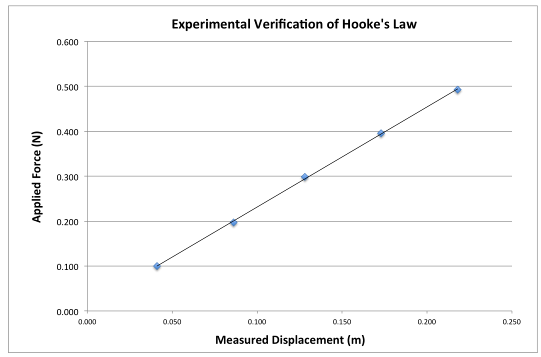
Example 1: Student Work from Spring 2021 © Jeffrey A. Anderson **Class Number:** 1C

Spring 2021, Math 1C, Quiz 1
Due: Tuesday 4/20/2021 at 1:30pm (via CANVAS)

Hooke's Law is a principle of physics stating that the force required to stretch a spring u units from the equilibrium position is given by $F(u) = k \cdot u$, where the positive spring constant k measures the stiffness of the spring. Recall from class that we can set up an experiment to verify Hooke's law using a spring, masses of various size, a scale, and a measuring stick. Below are five collected data points relating to Hooks Law. This data is plotted on a graph in the figure next to the table below. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

Displacement u in Meters (m)	Applied force f in Newtons (N)
0.041	0.100
0.086	0.197
0.128	0.298
0.173	0.395
0.218	0.492



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1. Set up a model for the error e_i between the ith data point (u_i, f_i) and any associated linear model

x y

$$f(u) = b + k \cdot u$$ slope

In this case, the parameters $b, k \in \mathbb{R}$ are unknown and the linear function $f(u)$ is the modeled internal force of the spring in Newtons corresponding to a measured displacement u in meters. Then, using the model for the errors, set up the least-squares problem for this input data. In particular, create a two variable function $E(b, k)$ that you can use to solve create the "best-fit" model for this data. Explain in detail the choices that you made to construct your error function E.

To create a model for the error, the idea is to subtract the actual data from its modeled values.

$$e_i = f_i - \hat{f}_i$$

The modeled data is given from $f(u) = b + k \cdot u$
we can sub in this information to the equation for error

$$\rightarrow e_i(k,b) = f_i - (ku_i + b)$$

To find the error, you want to take the sum of the squared error values.

5 ← points in data set.

$$E(k,b) = \sum_{i=1}^{5} [e_i(k,b)]^2$$

sum

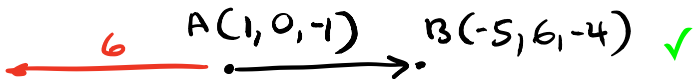
squared

we want to square the values rather than take the absoluted value because it makes calculations more complicated down the line, & only summing up the values will produce nonsense

We can sub in the error model to get the following equation that computes the error for the data set.

$$E(k,b) = \sum_{i=1}^{5} (f_i - ku_i - b)^2$$

Can you explain why it's easier to use calculus to optimize a square function versus absolute value? How is this related to the first versus second derivative tests?



2. Suppose that $\mathbf{v}$ is a vector in that starts a point $A(1,0,-1)$ and ends at point $B(-5,6,-4)$. Find a vector of length 6 that is in the same direction but opposite orientation of the vector $\mathbf{v}$. You can assume that the new vector you create has an initial point at the origin $O(0,0,0)$.

First, lets construct vector v from $\vec{A}$ & $\vec{B}$.

$$\vec{v} = \langle -5-1, 6-0, -4--1 \rangle$$
$$\vec{v} = \langle -6, 6, -3 \rangle$$

Next, let's determine ~~the~~ magnitude of $\vec{v}$

$$\|\vec{v}\|_2 = \sqrt{(-6)^2 + 6^2 + (-3)^2}$$
$$= \sqrt{36+36+9} = \sqrt{81} = 9$$

We can now create a unit vector, $\vec{u}$, by normalizing $\vec{v}$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|_2} = \frac{\langle -6, 6, -3 \rangle}{9} = \left\langle -\frac{6}{9}, \frac{6}{9}, -\frac{3}{9} \right\rangle = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

Let's reverse this vector to point it in the opposite direction.

$$\vec{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

Finally, we scale the vector up by 6 to get the desired vector

$$6 \cdot \vec{u} = 6 \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle$$

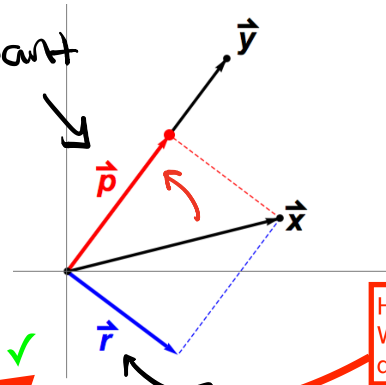
final vector $= \langle 4, -4, 2 \rangle$

Nice combination of verbal, visual, and symbolic representations: I see you making connections between various parts of your work! This is a great start to the type of evidence for learning that I am looking for!

3. (8 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and consider the diagram below

According to the question, we want to find $\vec{p} = \text{Proj}_{\vec{y}}(\vec{x})$

To find $\vec{p}$, we must find a scalar α such that $\vec{p} = \alpha \cdot \vec{y}$



How is this related to the idea of orientation? What is the difference between orientation and direction?

Derive an equation for the projection of vector $\mathbf{x}$ onto $\mathbf{y}$. Be sure to specifically DEFINE the vector $\mathbf{p}$ and the vector $\mathbf{r}$ from the diagram above. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

We can determine α in terms of $\vec{y}$ & $\vec{x}$ by using the residual vector $\vec{r} = \vec{x} - \vec{p}$

$$\vec{r} = \vec{x} - (\alpha \cdot \vec{y})$$

plug in equation for $\vec{p}$

$\vec{y}$ & $\vec{r}$ are orthogonal iff $\vec{y} \cdot \vec{r} = 0$, we can plug in the equation for $\vec{r}$ so that we only have variables $\vec{x}$, $\vec{y}$, & α in an equation

$$\left.\begin{aligned} 0 &= \vec{y} \cdot \vec{r} \\ 0 &= \vec{y} \cdot (\vec{x} - (\alpha \cdot \vec{y})) \\ 0 &= \vec{y} \cdot \vec{x} - \alpha\, \vec{y} \cdot \vec{y} \end{aligned}\right\} \text{then isolate } \alpha!$$

$$\alpha\, \vec{y} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

$$\alpha = \frac{\vec{y} \cdot \vec{x}}{\vec{y} \cdot \vec{y}}$$

by definition, $\vec{y} \cdot \vec{y} = \|\vec{y}\|_2$

$$\Rightarrow \alpha = \frac{\vec{y} \cdot \vec{x}}{\|\vec{y}\|_2}$$

Now that we have α, we can plug it into the equation for $\vec{p}$.

$$\vec{p} = \alpha \cdot \vec{y} = \vec{p} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|_2} \cdot \vec{y}$$

this $\vec{y}$ must have a magnitude of 1, so we can convert it to its unit vector with $\frac{\vec{y}}{\|\vec{y}\|_2}$

$$\Rightarrow \vec{p} = \frac{\vec{y} \cdot \vec{x}}{\|\vec{y}\|_2} \cdot \frac{\vec{y}}{\|\vec{y}\|_2}$$

$=y$ $\quad =x$ $\quad =x$

✓ 4. Let $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle 0, 1, -1\rangle$. Express $\mathbf{b}$ as the sum

$$\mathbf{b} = \mathbf{p} + \mathbf{r}$$

where $\mathbf{p}$ is parallel to $\mathbf{a}$ and $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Then, use the cosine formula for the dot product to show that $\mathbf{r}$ is orthogonal to $\mathbf{a}$. Explain your work. Why does this make sense?

We want to find vector $\vec{p} = \text{Proj}_{\vec{a}}(\vec{b})$, which is an orthogonal projection of $\vec{b}$ onto $\vec{a}$. We can use the following formula to determine $\vec{p}$.

$$\vec{p} = \text{Proj}_{\vec{a}}(\vec{b}) = \left[\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|_2^2}\right]\cdot\vec{a}$$

To calculate this, let's find $\vec{b}\cdot\vec{a}$ & $\|\vec{a}\|_2^2$.

$$\vec{b}\cdot\vec{a} = \langle 0,1,-1\rangle\cdot\langle 1,0,1\rangle = 0\cdot 1 + 1\cdot 0 + -1\cdot 1 = -1$$

$$\|\vec{a}\|_2^2 = (1)^2 + (0)^2 + (1)^2 = 2$$

We plug these values back into the equation for $\vec{p}$

$$\vec{p} = \left[\frac{-1}{2}\right]\cdot\langle 1,0,1\rangle = \langle -\tfrac{1}{2}, 0, -\tfrac{1}{2}\rangle$$ ✓

Now that we have $\vec{p}$, we can find $\vec{r}$.

$$\langle 0,1,-1\rangle = \langle -\tfrac{1}{2}, 0, -\tfrac{1}{2}\rangle + \langle ?, ?, ?\rangle$$

$0 = -\frac{1}{2} + x$

$x = \frac{1}{2}$

$1 = 0 + y$

$y = 1$

$-1 = -\frac{1}{2} + z$

$-\frac{1}{2} = z$

$$\vec{r} = \langle \tfrac{1}{2}, 1, -\tfrac{1}{2}\rangle$$ ✓

We can plug $\vec{r}$ & $\vec{a}$ into the cosine formula to check whether they are orthogonal

$$\vec{r}\cdot\vec{a} = \|\vec{r}\|_2 \cdot \|\vec{a}\|_2 \cdot \cos(\theta)$$

$$= \sqrt{(\tfrac{1}{2})^2 + 1^2 + (-\tfrac{1}{2})^2} = \sqrt{\tfrac{1}{4} + 1 + \tfrac{1}{4}}$$

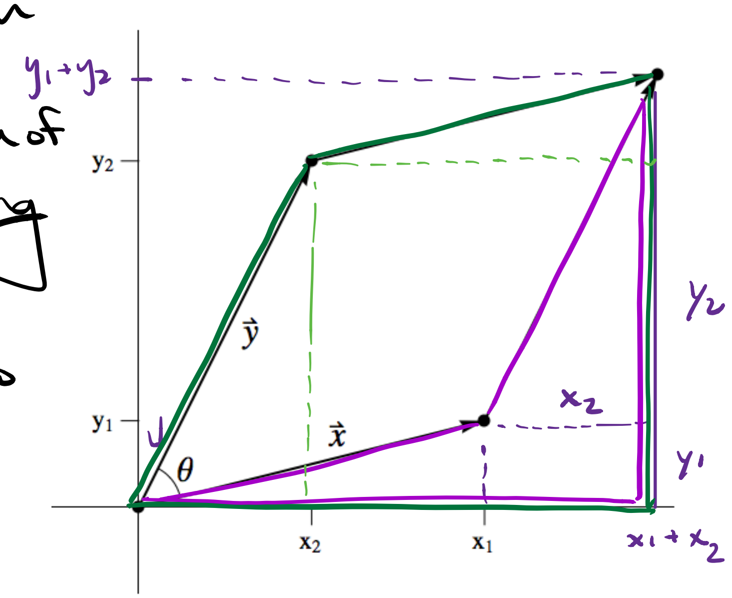
$$= \sqrt{\tfrac{3}{2}} \cdot \sqrt{2} \cdot \cos(90)$$

$$\vec{r}\cdot\vec{a} = 0$$ ✓

$\vec{r}$ & $\vec{a}$ must be orthogonal if the angle between them is $90°$

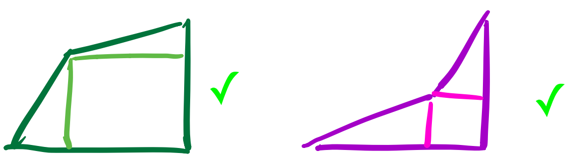
Below, please explain your understanding of the cross product between two vectors in $\mathbb{R}^3$ by answering each of the questions 5, 6, and 7 below.

5. Let $\mathbf{x} = \langle x_1, y_1 \rangle$ and $\mathbf{y} = \langle x_2, y_2 \rangle$ be two vectors in $\mathbb{R}^2$. Using the diagram below, derive an equation for the area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ based only on the components of these vectors (note: this equation should NOT be based on the angle θ between these vectors). Please explain your answer and specifically identify the steps you took to arrive at your final answer.



The area of the parallelogram in the figure can visually be found by taking the area of shape and subtracting the area of from it.

To find the areas of these two shapes, we can break them down into smaller parts!



The area of the larger shapes will be the sum of the areas of its smaller parts, which we can find based on the components of vectors $\vec{x}$ & $\vec{y}$.

For , $= \frac{1}{2} x_2 \cdot y_2$, notice these dudes are the same $= \frac{1}{2} x_1 \cdot y_1$, $= y_2 \cdot x_1$ same here

For , $= \frac{1}{2} x_2 \cdot y_2$, $= \frac{1}{2} x_1 \cdot y_1$, $= x_2 \cdot y_1$

these losers cancel out cuz they the same

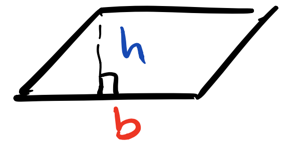
plug in

Area of the parallelogram $= y_2 \cdot x_1 - x_2 \cdot y_1$

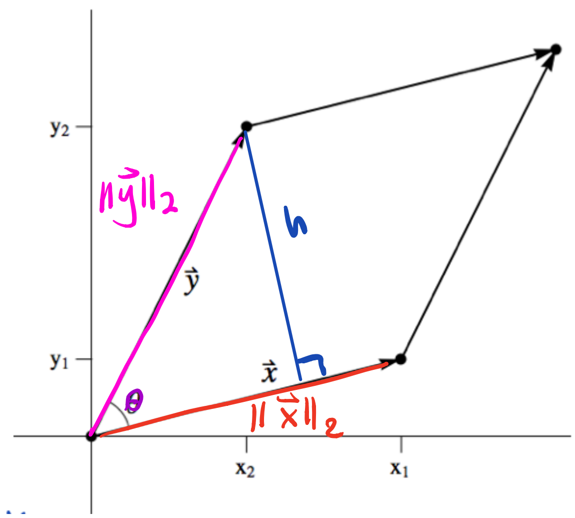
How do you know this is true?
Can you justify this visually?

6. (4 points) Under the same assumptions in problem 3 above, suppose that the variable θ denotes the angle between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Derive a formula for the area of the parallelogram parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ as a function of θ and the two norms of these vectors. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

The area of a parallelogram is $b \cdot h$ where



We are given the parallelogram:



In this figure, b of the parallelogram is $\|\vec{x}\|_2$ & h is unknown at the moment.

To find h, we can use the angle θ and $\|\vec{y}\|_2$ in

$$\sin(\theta) = \frac{h}{\|\vec{y}\|_2}$$

now, solve for h

$$h = \|\vec{y}\|_2 \cdot \sin(\theta)$$

Now we have b & h, we can plug into the area of parallelogram equation.

$$b = \|\vec{x}\|_2 \quad \& \quad h = \|\vec{y}\|_2 \cdot \sin(\theta)$$

Area of the parallelogram $= b \cdot h = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \sin(\theta)$

7. Explain how we can use our work on problems 3 and 4 above to derive the component form of the cross product between the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ where

$$\mathbf{a} = \langle x_1, y_1, z_1 \rangle, \qquad \mathbf{b} = \langle x_2, y_2, z_2 \rangle,$$

Make sure to explicitly state the component form of the cross product in your explanation. Please explain your work. For top scores, please demonstrate multiple dimensions of your concept image associated with this derivation.

✓ The components of the cross product of $\vec{a}$ & $\vec{b}$ are orthogonal to respective components of vectors $\vec{a}$ & $\vec{b}$.

✓

- The first component of the cross product, will be orthogonal to the components of $\vec{a}$ & $\vec{b}$ on the y, z plane (since the 1 component is in the $\vec{i}$ direction.)

The perpendicularity of the y, z components of $\vec{a}$ & $\vec{b}$ can be measured by taking the area of the parallelogram.

✓ $\text{Area}_{yz} = y_1 \cdot z_2 - y_2 \cdot z_1$ ✓ ← missing x, so in $\vec{i}$ direction

- The second component of the cross product is orthogonal to the components of $\vec{a}$ & $\vec{b}$ in the xz plane.

$\text{Area}_{xz} = x_1 \cdot z_2 - x_2 \cdot z_1$ ✓ ← missing y & goes in the $-\vec{j}$ direction according to right hand rule.

- The third component of the cross product is orthogonal to the components of $\vec{a}$ & $\vec{b}$ in the xy plane.

$\text{Area}_{xy} = x_1 \cdot y_2 - x_2 \cdot y_1$ ✓ ← missing z, goes in the $+\vec{k}$ direction

Plug in the components:

$$\vec{a} \times \vec{b} = \langle (y_1 \cdot z_2 - y_2 \cdot z) \cdot (+\vec{i}),\ (x_1 \cdot z_2 - x_2 \cdot z_1) \cdot (-\vec{j}),\ (x_1 \cdot y_2 - x_2 \cdot y_1) \cdot (+\vec{k}) \rangle$$

✓ $\vec{a} \times \vec{b} = \langle y_1 \cdot z_2 - y_2 z_1,\ x_2 z_1 - x_1 \cdot z_2,\ x_1 \cdot y_2 - x_2 \cdot y_1 \rangle$