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Math 48B, Lesson 9: Rational Functions, Part 1

In Math 48B Lessons 8, 9, and 10, we study rational functions in the form:

one polynomial on top and one polynomial on bottom

Rational functions
polynomial over polynomial

Numerator: Standard form of an nth degree polynomial

$$\boxed{R(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0 x^0}$$

Rational Function (RATIO of polynomials)

Denominator: Standard form of an mth degree polynomial

Rational functions are two different polynomials one in numerator and one in denominator

To begin our exploration, we explore some fundamental properties of division.

1. WHAT ARE RULES OF FRACTIONS?

Recall each of the following rules for fractions:

Divide zero by a number: $\boxed{\frac{0}{K} = 0}$

zero divided by anything nonzero is always zero (the denominator can never be zero)

Division by zero: $\boxed{\frac{K}{0} = \text{Error (Not possible)}}$

Any time the denominator is zero it results in an error (Not possible)

Divide an expression by itself: $\boxed{\frac{A}{A} = 1 \text{ as long as denominator } A \neq 0}$

when top & bottom are identical, fraction equals one (bottom can't be zero)

Multiplication of Fractions: $\boxed{\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}}$

when we multiply left fraction by right fraction, the top pair up and the bottom pair up

Division of Fractions: $\boxed{\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}}$

when we divide left fraction by right fraction, we swap C and D (D on top and C on bottom) and division sign becomes multiplication

when we divide two fractions, multiply by reciprocal

How to A's on math tests:

- Read the problem in full every time before you start
- work from top to bottom reading everything
- usually all information we need is on top of problem

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1B. Let's explore the following patterns for positive real numbers:

$$\boxed{\frac{1}{\text{HUGE NUMBER}} = \text{tiny number}}$$

To do so, let $f(x) = \frac{1}{x}$ and fill out the table below.

Input	*Output*
x	$f(x) = \frac{1}{x}$
$10^1 = 10$	$\frac{1}{10} = 0.1$
$10^2 = 100$	$\frac{1}{100} = 0.01$
$10^3 = 1{,}000$	$\frac{1}{1000} = 0.001$
$10^4 = 10{,}000$	$\frac{1}{10000} = 0.0001$
$10^5 = 1\boxed{00{,}000}$	$\frac{1}{100{,}000} = 0.00001$
1,000,000	$\frac{1}{1000000} = 0.000001$
10,000,000	$\frac{1}{10000000} = 0.0000001$

standard form of degree 0 polynomial

$$f(x) = \frac{1}{\underbrace{x}_{}} = \frac{\boxed{a_0 \cdot x^0}}{\boxed{b_1 x^1 + b_0 x^0}}$$

this is a rational function (polynomial over polynomial)

standard form of degree 1 polynomial

eg: $x = 10 \Rightarrow f(x) = f(10) = \frac{1}{10} = 0.1$

eg: $x = 100 \Rightarrow f(100) = \frac{1}{100} = 0.01$

eg: $x = 1000 \Rightarrow f(1000) = \frac{1}{1000}$ 0.001

- Every time we add a zero on input we add a zero behind the decimal point

- If we take

$$\frac{1}{10^5} = .00001.$$

the first slot is held by the number 1

we shift decimal point 5 slots to the left

Conjecture : $\frac{1}{10^n}$ when written in decimal form will have $n-1$ zeros between decimal point and the 1

$$\frac{1}{10^n} = 0.\boxed{0 \cdots 0}1$$

$n-1$ zeros here

As $n \to \infty$, $\underbrace{x = 10^n}_{\text{these are huge numbers}} \longrightarrow \infty$

$$\Rightarrow f(x) = \frac{1}{x} = \frac{1}{10^n} \longrightarrow 0^+$$

one over x approaches zero from above since $x > 0$ & $\frac{1}{x} > 0$

we notice that $\frac{1}{10^n} > 0$ is positive direction

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1C. Look back at your table from problem 1B. Now look at the notation below:

$$\lim_{x \to \infty} f(x) = 0^+$$

Please describe using VANVS how the notation is related your table. In other words, what ideas are captured in this notation?

Let's write this out slowly

$$\lim f(x) = 0^+$$

$x \to \infty$

★ this is asymptote (horizontal asymptote)

Nerdy language

- "the limit as x approaches positive infinity of f of x equals zero"
- "the limit of f of x as x approaches positive infinity is zero from the positive side"

Interpret :

$$\boxed{\lim_{x \to +\infty} f(x) = 0^+}$$

symbols (notation)

- This means as input x gets really large (closer to infinity) then the function output f(x) as a whole will approach zero from positive side

How to Thrive on a math test : read problem fully

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1D. Let's explore the following patterns for positive real numbers:

$$\boxed{\frac{1}{\text{tiny number}} = \text{HUGE NUMBER}}$$

To do so, let $f(x) = \frac{1}{x}$ and fill out the table below.

Input	*Output*
x	$f(x) = \frac{1}{x}$
$\frac{1}{10}$	$f(\frac{1}{10}) = 10$
$\frac{1}{100}$	$f(\frac{1}{100}) = 100$
$\frac{1}{1,000}$	$f(\frac{1}{1000}) = 1000$
$\frac{1}{10,000}$	10,000
$\frac{1}{100,000}$	100,000
$\frac{1}{1,000,000}$	1,000,000
$\frac{1}{10,000,000}$	10,000,000

eg: $x = \frac{1}{10} \Rightarrow f(x) = f(\frac{1}{10})$

$= \frac{1}{(1/10)}$

Consider: $\frac{1}{(1/10)} = \frac{1}{1} \div \frac{1}{10}$

$= \frac{1}{1} \cdot \frac{10}{1}$

$= \frac{1 \cdot 10}{1 \cdot 1}$

$= 10$

eg: $x = \frac{1}{100} \Rightarrow f(x) = \frac{1}{(1/100)}$

consider $\frac{1}{1/100} = \frac{1}{1} \div \frac{1}{100}$

$= \frac{1}{1} \cdot \frac{100}{1}$

$= \frac{1 \cdot 100}{1 \cdot 1}$

$= 100$

- ☐ Every time number in denominator increases, the result increases.
- ☐ As the input gets closer to zero the output gets closer to infinity

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1E. Look back at your table from problem 1B. Now look at the notation below:

$$\lim_{x\to 0^+} f(x) = \boxed{+\infty}$$

Please describe using VANVS how the notation is related your table. In other words, what ideas are captured in this notation?

make sure to mention function name

NOT QUITE

the limit as x approaches zero ~~from the postive direction equals~~

Formal language

"the limit as x approaches zero from positive side of f of x equals positive infinity"

"the limit of f of x as x approach zero from positive side is positive infinity"

Notice: in the two way to read this, we swap input and output locations.

this is calculus

Interpret : $\lim_{x \to 0^+} f(x) = +\infty$

- This means when we take one divided by a tiny number (when denominator is very close to zero), then the output will be a huge number closer to infinity

- we use limits as a mathematical tool to discuss behavior of function quickly/efficient

- limits are the mathematical tool to write equations for motion.

ANALOGY : ALGEBRA EQUATION ⟷ PHOTO (SINGLE SNAP SHOT IN TIME w/ NO MOVEMENT

ALGEBRA EQUATION → variable values (solutions) are constants

$\boxed{\text{DIFFERENTIAL (CALCULUS) EQUATIONS}} \longleftrightarrow \boxed{\text{MOVIES (A DYNAMIC PICTURE WITH MOVEMENT)}}$

solutions are functions that describe motion

In pre-calculus, we study $\boxed{\text{functions}}$ so we can use those in calculus

- [ ] algebra = constants | calculus = motion
- [ ] Calculus allows us to write equation that model thing that change
- [ ] limits are the central tools to make these equations make sense math-wise
- [ ]

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1F. Below is a table of symbols, known as arrow notation. Look over this table:

Symbol	*How to read this?*
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to -\infty$	x approaches negative infinity x decreases without bound
$x \to +\infty$	x approaches negative infinity x increases without bound

For each of these symbols, draw a diagram to describe the relationship being

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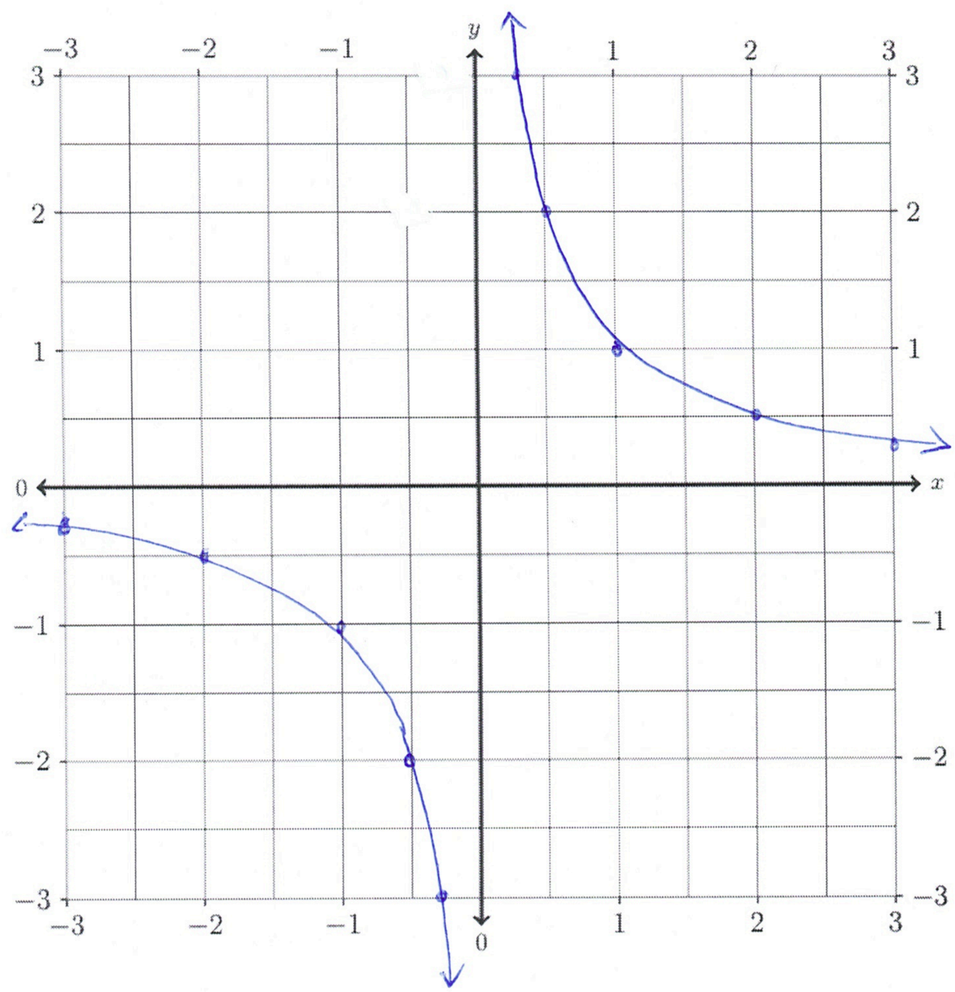
2. GRAPH THE SIMPLEST RATIONAL FUNCTION?

Consider the rational function $f(x) = \frac{1}{x}$

A. Fill in the table below
B. Plot these points on the axis provided
C. Interpolate between the points you plotted to create the graph of this function
D. Find the domain and range for this function.

Use Desmos.com to confirm you graph.

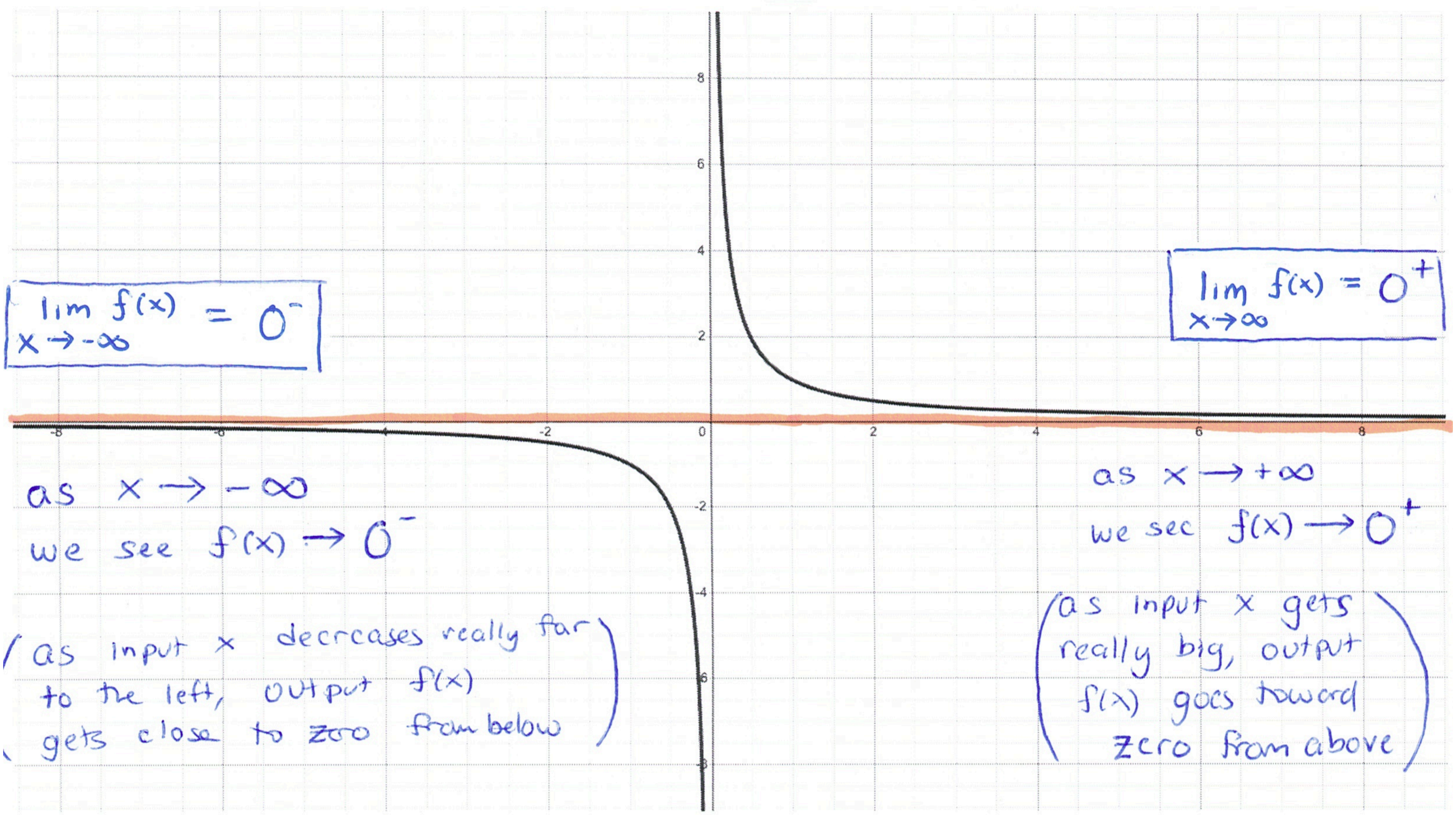
Input	*Output*
x	$f(x) = \frac{1}{x}$
-3	$\frac{1}{-3} = -\frac{1}{3}$
-2	$\frac{1}{-2} = -\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
0	Error
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	$1/1 = 1$
2	$\frac{1}{2} = 2$
3	$1/3 = 3$



Notice below we plot function $f(x) = \frac{1}{x}$ ←

☐ The horizontal line $\boxed{y=0}$ is a $\boxed{\text{horizontal asymptote}}$

(this is a value that output $y = f(x)$ gets really close to but doesn't touch in end behavior as $x \to \pm\infty$)



$\boxed{\lim_{x \to -\infty} f(x) = 0^-}$

$\boxed{\lim_{x \to \infty} f(x) = 0^+}$

as $x \to -\infty$

we see $f(x) \to 0^-$

(as input x decreases really far to the left, output $f(x)$ gets close to zero from below)

as $x \to +\infty$

we see $f(x) \to 0^+$

(as input x gets really big, output $f(x)$ goes toward zero from above)

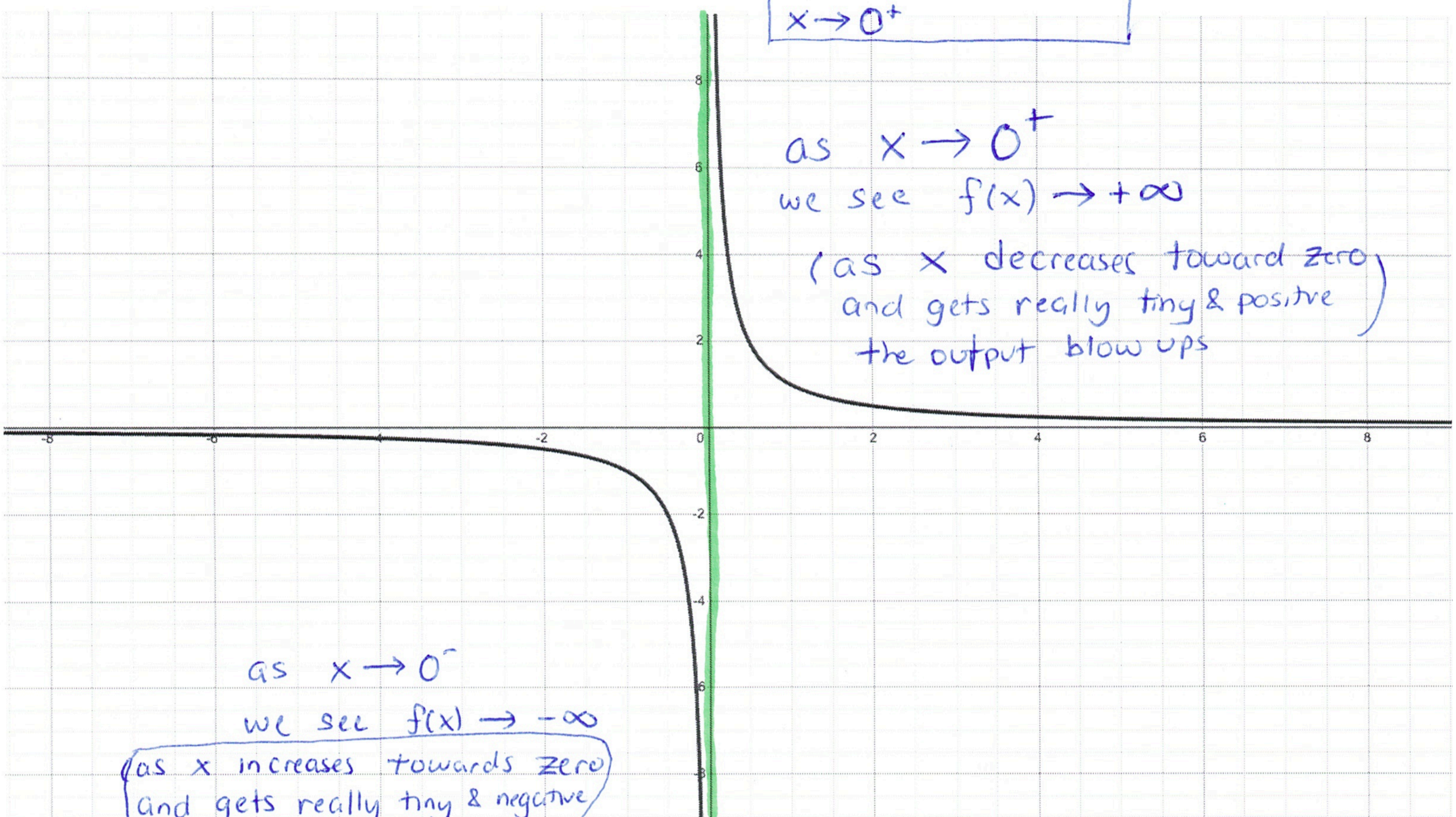
"when gets smaller (really negative) $f(x)$ goes to zero"

"when x gets big $f(x)$ goes to zero"

☐ The vertical line $\boxed{x = 0}$ is a $\boxed{\text{vertical asymptote}}$

$\frac{1}{\text{tiny number}}$ = HUGE NUMBE

$$\boxed{\lim_{x \to 0^+} f(x) = +\infty}$$



as $x \to 0^+$

we see $f(x) \to +\infty$

(as x decreases toward zero and gets really tiny & positive the output blow ups)

as $x \to 0^-$

we see $f(x) \to -\infty$

(as x increases towards zero and gets really tiny & negative the output decreases w/out bound)

$$\boxed{\lim_{x \to 0^-} f(x) = -\infty}$$