f(a,b)$ has maximum value when $\theta = 0$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is a function of which unit vector we choose. The dot product version of the directional derivative indicates that if we want to ascend our surface as quickly as possible, we will get the largest rise over run when $\mathbf{u}$ is the unit vector in the same direction and orientation as the gradient vector $\nabla F(a,b)$. In other words, the slope of this tangent line is maximum when we move in the direction of the gradient vector.

Case 2: $\theta = \dfrac{\pi}{2} \implies \cos(\theta) = 0$

We know that the cosine curve has a zero output value when $\theta = \frac{\pi}{2}$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}} f(a,b)$ is zero when $\theta = \frac{\pi}{2}$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is zero in this case. In other words, if we travel $90°$ from the gradient, we will get no upward or downward motion on the surface. This is equivalent to moving along the contour curve on the surface. Indeed, this unit vector is in the same direction as the tangent line to the level curve of the surface at this point of tangency.

Case 3: $\theta = \pi \implies \cos(\theta) = -1$

We know that the maximum value of the cosine curve is -1 and this occurs when $\theta = \pi$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}} f(a,b)$ has maximum value when $\theta = \pi$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is smallest when we travel in the same direction but opposite orientation as the the gradient vector $\nabla F(a,b)$. In other words, we can descend our surface fastest in the negative direction of the gradient vector.

For problems 5 - 6, let $f(x, y) = 5 - x^2 - y^2 - 8y$.

5. (8 points) Find a vector-valued equation for the tangent line to the level curve

$$L_{12}(f) = \{(x, y) \,:\, f(x, y) = 12\}$$

at the point $(3, -4)$.

Solution: If $D = \text{Dom}(f)$, then we notice that the level curve $L_{12}(f) \subseteq D \subseteq \mathbb{R}^2$. We begin our work by considering the geometry of this level curve. We notice

$$\begin{aligned} 5 - x^2 - y^2 - 8y = 12 \quad &\Longrightarrow \quad x^2 + y^2 + 8y = -7 \\ &\Longrightarrow \quad x^2 + y^2 + 8y + 16 = 9 \\ &\Longrightarrow \quad x^2 + (y+4)^2 = 3^2 \end{aligned}$$

This is a circle with radius $r = 3$ and center point $(h, k) = (0, -4)$.We notice that the given point is on the edge of the circle. To find the vector-valued equation of the tangent line to $L_{12}(f)$ given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v}$$

where $\mathbf{r}_0 \in \mathbb{R}^2$ is a point on the line and $\mathbf{v} \in \mathbb{R}^2$ represents the direction of the line. By the problem statement, we know that $\mathbf{r}_0 = \langle 3, -4 \rangle$. To find the "direction" of this line, we will use implicit differentiation:

$$\begin{aligned} \frac{d}{dy}\Big[5 - x^2 - y^2 + 6x\Big] = \frac{d}{dy}\Big[10\Big] \quad &\Longrightarrow \quad -2x \cdot \frac{dx}{dy} - 2y - 8 = 0 \\ &\Longrightarrow \quad \frac{dx}{dy} = -\frac{y+4}{x} \\ &\Longrightarrow \quad \frac{dx}{dy}\Big|_{(3,-4)} = -\frac{0}{3} \\ &\Longrightarrow \quad \mathbf{v} = \langle 0, 1 \rangle \end{aligned}$$

Using this calculation, we find

$$\mathbf{r}(t) = \langle 3, -4 \rangle + t \cdot \langle 0, 1 \rangle = \langle 3\,,\, -4 + t \rangle$$

6. (6 points) On the axes below, sketch the level curve $L_{12}(f)$ and it's the tangent line from problem 5 above. Also, sketch the vector $\mathbf{u} \in \mathbb{R}^2$ with tail at point $(3, -4)$ where $\mathbf{u}$ is the unit vector in the direction of the gradient vector $\nabla f(3, -4)$ given by

$$\mathbf{u} = \frac{\nabla f(3, -4)}{\|\nabla f(3, -4)\|_2}$$

Solution: We begin this problem by finding the gradient of our function at the given point:

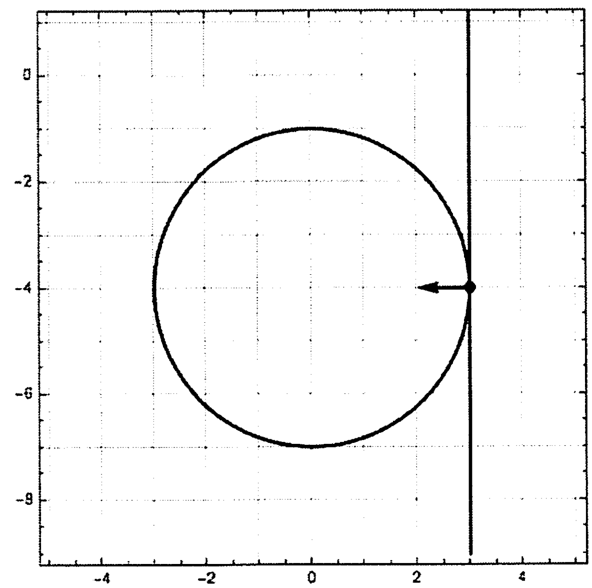
$$\nabla f(3,-4) = \langle -2x\,,\, -2y-8 \rangle \big|_{(3,-4)}$$

$$= \langle -6\,,\, 0 \rangle$$

Then, we can see that

$$\mathbf{u} = \frac{\nabla f(3,-4)}{\|\nabla f(3,-4)\|_2} = \langle -1\,,\, 0 \rangle$$

We graph this vector with tail $(3,-4)$ below.



Now, use full sentences to explain how your graph above relates your knowledge about the shape of the surface $f(x,y)$ and your solution to problem 4 above.

Solution: Notice that the surface is a downward facing elliptic parabola. The vertex of this surface is at the point $(0,-4,23)$. Based on the shape of the surface, we know that at the input point $(3,-4)$, the direction of fastest ascent is directly toward the vertex. Indeed, this is what we see with our gradient vector at this point.

7. (6 points) Traveling waves (like water waves or electromagnetic waves) show signs of periodic motion in both time and space. The dynamics of some types of waves can be modeled by the "one-dimensional" wave equation, given by

$$\frac{\partial^2}{\partial t^2}\left[u(x,t)\right] = c^2\,\frac{\partial^2}{\partial x^2}\left[u(x,t)\right]$$

where the two-variable function $u(x,t)$ is the height of the eave surface at position x and time t, and $c \in \mathbb{R}$ is a constant "speed" of the wave. Show that the function

$$u(x,t) = \cos\Big(2\,(x+ct)\Big)$$

is a solution to the wave equation. Assume that $c \in \mathbb{R}$ is a constant scalar. In other words, take the second partial derivatives of the given function $u(x,t)$ and verify that $u_{tt} = c^2 \cdot u_{xx}$

Solution: We want to show $u_{tt} = c^2 \, u_{xx}$

To this end consider

$$u_{tt} = \frac{d^2}{dt^2}\left[u(x,t)\right]$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left[\cos(2x+2ct)\right]\right]$$

$$= \frac{d}{dt}\left[-\sin(2x+2ct)\cdot\frac{d}{dt}\left[2x+2ct\right]\right]$$

$$= \frac{d}{dt}\left[-2c\sin(2x+2ct)\right]$$

$$= -2c\cos(2x+2ct)\cdot\frac{d}{dt}\left[2x+2ct\right]$$

$$\Rightarrow \quad u_{tt} = -4c^2 \cdot \cos(2x + 2ct)$$

On the other hand we have

$$u_{xx} = \frac{\partial^2}{\partial x^2}\left[u(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\cos(2x + 2ct)\right]\right]$$

$$= \frac{\partial}{\partial x}\left[-2\sin(2x + 2ct)\right]$$

$$= -4\cos(2x + 2ct)$$

$$\Rightarrow \quad c^2 u_{xx} = -4c^2 \cos(2x + 2ct) = u_{tt}$$

This is what we wanted to show.