

The Geometry of Elementary Operations

In this section, we study a variety of transformations used to manipulate two-dimensional shapes. We start by defining a begin polygon V using our wireframe model described in Example ???. We then apply various transformations to produce an end polygon W . Each transformation results from a matrix-matrix multiplication in the form

$$W = E \cdot V,$$

where V is a constant vertex matrix and the vertex matrix W depends on the choice of matrix E . Based on the choice of E , we can slant, scale, reflect, rotate, or project particular components of our begin polygon V in order to create our end polygon W . By studying geometric interpretations of matrix-matrix multiplication, we develop key insights used in later chapters.

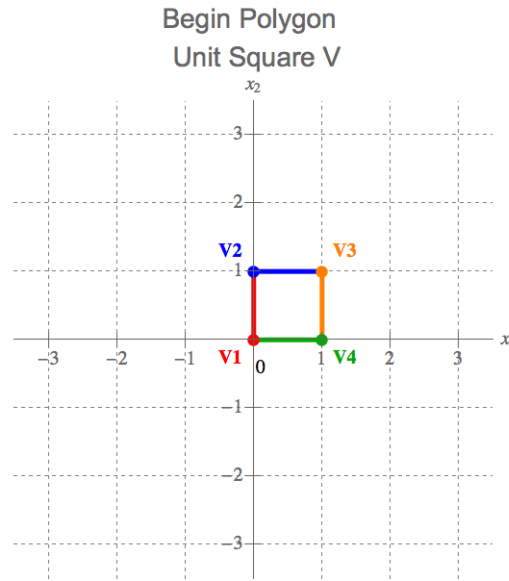
To create a two-dimensional wireframe model, we need to specify a vertex matrix and an edge table. For the remainder of this section, we define the vertex matrix for our begin polygon V to be

$$V = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The k th column of V encodes the first and second coordinates of the k th vertex of our polygon, for $k = 1, 2, 3, 4$. We define our edge table as follows

Edge #	Start Vertex	End Vertex
1	1	2
2	2	3
3	3	4
4	4	1

We visualize this wireframe model in \mathbb{R}^2 as a unit square on the coordinate axis:



In this diagram, vertex 1 is drawn in red, vertex 2 in blue, vertex 3 in orange and vertex 4 is given in green. We also notice that edge 1 is highlighted in red, edge 2 in blue, edge 3 in orange and edge four is highlighted in green. These color relations remain consistent throughout this section to enable easy comparisons between the begin polygon V and the various end polygons presented.

We limit our discussion to transformations on the vertex matrix V . We will not change the edge table nor will we change the enumeration scheme for the vertices. In other words, to go from the begin polygon V to our desired end polygon W , we change only the entries of matrix V . We do not change the interconnections between the vertices nor do we swap the order of the columns of V . For example, the red edge 1 will always connect the red vertex 1 and the blue vertex 2. While the locations of these vertices and the orientation of the edges may differ, the general connectivity pattern will remain constant.

Shears in the First Coordinate Axis

We begin our investigation of geometric transformations by considering 2×2 shear matrices $S_{ik}(c)$, as described in Definition ???. We start by setting $i = 1$ and $k = 2$ and left multiplying our vertex matrix V by

$$S_{12}(c) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

Let's study the effect of left multiplication by this shear matrix on a single vertex in \mathbb{R}^2 . Define the begin vertex

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and calculate the end vertex \mathbf{w} given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Using the dot-product version of matrix-vector multiplication, the individual components of \mathbf{w} are

$$w_1 = v_1 + c \cdot v_2 \quad \text{and} \quad w_2 = v_2.$$

The second coordinate, w_2 , of the end vertex remains the same as v_2 . The first coordinate, w_1 , is shifted away from v_1 along the first coordinate axis by c times the value of v_2 . Hence, left multiplication by shear matrix $S_{12}(c)$ produces a shift in the first coordinate axis with magnitude $|c \cdot v_2|$. The direction of this shift, either left or right, depends on the sign of both c and v_2 .

EXAMPLE 4.4.1

Let's shift blue vertex 2 and orange vertex 3 of the begin polygon V two units to the left. We leave red vertex 1 and green vertex 4 in place. To do so, we change the lengths of the red and orange edges by slanting these connection pieces while leaving the lengths of the blue and green edges unchanged. Set $E = S_{12}(-2)$ and consider:

$$W = E \cdot V = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The first row of W , given by the row-partition version of matrix-matrix multiplication, is

$$\begin{aligned} W(1,:) &= [1 \quad -2] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 1 \cdot [0 \quad 0 \quad 1 \quad 1] + -2 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad -2 \quad -1 \quad 1]. \end{aligned}$$

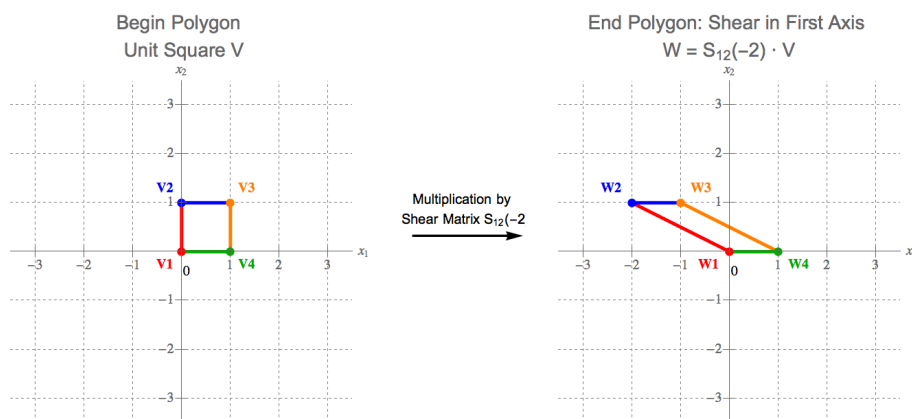
Similarly, the second row of W is calculated as

$$\begin{aligned} W(2,:) &= [0 \quad 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 0 \cdot [0 \quad 0 \quad 1 \quad 1] + 1 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 1 \quad 1 \quad 0]. \end{aligned}$$

Together, these vectors compose the matrix

$$W = \begin{bmatrix} 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$


The diagram below illustrates the effect of left multiplication by $S_{12}(-2)$.



Let's compare and contrast the our begin polygon V with our end polygon W . We focus on the location of the vertices and lengths of the edges. We will not mentions similarities in connectivity between vertices since the edge table of W is identical to that of V .

There are many similarities between V and W . First, the vertices of W are labeled in the same manner as the vertices of V : vertex 1 is in red, vertex 2 in blue, vertex 3 in orange and vertex 4 in green. Further, the lengths of blue edge 2 and green edge 4 remain unchanged. We also see that vertices 1 and 4 of our end polygon are in the exact same location as vertices 1 and 4 of our begin polygon since $W(:, 1) = V(:, 1)$ and $W(:, 4) = V(:, 4)$. Finally, the 2nd coordinates of vertices 2 and 3 have remained constant with $w_{22} = v_{22}$ and $w_{23} = v_{23}$.

The only differences between V and W are in the first coordinate entries of vertices 2 and 3. In particular, w_{12} is two units to the left of v_{12} while w_{13} is two units to the left of v_{13} . The left shift of these vertices implies that red edge 1 and orange edge 2 must be slanted in the end polygon W . Hence, edges 1 and 3 in W have different lengths than edges 1 and 3 in V .

Be careful about broad generalizations from this one example. The fact that we started with entry $v_{22} = 1$ and $v_{23} = 1$ is directly related to the shift of $(-2 \cdot 1)$ units, where the negative sign represents movement to the left. If we change the values of the second coordinates v_{2k} , the shift in the first coordinate entries of W produced by left multiplication with $S_{12}(c)$ will have a different magnitude. More about this can be found in the exercises at the end of this section. 

EXAMPLE 4.4.2

Let's use the same idea to move vertices 1 and 3 one unit to the right. To achieve this goal, set $E = S_{12}(1)$ and consider

$$W = E \cdot V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Using the row-partition version of matrix-matrix multiplication, calculate

$$\begin{aligned} W(1, :) &= [1 \quad 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 1 \cdot [0 \quad 0 \quad 1 \quad 1] + 1 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 1 \quad 2 \quad 1]. \end{aligned}$$

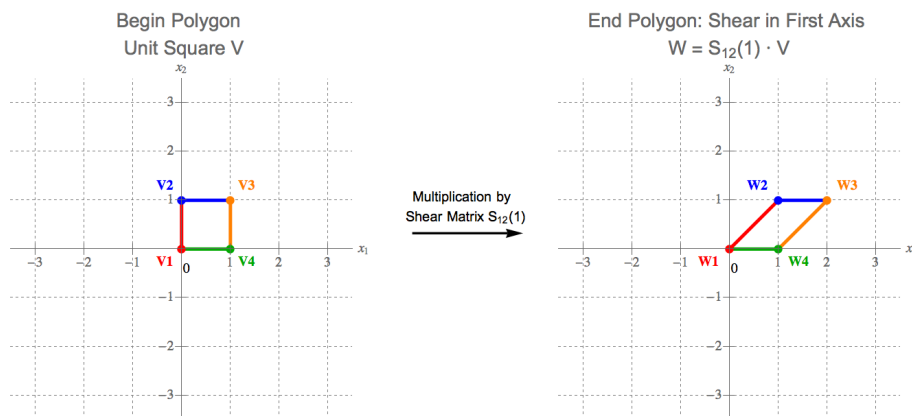
The row vector of second coordinates of W is

$$\begin{aligned} W(2, :) &= [0 \quad 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 0 \cdot [0 \quad 0 \quad 1 \quad 1] + 1 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 1 \quad 1 \quad 0]. \end{aligned}$$

Combining these two vectors together gives

$$W = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The diagram below portrays this transformation.



Many of the same patterns observed in the left-shear Example 4.4.1 also hold true in this case. The major difference when multiplying by $S_{12}(1)$ is the right shift of vertices 2 and 3 resulting from our choice of positive $c = 1$.

For the particular choice of begin polygon V in this section, left multiplication by $S_{12}(c)$ represents a left shift if $c < 0$ and a right shift if $c > 0$.

Shears in the Second Coordinate Axis

Let's continue by investigating the effects of multiplication by shear matrix

$$S_{21}(c) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}.$$

Left multiplication by $S_{21}(c)$ defines a map from a single begin vertex \mathbf{v} to a single end vertex \mathbf{w} given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Using the dot-product version of matrix vector multiplication, the individual entries of \mathbf{w} are

$$w_1 = v_1 \quad \text{and} \quad w_2 = c \cdot v_1 + v_2.$$

This map leaves the first coordinate entry constant and creates a shift of magnitude $|c \cdot v_1|$ in the second coordinate entry. Depending on the sign of both c and v_1 , this shift will be either upward or downward along the second coordinate axis.

EXAMPLE 4.4.3

Let $E = S_{21}(-3)$ and consider

$$W = E \cdot V = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The first row of W is

$$\begin{aligned} W(1,:) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 1 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

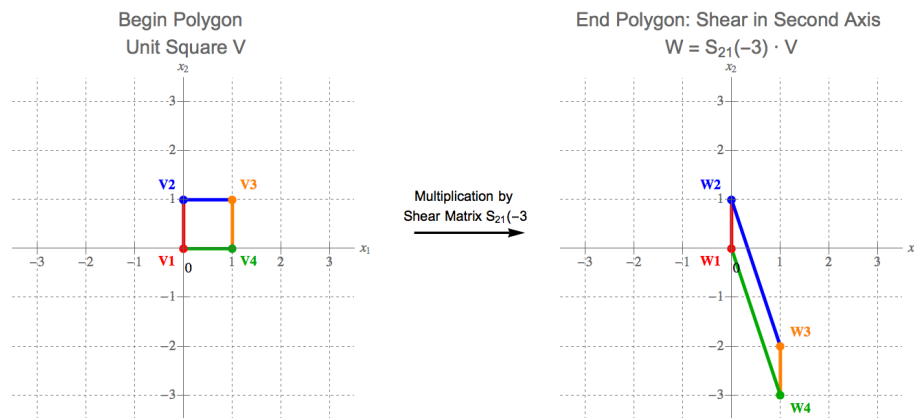
The vector of second coordinate entries of W is

$$\begin{aligned} W(2, :) &= [-3 \quad 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= -3 \cdot [0 \quad 0 \quad 1 \quad 1] + 1 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 1 \quad -2 \quad -3]. \end{aligned}$$

Together, these vectors create matrix

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & -3 \end{bmatrix}.$$

Because the end polygon W has identical edge table as begin polygon V , left multiplication of V by $S_{21}(-3)$ produces the following geometric transformation:



This map resulted in a downward shift of a portion of our begin polygon V because we added the product

$$c \cdot v_1 = -3 \cdot 1,$$

which is a negative number, to the original value of v_{2k} to produce w_{2k} for $k = 3, 4$. The effect of $S_{21}(-3)$ on a polygon in the second quadrant would be an upward shift, since $v_1 < 0$ in quadrant two. Again, we need to be careful to make explicit connections between our knowledge of algebra with the geometry of each polygon as a part of the process of visualizing these maps.

EXAMPLE 4.4.4

Let's produce a shift in the second coordinate in the opposite direction as described in Example 4.4.3. Set $E = S_{21}(1.5)$ and consider

$$W = E \cdot V = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Again, we construct the vertices of our end polygon row-by-row, starting with our first row

$$\begin{aligned} W(1,:) &= [1 \quad 0] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 1 \cdot [0 \quad 0 \quad 1 \quad 1] + 0 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 0 \quad 1 \quad 1]. \end{aligned}$$

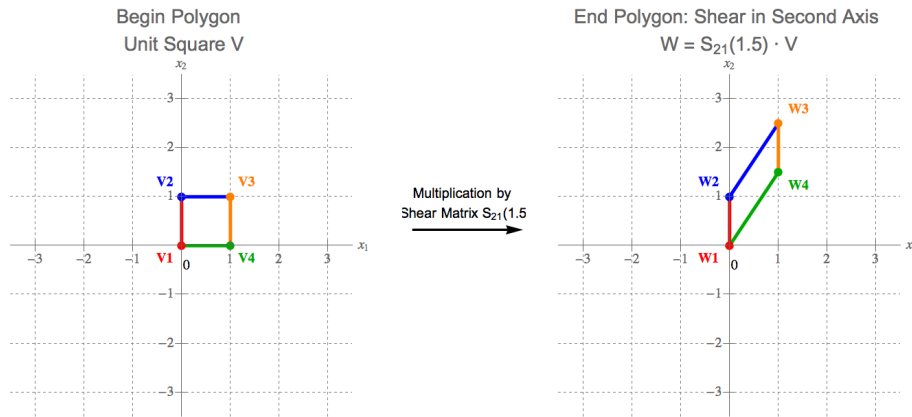
Row two of the vertex matrix for end polygon W is

$$\begin{aligned} W(2,:) &= [1.5 \quad 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 1.5 \cdot [0 \quad 0 \quad 1 \quad 1] + 1 \cdot [0 \quad 1 \quad 1 \quad 0], \\ &= [0 \quad 1 \quad 2.5 \quad 1.5]. \end{aligned}$$

With this, we have constructed the matrix

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 2.5 & 1.5 \end{bmatrix}.$$

We can visualize this transformation using the diagram below.



Left multiplication of V by $S_{21}(c)$ produces an upward slant of blue edge 2 and green edge 4 since we added

$$c \cdot v_1 = 1.5 \cdot 1$$

to both orange vertex 3 and green vertex 4.

Dilations in the First Coordinate Axis

We continue with 2×2 dilation matrices $D_j(c)$, given in Definition ???. Set $j = 1$ and left multiply vertex matrix V by

$$D_j(c) = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}.$$

This dilation matrix defines a map from between two vertices in \mathbb{R}^2 given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The individual components of the image of \mathbf{v} under this map are given by

$$w_1 = c \cdot v_1 \quad \text{and} \quad w_2 = v_2.$$

Assuming that $c > 0$, this map scales the first coordinate entry v_1 by a multiple of c to produce w_1 . The second coordinate w_2 remains equal to v_2 , producing no change in the second entry. If $0 < c < 1$, we expect W to be a compressed version of polygon V , where the compression by factor c occurs only along the first coordinate axis. If $c > 1$, we expect this map to expand polygon V along the first coordinate axis by a factor of c . We use the word dilation to indicate that some scaling is occurring without specifying if the scalar multiple increases or decreases size.

EXAMPLE 4.4.5

Define matrix $E = D_1(2.5)$ and consider

$$W = E \cdot V = \begin{bmatrix} 2.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again, we construct the vertices of our end polygon row-by-row. We begin by recognizing that

$$\begin{aligned} W(1,:) &= \begin{bmatrix} 2.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 2.5 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 2.5 & 2.5 \end{bmatrix} \end{aligned}$$

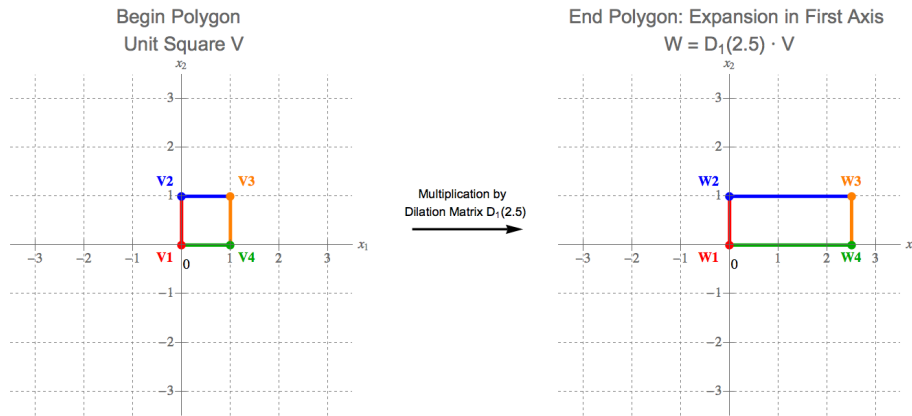
We find that the vector of second coordinates of the end polygon is calculated

$$\begin{aligned} W(2,:) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 0 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

These two vectors together compose the matrix W , and thus we have

$$W = \begin{bmatrix} 0 & 0 & 2.5 & 2.5 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Algebraically, left multiplication of V by $D_1(2.5)$ simply multiplies the entire first row of V by scalar $c = 2.5$ without changing any other entries. We interpret this operations as a expansion along the first coordinate axis by a factor of 2.5.

**EXAMPLE 4.4.6**

To compress the begin polygon V along our first coordinate axis, define $E = D_1\left(\frac{2}{5}\right)$ and consider

$$W = E \cdot V = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

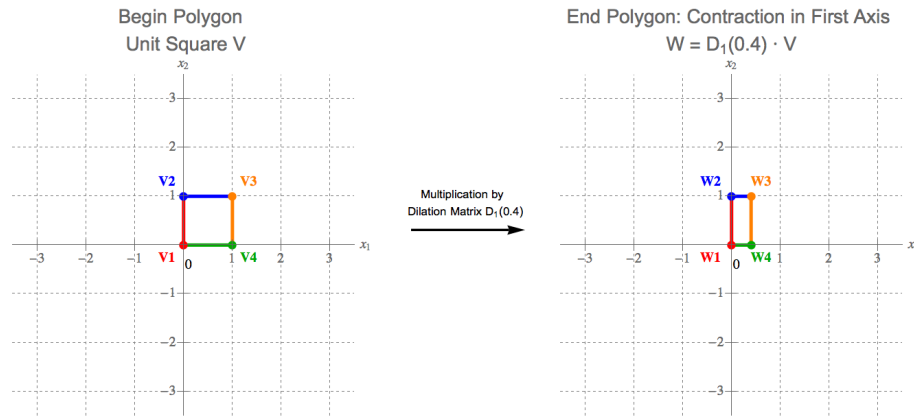
Again, we construct the vertices of our end polygon row-by-row. We begin by recognizing that

$$\begin{aligned} W(1,:) &= \begin{bmatrix} 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 0.4 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0.4 & 0.4 \end{bmatrix} \end{aligned}$$

To find the second row of W , we take a linear combinations of the rows of V . The scalar weights associated with row $V(i,:)$ is the i th column entry of vector $E(2,:) = \begin{bmatrix} 0 & 1 \end{bmatrix}$. In other words, we take 0 times $V(1,:)$ and add that to 1 times $V(2,:)$ to find $W(2,:) = V(2,:)$. We conclude

$$W = \begin{bmatrix} 0 & 0 & 0.4 & 0.4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

We can visualize this transformation using the diagram below



In both the expansion dilation Example 4.4.5 and in the contraction dilation Example 4.4.6, the end polygon W stayed in the same quadrant of the cartesian plane as the polygon V . The pure stretch or compression of polygon V without a change in orientation came from the assumption that $c > 0$. If $c < 0$, left multiplication by $D_1(c)$ will change the length of blue edge 2 and green edge 4 while also reflecting these edges about the second coordinate axis. More about reflections is found in the later parts of this section.

Dilations in the Second Coordinate Axis

To scale second coordinate entries, left multiply vertex matrix V by

$$D_2(c) = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}.$$

with our transformation between two vertices defined as

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This time, the entry-by-entry definition of the output of \mathbf{v} under this transformation is

$$w_1 = v_1 \quad \text{and} \quad w_2 = c \cdot v_2.$$

Once more, we assume $c > 0$ to scale the second coordinate entry v_2 without changing the parity of it's value. Compression along the second coordinate axis occurs when $0 < c < 1$ while expansion results from $c > 1$.

EXAMPLE 4.4.7

We continue with $E = D_2(3)$ and consider

$$W = E \cdot V = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

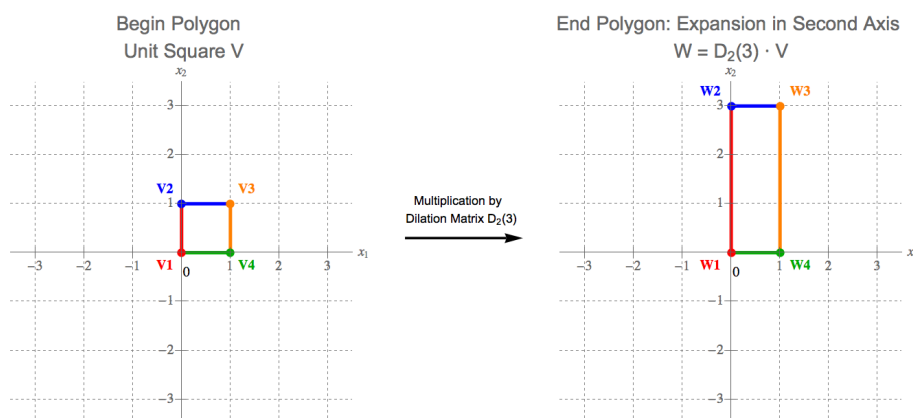
Since $E(1, :) = \begin{bmatrix} 1 & 0 \end{bmatrix}$ we see there is no change in row one with $W(1, :) = V(1, :)$. We calculate the second row as

$$\begin{aligned} W(2, :) &= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 0 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 3 & 3 & 0 \end{bmatrix}. \end{aligned}$$

The full end vertex matrix is

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 3 & 3 & 0 \end{bmatrix}.$$

The algebraic effect of left multiplication of V by $D_2(3)$ is to scale the entire second row by a factor of 3. Since the second coordinate entries of our vertex matrix are store in the second row, geometrically this transformation is an expansion along the second coordinate axis by a factor of 3.



EXAMPLE 4.4.8

For a contraction along the second coordinate axis, fix $E = D_2\left(\frac{3}{5}\right)$ and let

$$W = E \cdot V = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

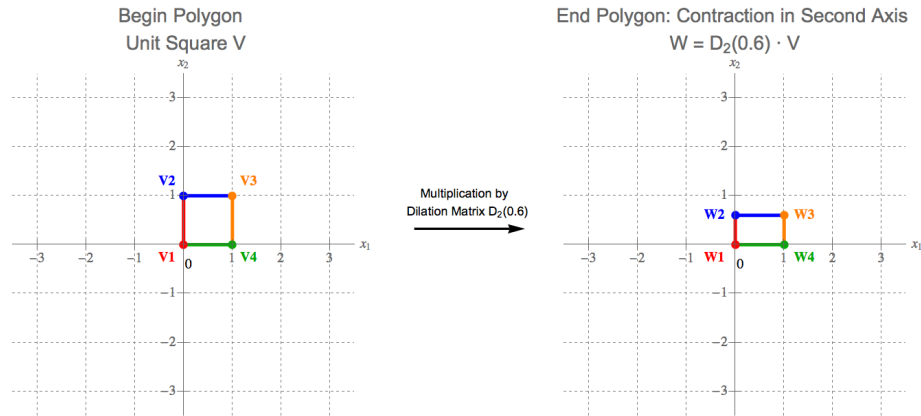
Because our dilation leaves row 1 untouched, we focus on row two to find

$$\begin{aligned} W(2, :) &= \begin{bmatrix} 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 0 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 0.6 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 0.6 & 0.6 & 0 \end{bmatrix}. \end{aligned}$$

The vertex matrix for our end polygon is

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0.6 & 0.6 & 0 \end{bmatrix}$$

where $W(2, :) = 0.6 \cdot V(2, :)$. The scalar $c = 0.6 < 1$ and the begin polygon V is compressed along the second axis to produce end polygon W .



Reflections about Special Axis

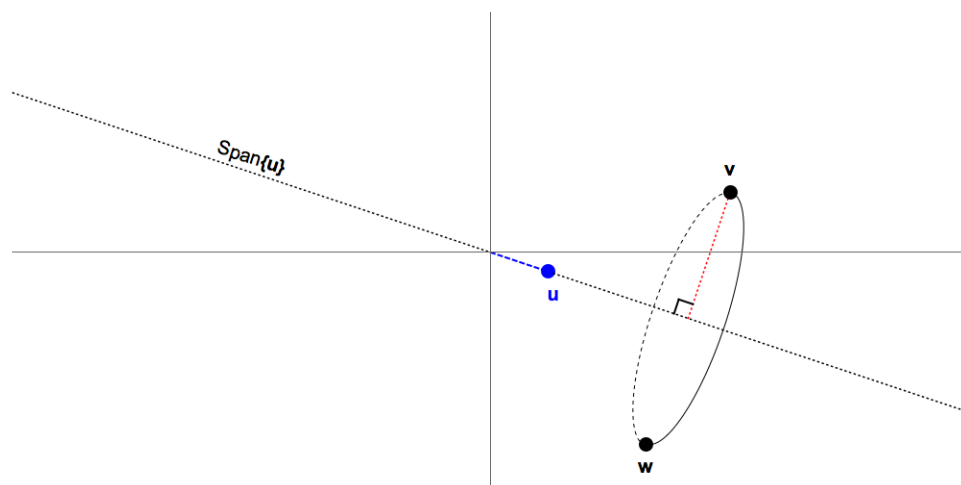
Both shear matrices $S_{ik}(c)$ and dilation matrices $D_j(c)$ alter the lengths of edges of the begin polygon V in order to produce end polygon W . These changes in length result from the choice of scalar c in both cases. For dilation matrices, the change in length is given by c times the original length. For shear matrices, finding the change in lengths for effected edges is a bit more involved and will be left as an exercise.

There is another class of transformations, known as reflections, that do not change the lengths of the edges of our polygon. Instead, reflections transform the coordinates of each vertex in a special manner. In order to define a reflection, we first choose a line in \mathbb{R}^2 about which to reflect the vertices. We call this the line of reflection. To define this line, we choose a single unit vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and say that the line of reflection is given by $\text{Span}\{\mathbf{u}\}$.

Any vertex that begins on the line of reflection will remain on this line at the end of the transformation. However, for any vertex that is not on this line, a reflection will rotate that vertex perpendicularly around this line to a point the same distance away from this line the other side of it.



We will begin our discussion of reflections by investigating the most common reflections. Then, we will create a very general paradigm from which we can create any line of reflection we desire.

EXAMPLE 4.4.9

For our first reflection, we set unit vector

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This vector defines the entire second coordinate axis as the line of reflection. In other words, we want to rotate our begin polygon 180° about the second coordinate axis to create our end polygon.

To accomplish such a translation, we set $E = D_1(-1)$ and define the map

$$W = E \cdot V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

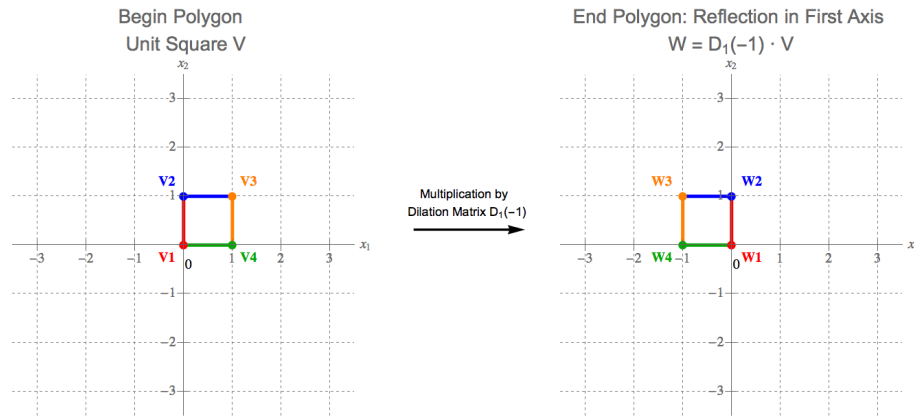
Base on this definition, we see

$$\begin{aligned} W(1, :) &= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= -1 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix}. \end{aligned}$$

The second coordinate entries remain unaltered, with $W(2, :) = V(2, :)$. Then, the reflected end polygon

$$W = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

has edges of equal length to the edges in V , yet all vertices that are not on the second coordinate axis have been reflected perpendicularly about this axis, as seen below.

**EXAMPLE 4.4.10**

Next, let's reflect about the first coordinate axis by defining unit vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, we rotate our begin polygon 180° about the first coordinate axis. Algebraically, we assign $E = D_2(-1)$ and calculate

$$W = E \cdot V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

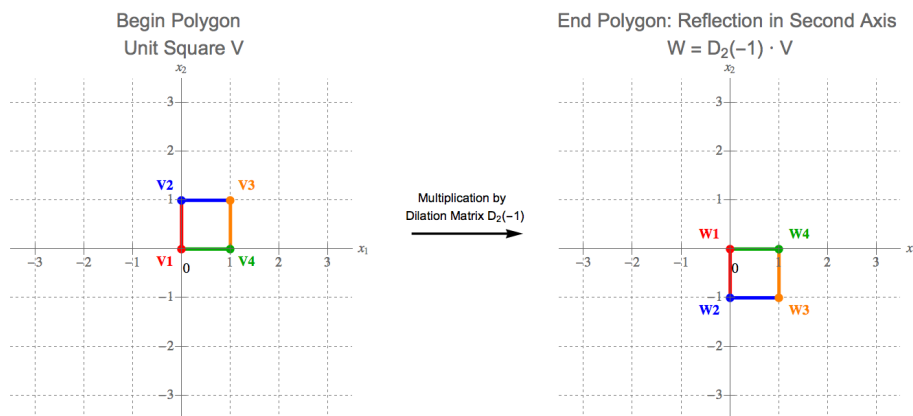
Once again, row one remains identical with $W(1,:) = V(1,:)$. The major change occurs in row two, given by

$$\begin{aligned} W(2,:) &= \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= 0 \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} 0 & -1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

The vertex matrix

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

stores end polygon W which results from reflecting V about the first coordinate axis.

**EXAMPLE 4.4.11**

Next, let's set

$$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and reflect about the line given by $\text{Span}\{\mathbf{u}\}$. This reflection will swap the first and second coordinates of each vertex. To do so, we let $E = P_{12}$ and consider

$$W = E \cdot V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again, we construct the vertices of our end polygon row-by-row. We begin by recognizing that

$$\begin{aligned} W(1,:) &= [0 \ 1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 0 \cdot [0 \ 0 \ 1 \ 1] + 1 \cdot [0 \ 1 \ 1 \ 0] \\ &= [0 \ 1 \ 1 \ 0] \end{aligned}$$

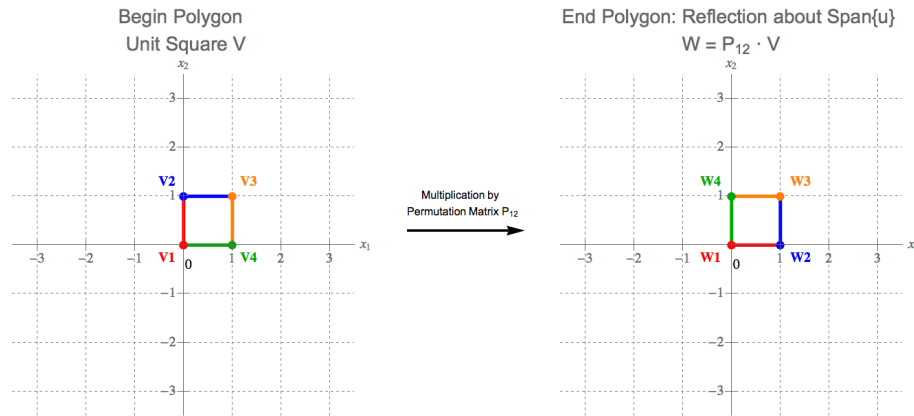
We find that the vector of second coordinates of the end polygon is calculated

$$\begin{aligned} W(2,:) &= [1 \ 0] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 1 \cdot [0 \ 0 \ 1 \ 1] + 0 \cdot [0 \ 1 \ 1 \ 0] \\ &= [0 \ 0 \ 1 \ 1] \end{aligned}$$

These two vectors together compose the matrix W , and thus we have

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We can visualize this transformation using the diagram below

**EXAMPLE 4.4.12**

We continue with $E = -P_{12}$ and consider

$$W = E \cdot V = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again, we construct the vertices of our end polygon row-by-row. We begin by recognizing that

$$\begin{aligned} W(1,:) &= [0 \quad -1] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= 0 \cdot [0 \quad 0 \quad 1 \quad 1] + (-1) \cdot [0 \quad 1 \quad 1 \quad 0] \\ &= [0 \quad -1 \quad -1 \quad 0] \end{aligned}$$

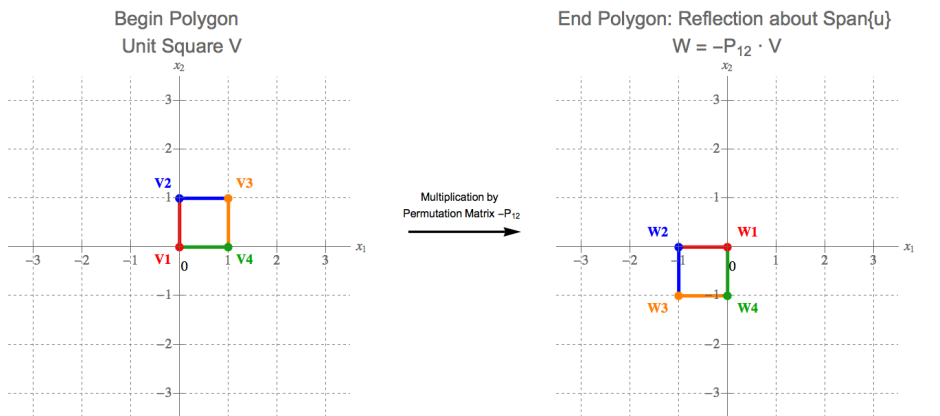
We find that the vector of second coordinates of the end polygon is calculated

$$\begin{aligned} W(2,:) &= [-1 \quad 0] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= (-1) \cdot [0 \quad 0 \quad 1 \quad 1] + 0 \cdot [0 \quad 1 \quad 1 \quad 0] \\ &= [0 \quad 0 \quad -1 \quad -1] \end{aligned}$$

These two vectors together compose the matrix W , and thus we have

$$W = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

We can visualize this transformation using the diagram below



Suggested Problems:

For all the problems below, be sure to explicitly state the dimensions of the vectors you use for each model.

1. Investigate multiplication of begin vertex V from this section by $E = S_{ik}(c) \in \mathbb{R}^{4 \times 4}$. Determine the effects of each type of shear matrix under right multiplication.
2. Rotation around an arbitrary point
3. Using Givens rotation to zero out arbitrary coefficients in a vector (Datta p. 195)
4. Redo all the examples from this section with a different begin polygon

$$V = \begin{bmatrix} -2 & -2 & 1 & \end{bmatrix}$$

Suppose this polygon has edge table HERE. What do you notice about the transformations on this polygon versus those discussed in this section? Explain your observations and figure out what is happening here. Use this to make general statements about the effect of each of the transformation matrices on polygons in \mathbb{R}^2 depending on the initial location of the polygon vertices.

5. Recreate all of these geometric transformations in \mathbb{R}^3