© Jeffrey A. Anderson **Class Number:** ____

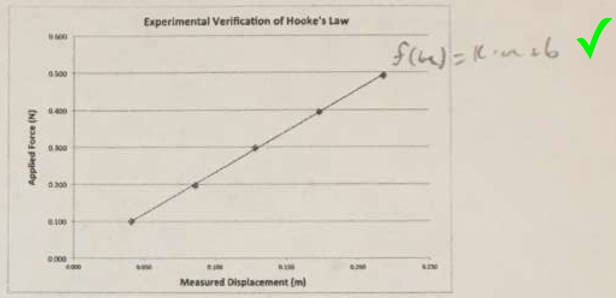
Spring 2021, Math 1C, Quiz 1

Due: Tuesday 4/20/2021 at 1:30pm (via CANVAS)

Hooke's Law is a principle of physics stating that the force required to stretch a spring u units from the equilibrium position is given by $F(u) = k \cdot u$, where the positive spring constant k measures the stiffness of the spring. Recall from class that we can set up an experiment to verify Hooke's law using a spring, masses of various size, a scale, and a measuring stick. Below are five collected data points relating to Hooks Law. This data is plotted on a graph in the figure next to the table below. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

5 data points ($1 \le i \le 5$)

	Displacement u in Meters (m)	Applied force f in Newtons (N)
1	0.041	0.100
2	0.086	0.197
3	0.128	0.298
4	0.173	0.395
5	0.218	0.492



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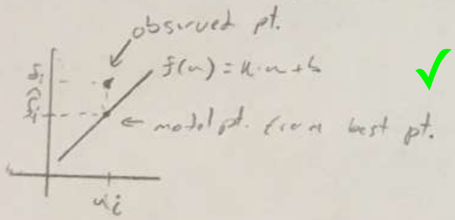
1. Set up a model for the error e_i between the ith data point (u_i, f_i) and any associated linear model

$$f(u) = \underbrace{b + k}_{\text{unknown}} \cdot u \rightarrow \text{known}$$

In this case, the parameters $b, k \in \mathbb{R}$ are unknown and the linear function $f(u)$ is the modeled internal force of the spring in Newtons corresponding to a measured displacement u in meters. Then, using the model for the errors, set up the least-squares problem for this input data. In particular, create a two variable function $E(b, k)$ that you can use to solve create the "best-fit" model for this data. Explain in detail the choices that you made to construct your error function E.



ith model output

$\hat{f}_i = k \cdot u_i + b$

error between model data & actual data

$e_i = f_i - \hat{f}_i$

$= f_i - (k \cdot u_i + b)$ unknown

$\Rightarrow e_i(k, b) = f_i - (k \cdot u_i + b)$

given given

total Square error

$$E_2(b, k) = \overbrace{\sum_{i=1}^{5} [e_i(b, k)]}^{\text{sum of the errors squared}}{}^2 \leftarrow \text{works well in calculus}$$

guarantees positivity

$$= \sum_{i=1}^{5} [f_i - (k \cdot u_i + b)]^2$$

the reason you use total square error to find b & k for the best fit model is that it guarantees positivity & work well with calculus. If you use the sum of individual errors the positive & negative errors can cancel each other out. But if you use sum of absolute value errors while your errors are positive & won't cancel out, it is hard to find the derivatives of absolute values.

Can you explain why it's easier to use calculus to optimize a square function versus absolute value? How is this related to the first versus second derivative tests?

2. Suppose that $\vec{v}$ is a vector in that starts a point $A(1,0,-1)$ and ends at point $B(-5,6,-4)$. Find a vector of length 6 that is in the same direction but opposite orientation of the vector **v**. You can assume that the new vector you create has an initial point at the origin $O(0,0,0)$.

$A(1,0,-1) = (x_1, y_1, z_1)$

$B(-5,6,-4) = (x_2, y_2, z_2)$

$\vec{V} = \vec{AB}$

Finding components of $\vec{v}$ $\rightarrow$ $\vec{V} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \langle -5-1, 6-0, -4-(-1) \rangle = \langle -6, 6, -3 \rangle$ ✓

using 2-Norm to find length of $\vec{v}$ $\rightarrow$ $\|\vec{v}\|_2 = \sqrt{(-6)^2 + (6)^2 + (-3)^2} = \sqrt{36+36+9} = \sqrt{81} = 9$ ✓

making unit vector $\rightarrow$ $\vec{u} = \frac{1}{9} \cdot \langle -6, 6, -3 \rangle = \langle \frac{1}{9} \cdot -6, \frac{1}{9} \cdot 6, \frac{1}{9} \cdot -3 \rangle = \langle \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3} \rangle$ ✓ $\leftarrow$ in the same direction as $\vec{v}$

Show length of $\vec{u}$ is 1 $\rightarrow$ $\|\vec{u}\|_2 = \sqrt{(-\frac{2}{3})^2 + (\frac{2}{3})^2 + (-\frac{1}{3})} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{\frac{9}{9}} = \sqrt{1} = 1$ ✓

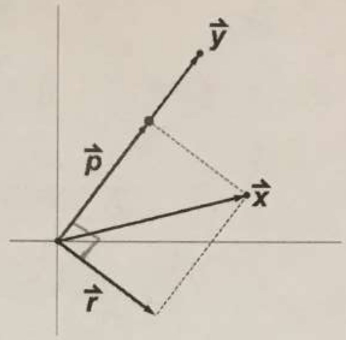
making a vector in the same direction as $\vec{v}$ but w/ the opposite orientation ✓ $\rightarrow$ $\vec{x} = -1 \cdot 6 \cdot \vec{u} = -6 \cdot \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle = \langle 4, -4, 2 \rangle$ ✓

makes the length of $\vec{x}$ 6

makes $\vec{x}$ have the opposite orientation of $\vec{v}$ ✓

Nice combination of visual, and symbolic representations: I see you making connections between various parts of your work! I'd love to see you expand on the verbal part but overall, I can follow you reasoning here.

3. (8 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and consider the diagram below



Derive an equation for the projection of vector $\mathbf{x}$ onto $\mathbf{y}$. Be sure to specifically DEFINE the vector $\mathbf{p}$ and the vector $\mathbf{r}$ from the diagram above. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

$\vec{p}$ = Projection of $\vec{x}$ onto $\vec{y}$, $Proj_{\vec{y}}(\vec{x})$

$\vec{r}$ = residual vector, it is orthogonal to $\vec{y}$

we want to find a scalar α such that

$\vec{p} = \alpha \cdot \vec{y}$ → this works as $\vec{p}$ is in the same direction as $\vec{y}$

to do this orthogonal

$\vec{r} = \vec{x} - \vec{p}$

$= \vec{x} - \alpha \cdot \vec{y}$ ← we do as we want to find α

$\vec{y} \cdot \vec{r} = 0$ ← for orthogonal vectors the dot product = 0

$\vec{y} \cdot (\vec{x} - \alpha \cdot \vec{y}) = 0$ ← homogeneity of dot product

linearity in Right argument ⇒ $(\vec{y} \cdot \vec{x}) - \alpha(\vec{y} \cdot \vec{y}) = 0$

$\alpha(\vec{y} \cdot \vec{y}) = \vec{x} \cdot \vec{y}$

$\alpha = \dfrac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|_2^2}$ ← Scalar

← length via dot product

$\vec{p} = Proj_{\vec{y}}(\vec{x}) = \alpha \cdot \vec{y}$

$= \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|_2^2} \cdot \vec{y}$

$= \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|_2} \cdot \dfrac{\vec{y}}{\|\vec{y}\|_2}$ ← unit vector in direction of $\vec{y}$

4. Let $\vec{\mathbf{a}} = \langle 1, 0, 1\rangle$ and $\vec{\mathbf{b}} = \langle 0, 1, -1\rangle$. Express $\vec{\mathbf{b}}$ as the sum

$$\vec{\mathbf{b}} = \vec{\mathbf{p}} + \vec{\mathbf{r}}$$

where $\mathbf{p}$ is parallel to $\mathbf{a}$ and $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Then, use the cosine formula for the dot product to show that $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Explain your work. Why does this make sense?

Projection $\left\{ \begin{array}{l} \vec{p} \text{ is in the direction of } \vec{a} \\ \vec{r} \text{ is orthogonal to } \vec{a} \end{array} \right.$

this is a projection of $\vec{b}$ onto $\vec{a}$

So, $\vec{p} = \mathrm{Proj}_{\vec{a}}(\vec{b}) = \alpha \cdot \vec{a}$ we know that $\alpha = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|_2}$

$= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|_2} \cdot \dfrac{\vec{a}}{\|\vec{a}\|_2}$

$= \dfrac{-1}{\sqrt{2}} \cdot \dfrac{\langle 1,0,1\rangle}{\sqrt{2}}$

$= -\dfrac{1}{2} \cdot \langle 1,0,1\rangle$

$= \langle -\tfrac{1}{2}, 0, -\tfrac{1}{2}\rangle$ ✓

$\|\vec{a}\|_2^2 = \vec{a}\cdot\vec{a}$
$= \langle 1,0,1\rangle \cdot \langle 1,0,1\rangle$
$= 1\cdot 1 + 0\cdot 0 + 1\cdot 1$
$= 1+1$
$= 2$
$\|\vec{a}\|_2 = \sqrt{2}$

$\vec{a}\cdot\vec{b} = \langle 1,0,1\rangle \cdot \langle 0,1,-1\rangle$
$= 0 + 0 - 1$
$= -1$

$\vec{b} = \vec{p} + \vec{r}$

$\vec{r} = \vec{b} - \vec{p}$

$\vec{r} = \langle 0,1,-1\rangle - \langle -\tfrac{1}{2}, 0, -\tfrac{1}{2}\rangle$

$= \langle \tfrac{1}{2}, 1, \cancel{-\tfrac{3}{2}}\rangle$

Careful: Arithmetic error here

$\|\vec{r}\|_2^2 = \vec{r}\cdot\vec{r}$
$= \frac{1}{4} + \frac{1}{4}$
$= \frac{2}{4} = \frac{1}{2}$
$\|\vec{r}\|_2 = \sqrt{\tfrac{1}{2}}$

$\vec{a}\cdot\vec{r} = \|\vec{a}\|_2 \cdot \|\vec{r}\|_2 \cdot \cos(\theta)$

$\cos(\theta) = \dfrac{\vec{a}}{\|\vec{a}\|_2} \cdot \dfrac{\vec{r}}{\|\vec{r}\|_2} = \dfrac{\langle 1,0,1\rangle}{\sqrt{2}} \cdot \dfrac{\langle -\frac{1}{2}, 0, -\frac{1}{2}\rangle}{\sqrt{\frac{1}{2}}}$

$= \langle \tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\rangle \cdot \langle \tfrac{-1}{2\sqrt{2}}, 0, \tfrac{-1}{2\sqrt{2}}\rangle = \tfrac{1}{\sqrt{2}}\cdot\left(-\tfrac{1}{2\sqrt{2}}\right) + 0\cdot 0 + \tfrac{1}{\sqrt{2}}\cdot\left(-\tfrac{1}{2\sqrt{2}}\right)$

$= -\frac{1}{4} + 0 - \frac{1}{4}$

$= -\frac{2}{4} = -\frac{1}{2}$

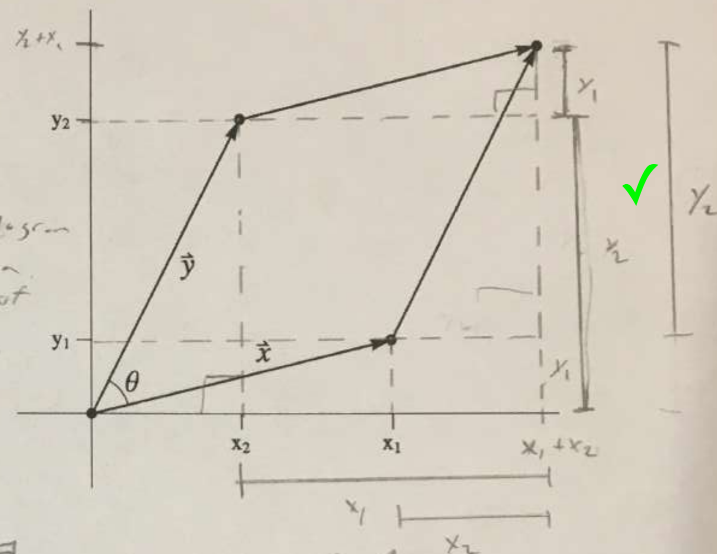
$\cos(\theta) \cancel{=} -\frac{1}{2}$

$\theta \cancel{=} \arccos(-\tfrac{1}{2})$

$\theta = \frac{2\pi}{3}$

Below, please explain your understanding of the cross product between two vectors in $\mathbb{R}^3$ by answering each of the questions 5, 6, and 7 below.

✓ 5. Let $\mathbf{x} = \langle x_1, y_1 \rangle$ and $\mathbf{y} = \langle x_2, y_2 \rangle$ be two vectors in $\mathbb{R}^2$. Using the diagram below, derive an equation for the area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ based only on the components of these vectors (note: this equation should NOT be based on the angle θ between these vectors). Please explain your answer and specifically identify the steps you took to arrive at your final answer.



Area of parallelogram = area of − area of ✓

You do it like this as you can use geometry to find the area of the parallelogram by finding the area of you get the area of the parallelogram easily as it's the area of a rectangle plus two triangles. To fix this you subtract from it to only get the area of the parallelogram, the area of is similarly easy to find as it is also two triangles & a square. ✓

$$\text{Area of parallelogram} = \overbrace{\underbrace{\tfrac{1}{2} \cdot x_2 \cdot y_2}_{\text{area of}} + \underbrace{\tfrac{1}{2} \cdot x_1 \cdot y_1}_{\text{area of}} + \underbrace{y_2 \cdot x_1}_{\text{area of}}}^{\text{area of}} \overbrace{- \underbrace{\tfrac{1}{2} \cdot x_1 \cdot y_1}_{\text{area of}} - \underbrace{\tfrac{1}{2} \cdot x_2 \cdot y_2}_{\text{area of}} - \underbrace{x_2 \cdot y_1}_{\text{area of}}}^{\text{area of}}$$ ✓

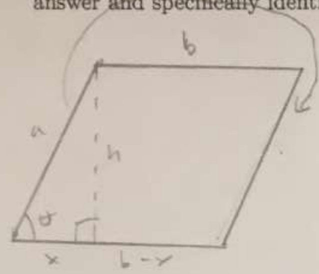
$$\boxed{= x_1 \cdot y_2 - x_2 \cdot y_1}$$ ✓

Once again: great work describing in words the ideas behind this math.

Area of a triangle $= \frac{1}{2} \cdot \text{base} \cdot \text{height}$

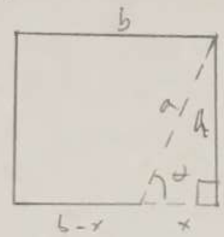
Area of a rectangle $= \text{width} \cdot \text{length}$

6. (4 points) Under the same assumptions in problem 3 above, suppose that the variable θ denotes the angle between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Derive a formula for the area of the parallelogram parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ as a function of θ and the two norms of these vectors. Please explain your answer ~~and specifically~~ identify the steps you took to arrive at your final answer.



Start w/ a parallelogram make a line go straight down one of the end points of b to establish a h for the parallelogram.

move the triangle form the height to the other side of the parallelogram & reconize that it creates a rectangle.



Recall the area of a rectangle is $b \cdot h$. Remember that $\sin(\theta)$ is opposite over hypotenuse. So in this case

$\sin(\theta) = \frac{h}{a}$

$h = \sin(\theta) \cdot a$

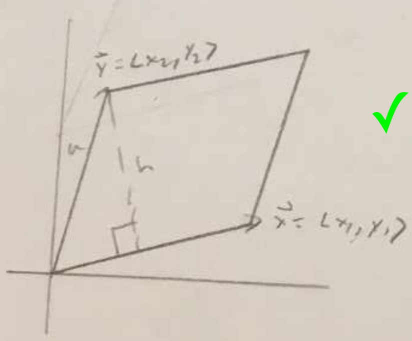
Since the parallelogram can be rearrange into a rectangle its area is also base times height.

So,

Area of parallelogram: $b \cdot h = b \cdot a \cdot \sin(\theta) = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \sin(\theta)$

$\|\vec{x} - \vec{y}\|_2 = \|\vec{x}\|_2 \|\vec{y}\|_2 \cdot |\sin(\theta)|$

2-norms of vectors is always positive, so $\sin(\theta)$ needs to be positive.



$a = \|\vec{y}\|_2$

$b = \|\vec{x}\|_2$

2-Norm so you get the length of the vectors

7. Explain how we can use our work on problems 3 and 4 above to derive the component form of the cross product between the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ where

$$\mathbf{x} = \langle x_1, y_1, z_1 \rangle, \qquad \mathbf{y} = \langle x_2, y_2, z_2 \rangle,$$

Make sure to explicitly state the component form of the cross product in your explanation. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

to generalize vectors $\vec{x}$ & $\vec{y}$ recall the area of a parallelogram. because we did it w/ 2 components, let's look at $\vec{x}$ & $\vec{y}$ in terms of xy plane, xz plane & yz plane.

Area$_{12}$ (Project vector on xy plane) $= x_1 \cdot y_2 - x_2 \cdot y_1$

Area$_{13}$ (Project vector on xz plane) $= x_1 \cdot z_2 - x_2 z_1$

store in different components of vector. store in component that corresponds w/ miss component

Area$_{23}$ (Project vector on yz plane) $= y_1 z_2 - y_2 z_1$

I'd love to push you a little more:

how is the work you're showing here related to the idea that we use the cross product to do two things:
1. As a "measurement of perpendicularity.
2. To take two vectors in 3D and produce a unique vector that is orthogonal to that vector

$$\vec{x} \times \vec{y} = (y_1 z_2 - y_2 z_1) \cdot (\vec{i}) + (x_1 z_2 - x_2 z_1) \cdot (-\vec{j}) + (x_1 y_2 - x_2 y_1) \cdot (\vec{k}) = \langle y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 \rangle$$

$$\Rightarrow \vec{x} \times \vec{y} = \langle y_1 z_2 - y_1 z_2, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 \rangle$$

relate w/ missing component

$$\begin{cases} \vec{j} \times \vec{k} = \vec{i} \\ \vec{i} \times \vec{k} = -\vec{j} \\ \vec{i} \times \vec{j} = \vec{k} \end{cases}$$

$\vec{i} = \langle 1, 0, 0 \rangle$
$\vec{j} = \langle 0, 1, 0 \rangle$
$\vec{k} = \langle 0, 0, 1 \rangle$