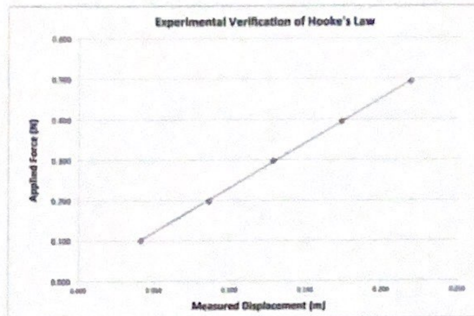
FINAL ANSWER BOXED IN RED

© Jeffrey A. Anderson Class Number: 1C

Spring 2021, Math 1C, Quiz 1
Due: Tuesday 4/20/2021 at 1:30pm (via CANVAS)

Hooke's Law is a principle of physics stating that the force required to stretch a spring u units from the equilibrium position is given by $F(u) = k \cdot u$, where the positive spring constant k measures the stiffness of the spring. Recall from class that we can set up an experiment to verify Hooke's law using a spring, masses of various size, a scale, and a measuring stick. Below are five collected data points relating to Hooks Law. This data is plotted on a graph in the figure next to the table below. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

Displacement u in Meters (m)	Applied force f in Newtons (N)
0.041	0.100
0.086	0.197
0.128	0.298
0.173	0.395
0.218	0.492



1. Set up a model for the error e_i between the ith data point (u_i, f_i) and any associated linear model

$$f(u) = b + k \cdot u$$

In this case, the parameters $b, k \in \mathbb{R}$ are unknown and the linear function $f(u)$ is the modeled internal force of the spring in Newtons corresponding to a measured displacement u in meters. Then, using the model for the errors, set up the least-squares problem for this input data. In particular, create a two variable function $E(b, k)$ that you can use to solve create the "best-fit" model for this data. Explain in detail the choices that you made to construct your error function E.

① recall, the error in the data is defined as the actual data subtracted by the modelled data

↳ $e_i = y_i - \hat{y}_i$ where e_i = error
y_i = actual data
$\hat{y}_i$ = modelled data

$e_i(b, k) = y_i - (b + k \cdot u_i)$

② using the error model we found (e_i), we will construct a two variable function $E(b, k)$ to "best fit" the model

$$E(b,k) = \sum_{i=1}^{5} [e_i(b,k)]^2$$

* we are using summation b/c we want to consider all of the points in the data set

$$\boxed{E(b,k) = \sum_{i=1}^{5} [y_i - (b + k \cdot u_i)]^2}$$

* we take the total squared function b/c it is

Page 2

easier to minimize

✓ * this creates positive values to work with while creating an easier derivation (in contrast to absolute value)

Can you explain why it's easier to use calculus to optimize a square function versus absolute value? How is this related to the first versus second derivative tests?

2. Suppose that $\mathbf{v}$ is a vector in that starts a point $A(1,0,-1)$ and ends at point $B(-5,6,-4)$. Find a vector of length 6 that is in the same direction but opposite orientation of the vector $\mathbf{v}$. You can assume that the new vector you create has an initial point at the origin $O(0,0,0)$.

① to begin, we want to find the vector produced by the points A $ B. recall, a vector from point A to point B is written as:

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$A(1, 0, -1)$ $ $B(-5, 6, -4)$

$x_1\ y_1\ z_1$ $\quad x_2\ y_2\ z_2$

Let's fill in the vector!

$$\vec{v} = \langle -5-1, 6-0, -4-(-1) \rangle$$

$$\vec{v} = \langle -6, 6, -3 \rangle \checkmark$$

② next, we want to find the two norm of $\vec{v}$. recall, the two norm of a vector is found with:

$$\|\cdot\|_2 = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

*using the pythagorean theorem

$$\|\vec{v}\|_2 = \sqrt{(-6)^2 + 6^2 + (-3)^2}$$

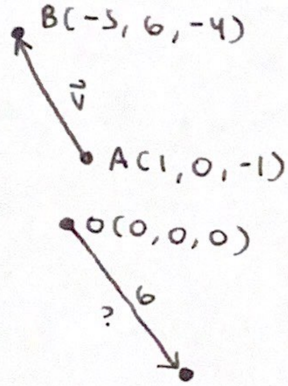
$$= \sqrt{36 + 36 + 9}$$

$$= \sqrt{81}$$

$$\|\vec{v}\|_2 = 9 \checkmark$$

✓ (3) now we want to find the unit vector of $\vec{v}$. why? this is b/c if we find the unit vector of $\vec{v}$, we can multiply a scalar we want to get the new vector we are looking for. changing $\vec{v}$ to a unit vector reduces the vector to a two norm of 1, which allows us to change the length and orientation.

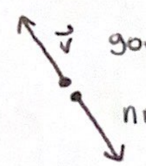
✓ we want to find another vector that has the same direction as $\vec{v}$, but different length and opposite orientation. what does that mean? let's look at it visually:



* the new vector has same direction as $\vec{v}$

* the new vector has length 6 ✓

* the new vector is in opposite direction as $\vec{v}$ ✓



Nice combination of verbal, visual, and symbolic representations: I see you making connections between various parts of your work! This is a great start to the type of evidence for learning that I am looking for!

okay, back to the unit vector. we find the unit vector by "dividing" the vector by it's magnitude.

$$\frac{\langle -6, 6, -3 \rangle}{\|\vec{v}\|_2} = \frac{1}{\|\vec{v}\|_2} \cdot \langle -6, 6, -3 \rangle$$

$$= \frac{1}{9} \cdot \langle -6, 6, -3 \rangle$$

$$= \left\langle \frac{-6}{9}, \frac{6}{9}, \frac{-3}{9} \right\rangle$$

unit vector $= \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$ ✓

④ using the unit vector $\vec{v}$, we scale it by -6. we use -6 because 6 is the length we want and the (-1) allows the vector to be facing the opposite orientation.

$$-6 \cdot \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$$

* scalar-vector multiplication results in a vector

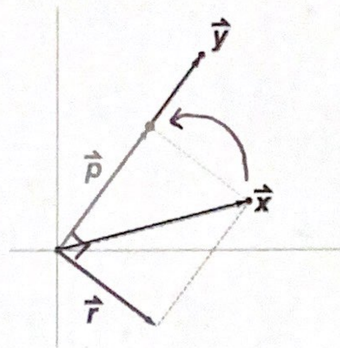
$\bullet : \mathbb{R} \times \mathbb{R}^3 \Rightarrow \mathbb{R}^3$

$$= \left\langle -6 \cdot \left(-\frac{2}{3}\right), -6 \cdot \frac{2}{3}, -6 \cdot \left(-\frac{1}{3}\right) \right\rangle$$

$$= \left\langle \frac{12}{3}, \frac{-12}{3}, \frac{6}{3} \right\rangle$$

$$= \boxed{\langle 4, -4, 2 \rangle}$$ ✓

3. (8 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and consider the diagram below



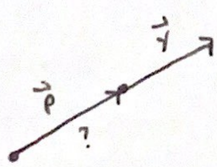
Derive an equation for the projection of vector $\mathbf{x}$ onto $\mathbf{y}$. Be sure to specifically DEFINE the vector $\mathbf{p}$ and the vector $\mathbf{r}$ from the diagram above. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

① before we derive an eq for $\text{Proj}_{\vec{y}}(\vec{x})$, we need to define some vectors

② from the diagram, $\vec{p}$ is parallel to $\vec{y}$. $\vec{p}$ shares the same direction ( ) and orientation( ) as $\vec{y}$ b/c they are on the same "line" going upwards. However, they have different magnitudes

③ let's define $\vec{p}$ algebraically. recall, a vector $\vec{p}$ sharing the same direction as another vector $\vec{y}$ will look like the following:

$\vec{p} = \alpha \cdot \vec{y}$



* where α is an unknown scalar that can adjust $\vec{p}$'s magnitude and orientation

↳ since $\vec{p}$ shares the same orientation as $\vec{y}$, α will be positive

④ so how do we find α? we use $\vec{r}$.

$\vec{r} = \vec{x} - \vec{p}$ * the distance b/t two vectors $\vec{x}$ & $\vec{p}$ is $\vec{x} - \vec{p}$

$\vec{r} = \vec{x} - (\alpha \cdot \vec{y})$

$\vec{y} \cdot \vec{r} = 0$ * the dot product of two vectors is 0 iff they are orthogonal

$\vec{y} \cdot (\vec{x} - \alpha \cdot \vec{y}) = 0$

$(\vec{y} \cdot \vec{x}) - \alpha(\vec{y} \cdot \vec{y}) = 0$ * distribute the $\vec{y}$

$\vec{y} \cdot \vec{x} = \alpha(\vec{y} \cdot \vec{y})$

$\frac{\vec{y} \cdot \vec{x}}{\vec{y} \cdot \vec{y}} = \alpha$ ✓ * isolate the α

(5) now that we found α, we know how much to apply to vector $\vec{y}$

$\vec{p} = \text{Proj}_{\vec{y}}(\vec{x}) = \alpha \vec{y}$

$= \left[\frac{\vec{y} \cdot \vec{x}}{\vec{y} \cdot \vec{y}}\right] \cdot \vec{y}$ * $\vec{y} \cdot \vec{y}$ gives the two norm of $\vec{y}^2$

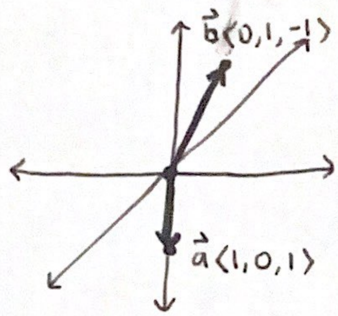
$= \frac{\vec{y} \cdot \vec{x}}{\|\vec{y}\|_2^2} \cdot \vec{y}$ ✓

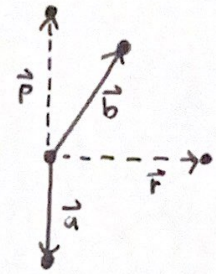
$= \boxed{\frac{\vec{y} \cdot \vec{x}}{\|\vec{y}\|_2} \cdot \frac{\vec{y}}{\|\vec{y}\|_2}}$ ✓ * if you multiply our found α scalar by $\vec{y}$ as a unit vector, we get the specific $\vec{p}$ vector with it's magnitude and orientation accounted for

4. Let $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle 0, 1, -1\rangle$. Express $\mathbf{b}$ as the sum

$$\mathbf{b} = \mathbf{p} + \mathbf{r}$$

where $\mathbf{p}$ is parallel to $\mathbf{a}$ and $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Then, use the cosine formula for the dot product to show that $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Explain your work. Why does this make sense?





possible visualization

① recall, two nonzero vectors $\vec{x}$ \$ $\vec{y}$ are orthogonal iff $\vec{x} \cdot \vec{y} = 0$.
also recall, the cosine formula for the dot product tells us that $\vec{x} \cdot \vec{y} = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \cos\theta$.

② since we want to show that $\vec{r}$ is orthogonal to $\vec{a}$ w/ the cosine formula for dot product, $\|\vec{r}\|_2 \cdot \|\vec{a}\|_2 \cdot \cos\theta$ should have a θ of $90°$ or $\pi/2$ radians.

③ but first, let's find what $\vec{p}$ is. since $\vec{p}$ is parallel to $\vec{a}$, that means they share the same direction, but may have different magnitude and orientation. we can use a projection of $\vec{b}$ onto $\vec{a}$, recall:

$$\vec{p} = \text{Proj}_{\vec{a}}(\vec{b})$$

$$\vec{p} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|_2} \cdot \underbrace{\frac{\vec{a}}{\|\vec{a}\|_2}}$$

unit vector of $\vec{a}$
to properly scale
$\vec{a}$ up to $\vec{p}$

let's find the magnitude of $\vec{a}$ to fill in the above eqv.

$$\|\vec{a}\|_2 = \sqrt{1^2+0^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

now let's try to solve for $\vec{p}$

$$\vec{p} = \frac{\langle 1,0,1\rangle \cdot \langle 0,1,-1\rangle}{\sqrt{2}} \cdot \frac{\langle 1,0,1\rangle}{\sqrt{2}}$$

*vector $\vec{p}$ using projections

$$= \frac{1}{\sqrt{2}} \cdot \langle 1,0,1\rangle \cdot \langle 0,1,-1\rangle \cdot \frac{1}{\sqrt{2}} \cdot \langle 1,0,1\rangle$$

$$= \frac{\sqrt{2}}{2} \cdot (1\cdot 0 + 0\cdot 1 + 1\cdot(-1)) \cdot \frac{\sqrt{2}}{2} \cdot \langle 1,0,1\rangle$$

$$= \frac{\sqrt{2}}{2} \cdot (0+0-1) \cdot \frac{\sqrt{2}}{2} \cdot \langle 1,0,1\rangle$$

$$= \frac{-\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \langle 1,0,1\rangle$$

$$= \frac{-2}{4} \cdot \langle 1,0,1\rangle$$

$$\vec{p} = \langle \tfrac{-1}{2}, 0, \tfrac{-1}{2} \rangle \checkmark$$

(4) now let's solve for r using the sum provided

$b = p + r$

$b - p = r$

$b + (-1)p = r$

we can use $\vec{b}$ and the $\vec{p}$ we found to find $\vec{r}$!

$$r = \langle 0,1,-1\rangle + (-1)\cdot\langle -\tfrac{1}{2}, 0, -\tfrac{1}{2}\rangle$$

$$= \langle 0,1,-1\rangle + \langle \tfrac{1}{2}, 0, \tfrac{1}{2}\rangle$$

$$r = \langle \tfrac{1}{2}, 1, -\tfrac{1}{2}\rangle \checkmark$$

(5) now that we have $\vec{r}$, we need to find the two norm of $\vec{r}$ to plug into the cosine formula

$$\|\vec{r}\|_2 = \sqrt{(\tfrac{1}{2})^2 + 1^2 + (-\tfrac{1}{2})^2}$$

$$= \sqrt{\tfrac{1}{4} + 1 + \tfrac{1}{4}}$$

$$= \sqrt{\tfrac{2}{4} + \tfrac{4}{4}}$$

$$\|\vec{r}\|_2 = \sqrt{\tfrac{6}{4}} = \sqrt{6 \cdot \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{6}$$

(6) let's plug our findings into the cosine formula for dot product.

$$\vec{r} \cdot \vec{a} = \|\vec{r}\|_2 \cdot \|\vec{a}\|_2 \cdot \cos\theta$$

$$\langle \tfrac{1}{2}, 1, -\tfrac{1}{2} \rangle \cdot \langle 1, 0, 1 \rangle = \tfrac{1}{2}\sqrt{6} \cdot \sqrt{2} \cdot \cos\theta$$

$$\tfrac{1}{2} \cdot 1 + 1 \cdot 0 + (-\tfrac{1}{2}) \cdot 1 = \tfrac{1}{2}\sqrt{12} \cdot \cos\theta$$

$$\tfrac{1}{2} + 0 - \tfrac{1}{2} = \tfrac{1}{2}\sqrt{4 \cdot 3} \cdot \cos\theta$$

$$0 = \tfrac{1}{2} \cdot 2 \cdot \sqrt{3} \cdot \cos\theta$$

$$0 = \sqrt{3}\cos\theta$$

$$0 = \cos\theta$$

$$\cos^{-1}(0) = \theta$$

$$\boxed{\theta = \tfrac{\pi}{2}}$$ ✓

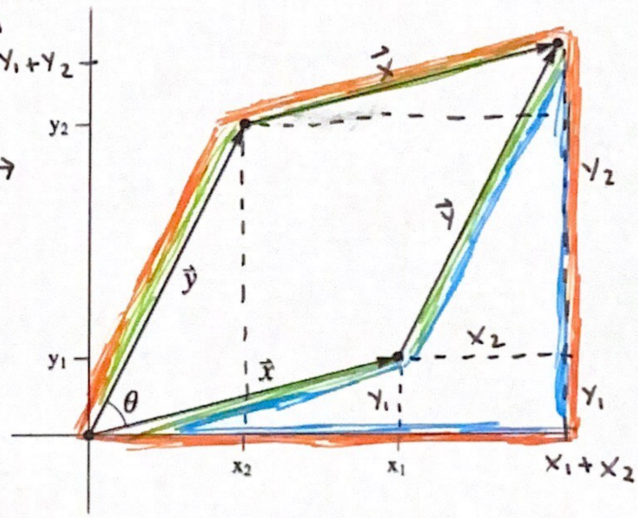
✓ [since $\theta = \frac{\pi}{2}$, we know that $\vec{r}$ is orthogonal to $\vec{a}$ w/ the cosine formula for dot product. since the dot product on the left side of the equation is 0, the two vectors are orthogonal because the cosine of $\pi/2$ is 0.]

Below, please explain your understanding of the cross product between two vectors in $\mathbb{R}^3$ by answering each of the questions 5, 6, and 7 below.

✓ 5. Let $\mathbf{x} = \langle x_1, y_1 \rangle$ and $\mathbf{y} = \langle x_2, y_2 \rangle$ be two vectors in $\mathbb{R}^2$. Using the diagram below, derive an equation for the area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ based only on the components of these vectors (note: this equation should NOT be based on the angle θ between these vectors). Please explain your answer and specifically identify the steps you took to arrive at your final answer.

✓ ① to find the area of a parallelogram formed by vectors $\vec{x}$ & $\vec{y}$, we need to look at it geometrically to form algebraic setences ⟶

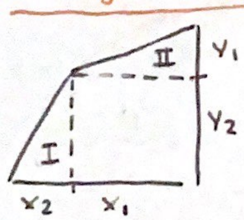
↳ we notice that there can be triangles formed inside/outside of the parallelogram, and their side lengths have some sort to do w/ the components of the vectors $\vec{x}$ & $\vec{y}$



✓ ② we can find the area of the parallelogram by subtracting out what we dont need! the orange shape subtracted by the blue shape gives us the parallelogram we want.

✓ ③ what we do now is find the areas of the orange shape and the blue shape

orange shape



triangle I : $\frac{1}{2} \cdot b \cdot h$

$= \frac{1}{2} \cdot x_2 \cdot y_2$

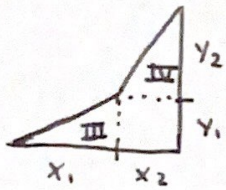
triangle II : $\frac{1}{2} \cdot x_1 \cdot y_1$

rectangle: $b \cdot h$

$= y_2 \cdot x_1$

$\boxed{\frac{1}{2} x_2 y_2 + \frac{1}{2} x_1 y_1 + y_2 x_1}$

blue shape



triangle III : $\frac{1}{2} \cdot x_1 \cdot y_1$

triangle IV : $\frac{1}{2} \cdot x_2 \cdot y_2$

rectangle : $y_1 \cdot x_2$

$$\boxed{\frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + y_1x_2}$$

④ now, we subtract the area of the orange shape from the area of the blue shape to get the area of the parallelogram! ☺

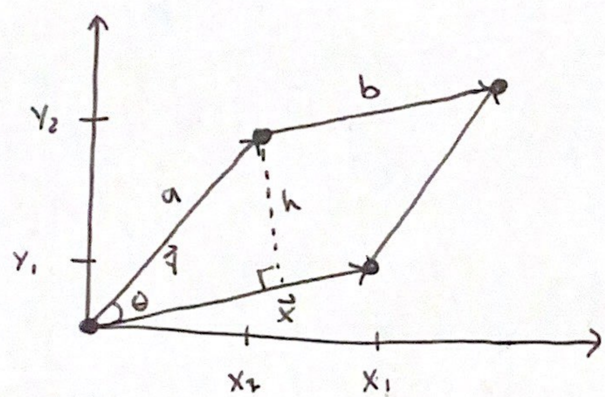
$$\cancel{\frac{1}{2}x_2y_2} + \cancel{\frac{1}{2}x_1y_1} + y_2x_1 - \left(\cancel{\frac{1}{2}x_1y_1} + \cancel{\frac{1}{2}x_2y_2} + y_1x_2\right)$$

$$= \boxed{x_1y_2 - x_2y_1 = \text{area of parallelogram!}}$$ ✓

6. (4 points) Under the same assumptions in problem 3 above, suppose that the variable θ denotes the angle between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Derive a formula for the area of the parallelogram ~~parallelogram~~ formed by vectors $\mathbf{x}$ and $\mathbf{y}$ as a function of θ and the two norms of these vectors. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

① recall, back in elementary school,
the area of a parallelogram is base times height

How do you know this is true?
Can you justify this visually?



area = b • h

* this is not the coordinate axis!

② using the components of the vectors $\vec{x}$ & $\vec{y}$, we can find the base and the height of the parallelogram

base

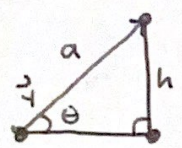
the base of the parallelogram will be the two norm of $\vec{x}$. recall, the two norm of a vector is it's magnitude aka length

$b = \|\vec{x}\|_2$

height

the height of the parallelogram is more tricky b/c it is not as clear. we will need to use trigonometric identities to find the height.

recall, $\sin(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$



using the triangle found inside of the parallelogram, we can apply this trig identity

$$\sin(\theta) = \frac{h}{a}$$

since we want to isolate the h (we are trying to find the area of the parallelogram, that is why we are here), we will rearrange the eauation

$$\sin(\theta) = \frac{h}{a}$$

$$a\sin(\theta) = h$$

* $a = \|\vec{y}\|_2$ b/c the two norm of $\vec{y}$ gives the length of a

$$\underline{\|\vec{y}\|_2 \sin(\theta) = h}$$ ☺ yayyyy

③ now that we found our base and height, we can multiply them together!!!

area of parallelogram $= b \cdot h$

$$= \boxed{\underbrace{\|\vec{x}\|_2}_{\text{base}} \cdot \underbrace{\|\vec{y}\|_2 \cdot \sin(\theta)}_{\text{height!}} \text{ units}^2}$$

✓ 7. Explain how we can use our work on problems [illegible] and 4 above to derive the component form of the cross product between the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ where

$$\mathbf{x} = \langle x_1, y_1, z_1 \rangle, \qquad \mathbf{y} = \langle x_2, y_2, z_2 \rangle.$$

Make sure to explicitly state the component form of the cross product in your explanation. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

① so, the formula for component form cross product is:

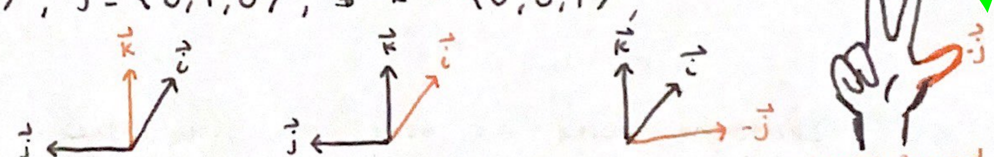
✓ $\vec{x} \times \vec{y} = (y_1 z_2 - y_2 z_1) \cdot \vec{i} + (x_1 z_2 - x_2 z_1) \cdot (-\vec{j}) + (x_1 y_2 - x_2 y_1) \cdot \vec{k}$

✓ ② now what did that come from? recall, if we have a set of vectors $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, & $\vec{k} = \langle 0,0,1 \rangle$,

$\vec{i} \times \vec{j} = +\vec{k}$ ✓

$\vec{j} \times \vec{k} = +\vec{i}$

$\vec{i} \times \vec{k} = -\vec{j}$

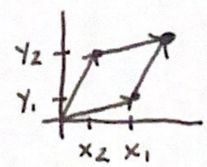


✓

*notice that each vector has 2 zero components, and 1 one component (the 1 is where we will apply the vector)

also, recall in question 5, we found that the area of a parallelogram is:

$x_1 y_2 - x_2 y_1$



✓ ③ since we have 3 components to deal with, we would need to split up the problem into 3 so that we will instead deal with 3 two component vectors

using the area of a parallelogram, we will disregard a different component for each set

Area without the z

$x_1 y_2 - x_2 y_1 \rightarrow \vec{k}$

Area without the y

$x_1 z_2 - x_2 z_1 \rightarrow -\vec{j}$

Area without the x

$y_1 \cdot z_2 - y_2 \cdot z_1 \rightarrow \vec{i}$

✓

✓ * since we noticed that the $\vec{i}, \vec{j}, \& \vec{k}$ vectors have a 1 in a certain component, we can match them up with their respective "areas w/o the blah"

* now it looks similar to the formula in ①!

~EPIC~ ✓

✓ ④ putting it all together, we should arrive at a vector in $\mathbb{R}^3$. the cross product of two vectors results in a vector.

$$\boxed{\vec{x} \times \vec{y} = \langle \underbrace{y_1 z_2 - y_2 z_1}_{\text{1st component}}, \underbrace{x_2 z_1 - x_1 z_2}_{\text{2nd component}}, \underbrace{x_1 y_2 - x_2 y_1}_{\text{3rd component}} \rangle}$$

✓

* 2nd component is flipped b/c it is multiplied by $-\vec{j}$