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Math 48B, Lesson 8: Rational Functions, Part 1

In Math 48B Lessons 8, 9, and 10, we study techniques to ~~find the zeros of a polynomial by factoring that polynomial into a form~~: understand rational functions:

Numerator: Standard form of an nth degree polynomial

$$\boxed{R(x)} = \frac{a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x^1 + a_0\, x^0}{b_m\, x^m + b_{m-1}\, x^{m-1} + \cdots + b_1\, x^1 + b_0\, x^0}$$

Rational Function (RATIO of polynomials)

Denominator: Standard form of an mth degree polynomial

(polynomial divided by polynomial)

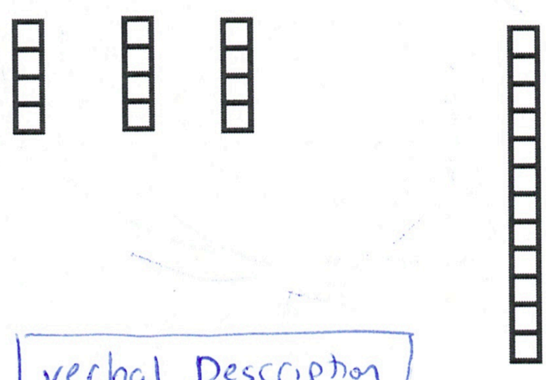
To begin our exploration, we explore some fundamental properties of division.

1. WHAT IS MULTIPLICATION?

1A. Use abuelita language (simple language that your grandma would understand) to describe what you see in multiplication problems below:

Visual representation

$4 + 4 + 4 = 3 \cdot 4 = 12$



Symbolic Representation

multiplication sign

$\boxed{3} \cdot \boxed{4} = \boxed{12}$

left factor — right factor — product

Verbal Description

- 3 group of 4 units is a total of 12 individual units

factor : • piece of a product
• part of a multiplication

- As we study <u>RATIO</u>nal Function (polynomial over polynomial) the zero values of the polynomial on the bottom have a big impact on the graph of our rational function!

- In other words, to understand the behavior of rational functions, we're going to look for location where the denominator is zero.

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3. WHAT IS GOING ON WHEN WE DIVIDE BY ZERO?

3A. Consider the following division problem:

$$1 \div 0 = \frac{1}{0} = q$$

Not possible
No solution
Nonsense

Translate this problem into the corresponding multiplication problem (like you did in problem 2D above. What do you notice? Given your observation, what is your answer to this division problem?

Division Problem

$$1 \div 0 = \frac{1}{0} = \boxed{q}$$

these are two representation of the same problem

quotient

$$\Rightarrow \quad \boxed{1 = 0 \cdot q}$$

multiplication Problem

find the quotient q s.t. when we multiply q by zero, we get 1

PROBLEM:
- Zero times anything equals zero
- Zero times any q can not equal one
- There is NO real answer to this problem

CONCLUSION: NOT Possible to divide a nonzero number by zero!

3B. Suppose that $N \in \mathbb{R}$ is any nonzero real number, with $N \neq 0$. Consider the following division problem:

not equal to zero

quotient

$$N \div 0 = \frac{\boxed{N}}{0} = \boxed{q}$$

Translate this problem into the corresponding multiplication problem (like you did in problem 2D above. What do you notice? Given your observation, what is your answer to this division problem? Put this observation into a full sentence to describe your findings.

Division Problem

$$N \div 0 = \frac{N}{0} = q$$

Zero on bottom

Multiplication Problem

$$N = 0 \cdot q$$

this value must be nonzero

- Any time I divide a nonzero number $N \neq 0$ by zero, I'm looking for quotient q such that zero times q equals N, which is nonzero.
- This is nonsense: q times zero is always zero ~~and~~ never equal to N.

3C. Consider the following division problem:

$$0 \div 0 = \frac{0}{0} = q$$

Translate this problem into the corresponding multiplication problem (like you did in problem 2D above. What do you notice? Given your observation, what is your answer to this division problem? Put this observation into a full sentence to describe your findings.

original problem

want to find this

Division Problem

$$0 \div 0 = \boxed{\frac{0}{0}} = q$$

to turn division into multiplication we multiply denominator by quotient

Multiplication Problem

$$0 = 0 \cdot q$$

☐ When we divide zero by zero, we want to find quotient q such that q times zero is the numerator, which in this case is zero.

Note: for $\frac{0}{0} = q$, we see this

is true when $0 = 0 \cdot q$.

we can satisfy this with

$\boxed{\text{Option 1}}$ $q = 0 \Rightarrow \frac{0}{0} = 0 \Leftrightarrow 0 = 0 \cdot 0$ ✓

$\boxed{\text{Option 2}}$ $q = 1 \Rightarrow \frac{0}{0} = 1 \Leftrightarrow 0 = 0 \cdot 1$

$\boxed{\text{Option 3}}$ $q = 52 \Rightarrow \frac{0}{0} = 52 \Leftrightarrow 0 = 0 \cdot 52$

In fact, we can say that $\frac{0}{0}$ is

any number we want since for

$$\frac{0}{0} = q \Leftrightarrow 0 = 0 \cdot q$$

no matter which q we choose, this is zero

Observations about division by zero

Observation 1 → **No solution**

We can't divide a nonzero number by zero:

$$\frac{N}{\boxed{0}} = q \iff \underbrace{N}_{\text{Nonzero}} = 0 \cdot q = \underbrace{0}_{\text{zero}}$$

Nonsense

Zero on bottom

Observation 2 → **Infinite Solution**

When we divide zero by zero, we have an infinite number of ways to do that:

$$\Rightarrow \frac{0}{\boxed{0}} = q \iff 0 = 0 \cdot q$$

Zero on bottom

- In both cases, zero shows up on bottom. In case 1, no answer is possible. In case 2, we have an infinite number of possible answers.

- We have a choice to make:

  Choice 1: remember the two seperate scenarios and try to agree on a single answer for case

  ★ **Choice 2: Don't divide by zero** ★

KO : Knock out (from boxing)

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3D. Look back on your work on problems 3A, 3B, and 3C above. Now, suppose that Suppose that $K \in \mathbb{R}$ is any real number (i.e. K might be nonzero or it may be zero). Consider the following division problem:

$$K \div 0 = \frac{K}{0} = D$$

What is your answer to this division problem? Come up with at least three different ways to remember your work here.

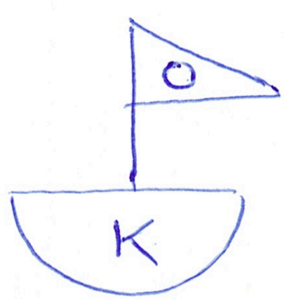
CONCLUSION: ☐ Any time we divide by zero, we say no solution

☐ We CANNOT divide by zero

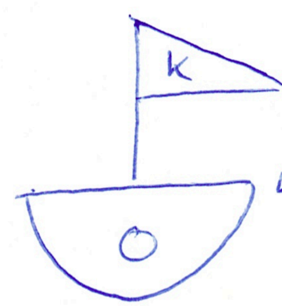
☐ It's not possible to divide by zero

☐ Way to remember:
When dividing by zero, were going to think of a boat

$$\frac{N}{D}$$



If you have a whole in sail but a full body you're OK



NO zero on bottom

If you have a full sail but a whole in your body you're KO'd (sunk).

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4. WHAT HAPPENS WHEN WE DIVIDE SOMETHING BY ITSELF?

Suppose that $A \in \mathbb{R}$ is any real number. Consider the fraction

$$\frac{A}{A}$$

What is this fraction equal to? When is that true? Why is that true?

Division Problem

$$\frac{A}{A} = q \iff A = A \cdot q$$

Multiplication Problem

Notice, as long as $A \neq 0$, then the only possible value of q is 1

$$\Rightarrow \boxed{\frac{A}{A} = 1 \text{ as long as } A \neq 0}$$

Any time I divide a number by itself I get 1 (as long as denom is not zero)

Example 4

$$R(x) = \frac{(x-4)}{(x-4)} = 1$$ as long as denominator is not zero

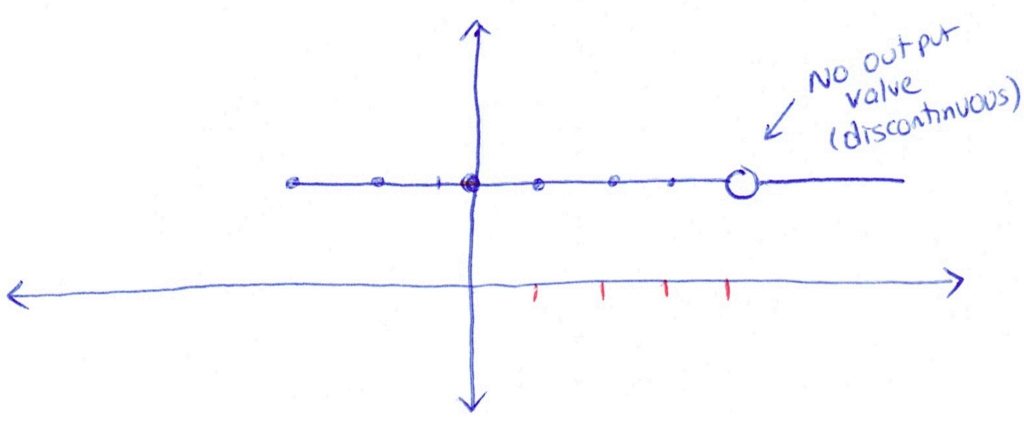
$$\Rightarrow \frac{(x-4)}{(x-4)} = 1$$ as long as $x - 4 \neq 0$

$$\Rightarrow \frac{x-4}{x-4} = 1$$ as long as $x \neq 4$

$$\Rightarrow R(x) = \frac{x-4}{x-4} = \begin{cases} 1 & \text{if } x \neq 4 \\ \text{Error} & \text{if } \underbrace{x = 4} \end{cases}$$

at $x = 4$
we see denominator
is zero
(we can't divide
by zero)

Graph this function $R(x) = \dfrac{x-4}{x-4} =$



x	R(x)
0	1
1	1
2	1
3	1
4	ERROR
5	1

$$x = 1 \Rightarrow R(x) = R(1) = \frac{1-4}{1-4} = \frac{-3}{-3} = 1$$

$$x = 2 \Rightarrow R(x) = R(2) = \frac{2-4}{2-4} = \frac{-2}{-2} = 1$$

$$x = 4 \Rightarrow R(x) = R(4) = \frac{4-4}{4-4} = \frac{0}{0}$$ NOT possible

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Example 4.2

quadratic (degree 2) polynomial ontop

$$R(x) = \frac{x^2+5x-6}{(x-1)}$$

polynomial over polynomial

Rational Function

RATIO: of polynomials

linear (degree 1) polynomial

$$\Rightarrow R(x) = \frac{x^2+5x-6}{(x-1)}$$

Note:

$x^2+5x-6 = (x+6)(x-1)$

$$\Rightarrow R(x) = \frac{(x+6)\cdot(x-1)}{1\cdot(x-1)}$$

$$\Rightarrow R(x) = \frac{(x+6)}{1}\cdot\boxed{\frac{(x-1)}{(x-1)}}$$

$$\frac{A\cdot C}{B\cdot D} = \frac{A}{B}\cdot\frac{C}{D}$$

$\Rightarrow R(x) = \frac{(x+6)}{1} \cdot 1$ as long as denominator $x - 1 \neq 0$

$\Rightarrow R(x) = x + 6$ as long as $x \neq 1$

Key take away:

When looking at the ratio of two polynomials, the location of zeros on the bottom make a big difference for behavior of the function

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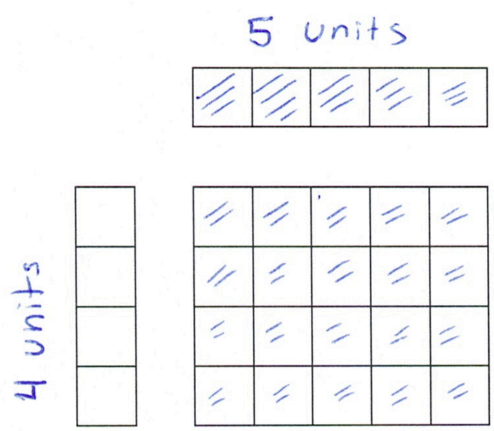
5. HOW DO WE MULTIPLY FRACTIONS?

5A. Consider the following multiplication problem:

$$4 \cdot 5 = 20$$

← let's think about this as an area problem

Develop a VANVS description of the answer to this multiplication problem. I've provided a visual below to help the development here.



- we have a length of four units times a width of five units
- 4 rows of 5 units for a total of twenty un

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5B. Consider the following multiplication problem:

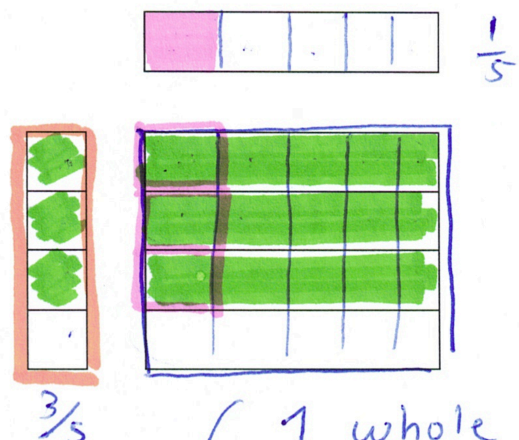
$$\boxed{\frac{3}{4}} \cdot \frac{1}{5} = \frac{3}{4} \div \frac{5}{1} = \frac{3}{4} \div 5$$

Develop a VANVS description of the answer to this multiplication problem. I've provided a visual below to help the development here.

take 3

$\boxed{\frac{3}{4}}$ = 4 分 の 3

out of four parts



(1 whole
cut into 20 pieces)

three fourths : □ take one whole
□ cut into four pieces
□ grab three of those 1/4 size pieces

counting problem : □ I have 3 pieces
where each piece comes
from slicing 1/4 into 5 pieces
(out of a total of 20 pieces)

$$\boxed{\frac{3 \cdot 1}{4 \cdot 5} = \frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}} = 20 \text{ 分 } \boxed{\text{の}}\ 3$$

out of 20 pieces
we grab 3
(where piece is of size $\frac{1}{20}$)

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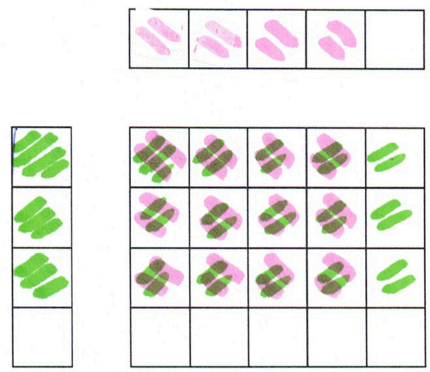
5C. Consider the following multiplication problem:

$$\frac{3\cdot 4}{4\cdot 5} = \frac{3}{4}\cdot\frac{4}{5} = \frac{12}{20} = \frac{3\cdot 4}{5\cdot 4} = \frac{3}{5}\cdot\cancel{\frac{4}{4}}$$

$$= \boxed{\frac{3}{5}}\ \checkmark$$

Develop a VANVS description of the answer to this multiplication problem. I've provided a visual below to help the development here.



☐ How many, out of all 20 pieces shown do we highlight?

☐ 20 分の 12 $= \frac{12}{20}$

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5D. Look back at your answers to problems 5ABC. Now consider the multiplication problem:

$$\boxed{\frac{A}{B}\cdot\frac{C}{D} = \frac{A\cdot C}{B\cdot D} = \frac{AC}{BD}}$$

What is your answer to this problem? Come up with at least three different ways to remember your work here.

Two-way street:

$\xrightarrow{\text{forward}}$

Option 1

$$\boxed{\frac{A}{B}\cdot\frac{C}{D}} = \boxed{\frac{AC}{BD}}$$

start here

end here

- combine two fractions together

Option 2

$$\boxed{\frac{AC}{BD}} = \boxed{\frac{A}{B}\cdot\frac{C}{D}}$$

start here

end here

- uncombine two fractions
- factor out parts of top & bottom

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6. HOW DO WE DIVIDE FRACTIONS?

Develop a VANVS description of the answer to the division problem:

left fraction

$$\boxed{\frac{A}{B}} \div \boxed{\frac{C}{D}} = \boxed{\frac{A}{B}} \cdot \boxed{\frac{D}{C}}$$

right fraction

What is your answer to this problem? Come up with at least three different ways to remember your work here. Note: I don't have a good visual for this one. I would love to learn from you. Please share what you discover.

Algebraic Argument:

$$\frac{A}{B} \div \frac{C}{D} = q$$

$$\Rightarrow \frac{\left[\frac{A}{B}\right]}{\left[\frac{C}{D}\right]} = q \qquad \frac{x}{y} = q$$

$$\Rightarrow \frac{A}{B} = \frac{C}{D} \cdot q \qquad x = y \cdot q$$

$$\Rightarrow \quad \frac{A}{B} \cdot D = \frac{\cancel{D}}{1} \cdot \frac{C}{\cancel{D}} \cdot q$$

$$\Rightarrow \quad \frac{A}{B} \cdot \frac{D}{C} = \frac{\cancel{C} \cdot q}{\cancel{C}}$$

$$\Rightarrow \quad q = \boxed{\frac{A}{B} \cdot \frac{D}{C} = \frac{A}{B} \div \frac{C}{D}}$$

$\boxed{\text{Community Challenge:}}$

Come up with a visual description for the formula

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$