Name: Solutions

How do we develop deep understanding of math ideas → VANVS

V: Verbal description
A: Abuelita language
N: Nerdy language
V: Visual
S: Symbolic

Math 48B, Lesson 6: Zeros of Polynomials, Part 2

In Math 48B Lessons 3, 4, 5, and 6, we study techniques to find the zeros of a polynomial by factoring that polynomial into a form:

Our math Notes are a 2nd brain (human memories are finite: taking good notes is really helpful to remember)

$$\boxed{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0} = \boxed{a(x - c_1) \cdot (x - c_2) \cdots (x - c_n)}$$

Standard Form of an nth Degree Polynomial (written in descending order)

Complete Zero Factorization Form (Factored completely into linear factors)

In this lesson, we continue our exploration from Lesson 4.

0. USE REMAINDERS TO FIND ZEROS

0A. Read the following theorem and "how to" sections:

Every time I read a math definition or theorem, I think WTF

THE REMAINDER THEOREM

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is the value $f(k)$.

HOW TO

Given a polynomial function f, evaluate $f(x)$ at $x = k$ using the Remainder Theorem.

1. Use long division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

Look back at your work from Math 48B, Lesson 5. Now, translate these theorems into abuelita (intuitive) language. What, exactly, is this saying about the zeros of a polynomial.

Key math idea: □ when studying a general principle, focus on understanding specific examples

□ use the intuition from the specific example to build understanding of the general principle

Example 1: The remainder theorem in action

General Principle: (Nerdy language)

If a polynomial $P(x)$ is divided by factor $(x - k)$, then the remainder of that division problem is the value output $P(x)$ evaluated at $x = k$.

Specific Example: (translate into intuition)

Let's look back on Problem 4A from Lesson 5 ...

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\
\hline
(x-5) & \boxed{x^3} - 6x^2 \underbrace{- x}\ \underbrace{+ 30} \\
 & -x^3 + 5x^2 \\
\hline
 & 0 - x^2 - x \\
 & + x^2 - 5x \\
\hline
 & 0 - 6x + 30 \\
 & + 6x - 30 \\
\hline
 & 0 + \boxed{0}
\end{array}
$$

remainder

side notes:

- $x^2(x-5) = x^3 - 5x^2$
- $-x(x-5) = -x^2 + 5x$

remember: a negative times a negative is a positive

- $-6(x-5) = -6x + 30$

If polynomial $\boxed{P(x) = x^3 - 6x^2 - x + 30}$ is divided by linear factor $(x - 5)$ with $k = 5$, then the remainder of the division problem is the value of the output $P(x)$ when $\boxed{x = 5 = k}$.

Note: this theorem claims that the final results

$$P(5) = 0$$

the remainder of our division problem

$$\frac{P(x)}{x - k}$$

Check: $P(5) = P(x)\Big|_{x=5}$

$$= (x^3 - 6x^2 - x + 30)\Big|_{x=5}$$

$$= 5^3 - 6 \cdot 5^2 - 5 + 30$$

$$= 125 - 6 \cdot 25 - 5 + 30$$

$$= 125 - 150 - 5 + 30$$

$$= 120 - 150 + 30$$

$$= 150 - 150$$

$$= \boxed{0 \leftarrow \text{the remainder of our division problem}}$$ ✓

this theorem rings true for this example...

To make math easier to read, focus on specific examples

General Approach:
Nerdy language

To use the remainder theorem, we run two steps:

Step 1: Use long division to divide the polynomial $P(x)$ by $x-k$

Step 2: The remainder of that division problem is the value $P(k) = P(x)\big|_{x=k}$

"P of k" = "P evaluated at k"

Specific Example
Abuelita language

Let's look back on Problem 2 from Lesson 5:

$$\frac{3x^2 - 5x - 17}{3x + 1} =$$

Step 1: Use long division to divide the polynomial

$$\boxed{x} \quad -2$$

$$(\underline{3x^1} + 1) \overline{\big)\ \boxed{3x^2} - 5x \quad \underbrace{-17}}$$

$$- \ \underline{3x^2 - 1x}$$

$$0 \quad \boxed{-6x - 17}$$

$$\underline{+6x + 2}$$

$$0 \quad \boxed{-15}$$

remainder

☐ How many times does $3x$ go into $3x^2$

☐ What do we need to multiply $3x$ by to get $3x^2$

☐ $3x^2 = 3 \cdot x \cdot x$

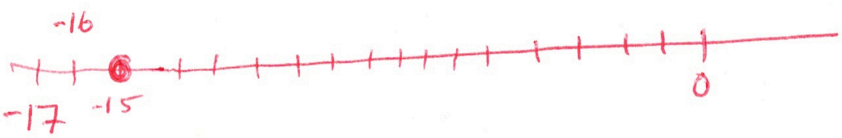
☐ $3x = 3 \cdot x$

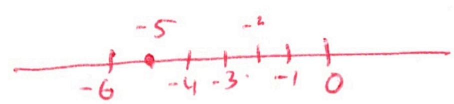
we need one more x

☐ $x^1 \cdot (3x^1 + 1) = 3x^{\boxed{2}} + x$

☐ $-5 - 1 = -6$

(subtraction)





☐ $-5 + 1 = -4$

☐ How many times $3x$ goes into $-6x$

☐ What do we need to multiply $3x$ by to get $-6x$

$$-6x = \boxed{-2}\, 3 \cdot x$$

☐ $-2 \cdot (3x + 1) = -6x - 2$

$$\frac{3x^2 - 5x - 17}{3x + 1} = \boxed{(x-2) R (-15)}$$

Long Division : $\frac{47}{3} = 47 \div 3$

multiply

$$\begin{array}{r} \boxed{1} \\ 3\overline{)\boxed{4}7} \end{array}$$

How many factors of 3 go into the number 4

put result down here

- Abuela : grandma

  Abuelita: cute way saying grandma

$$\begin{array}{r} 15 \\ 3\overline{)\boxed{4}\boxed{7}} \\ -3 \\ \hline \boxed{17} \\ -15 \\ \hline \boxed{2} \end{array}$$

how many times does 3 go into 17

remainder (what's left over)

$\Rightarrow \frac{47}{3} = 15 \text{ R } 2$

In this case, we see that we divided by

$$3x + 1$$

which is not in the form of $x - k$

$$3x + 1 = 3 \cdot \left(x + \tfrac{1}{3}\right)$$

$$= 3 \cdot \left(x - \tfrac{-1}{3}\right)$$

$$= 3 \cdot (x - k)$$

where $k = -1/3$

Notice: $P(x) = 3x^2 - 5x - 17$

$$= (3x + 1) \cdot (x - 2) + -15$$

$$= 3\left(x - \tfrac{-1}{3}\right) \cdot (x - 2) + \underbrace{-15}_{\text{remainder}}$$

$$\Rightarrow P(\underbrace{-1/3}) = 3 \cdot \left(\cancel{-1/3 - \tfrac{-1}{3}}\right)\left(-\tfrac{1}{3} - 2\right) + -15$$

cancel out

$x = -1/3$ $\quad = 3 \cdot 0 \cdot \left(-\tfrac{7}{3}\right) - 15 = 0 - 15 = \boxed{-15}$

remainder

Rookie

If we look at

$$\boxed{\text{Division problem}} \qquad \frac{P(x)}{(x-k)} = q(x) + \frac{r(x)}{(x-k)}$$

$$\Rightarrow \quad P(x) = (x-k)\cdot q(x) + r(x)$$

multiplication problem
(move denom to other side)

x = k

symbol pushing

$$\Rightarrow \quad P(k) = P(x)\Big|_{x=k}$$

$$= \Big[(x-k)\cdot q(x) + r(x)\Big]\Big|_{x=k}$$

$$= (\cancel{k-k})\cdot q(k) + r(k)$$

0 cancel out

$$= \underbrace{0 \cdot q(k)}_{\text{total annihilation}} + r(k)$$

$$= r \leftarrow \text{remainder}$$

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0B. Read the following theorem and "how to" sections:

THE FACTOR THEOREM

According to the **Factor Theorem**, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

HOW TO

Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial.

1. Use long division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

How are these sections related to your work on Problem 0A above?

Remember: In problem 1A, we claim that

$$\frac{P(x)}{(x-k)} = q(x) + \frac{r(x)}{(x-k)}$$

$$\Rightarrow P(x) = (x-k)\cdot q(x) + r(x)$$

eg 1: $$\frac{x^3-6x^2-x+30}{x-5} = (x^2-x-6) + \frac{\boxed{0}}{x-5}$$

$$\Rightarrow x^3-6x^2-x+30 = \boxed{(x-5)}(x^2-x-6) + \boxed{0}$$

this is a "pure" factor
(the remainder is zero)

$P(5) = 0$

eg 2: $\frac{3x^2-5x-17}{3x+1} = (x-2) + \frac{-15}{3x+1}$

$\Rightarrow 3x^2-5x-17 = \underbrace{(3x+1)}\cdot(x-2) - 15$

this is NOT
a "pure" factor
because the remande
is NOT zero
$P(-1/3) = -15 \neq 0$

Notice: when we divide

$\frac{P(x)}{x-k} = q(x) \quad R \; \boxed{r(x)}$

if this is
zero, we find
a PURE factor

we say $(x-k)$ is a pure factor of $P(x)$ or
($x=k$ is a zero of $P(x)$
that $P(k)=0$) if and only if
the remainder of the division is zero.

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1. CREATE A POLYNOMIAL WITH DESIRED ZERO VALUES

1A. Create a quadratic polynomial with two zero given by

$$x = \frac{2}{5} \qquad \text{and} \qquad x = -\frac{3}{4}$$

Math Memories Make Money

$x = k$ is a zero

$\Rightarrow x - k = 0$

$\Rightarrow x - k$ is a factor of polynomial

$\Rightarrow$ to figure out zeros in linear form get RHS equal to zero

$$x = \frac{2}{5} \Rightarrow x - \frac{2}{5} = 0$$

$$\Rightarrow 5 \cdot \left(x - \frac{2}{5}\right) = \underbrace{5 \cdot 0}$$

$$\Rightarrow \underbrace{5x - 2}_{\text{linear factor 1}} = 0$$

$$x = \frac{-3}{4} \Rightarrow x + \frac{3}{4} = 0$$

$$\Rightarrow 4\left(x + \frac{3}{4}\right) = 4 \cdot 0$$

$$\Rightarrow \underbrace{4x + 3}_{\text{linear factor 2}} = 0$$

To create our desired polynomial, we can multiply the linear factor together

$$P(x) = (5x - 2) \cdot (4x + 3)$$

$$= 5x \cdot (4x + 3) - 2(4x + 3)$$

$$= 20x^2 + 15x - 8x - 6$$

$$\boxed{= 20x^2 + 7x - 6} \leftarrow \text{option 1}$$

to get from option 1 to option 2, we divide by 20

Or, we could have messed with fractions

$$\left(x - \frac{2}{5}\right) \cdot \left(x + \frac{3}{4}\right) = x^2 - \frac{2}{5}x + \frac{3}{4}x - \frac{2}{5} \cdot \frac{3}{4}$$

$$= x^2 - \frac{2}{5}x + \frac{3}{4}x - \frac{6}{20}$$

$$\boxed{= x^2 + \frac{7}{20}x - \frac{6}{20}}$$

↑ option 2

Note:

$$-\frac{2}{5} + \frac{3}{4} = -\frac{2}{5} \cdot \frac{4}{4} + \frac{3}{4} \cdot \frac{5}{5}$$

$$= \frac{-8}{20} + \frac{15}{20} = \frac{7}{20}$$

Notice :

$$P(x) = 20x^2 + 7x - 6 = (5x-2)\cdot(4x+3)$$

$$P(x) = a_2 x^2 + a_1 x + a_0 \quad \text{where}$$

$$a_2 = 20, \quad a_1 = 7, \quad a_0 = -6$$

Let's remember our rational zeros of this polynomial are

$$x = \frac{\boxed{2}}{\boxed{5}} \quad \text{or} \quad x = -\frac{\boxed{3}}{\boxed{4}}$$

Look at factors of $a_2 = 20$ & $a_0 = -6$

Factors of $a_2 = 20$: $\pm 1, \pm 2, \pm\boxed{4}, \pm\boxed{5}, \pm 10, \pm 20$

Factor of $a_0 = -6$: $\pm 1, \pm\boxed{2}, \pm\boxed{3}, \pm 6$

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1C. What relationship do you notice between the rational zeros of this polynomial from Problems 1AB and the factors of the coefficients a_2 and a_0.

Assuming our quadratic has rational zero (zeros in fractional form), we see each zero is written

some factor of a_0

$$k = \frac{\boxed{N}}{\boxed{D}}$$

some factor of a_2

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2. USE DESMOS TO FIND THE ZEROS

2A. Use Desmos.com to find the zeros of the polynomial

$$P(x) = 2x^3 + x^2 - 4x + 1$$

If our goal is to find the zeros of this polynomial, how can we do that using the graph of our function.

Factors : $a_3 = 2$: $\pm 1, \pm 2$

Factor $a_0 = 1$: ± 1

Possible rational zeros

$\frac{N}{D}$: $\pm\frac{1}{1}$, $\pm\frac{1}{2}$

Only a few choices for zeros : ± 1 or $\pm 1/2$

$\Rightarrow$ $P(+1) =$

$P(-1) =$

$P(+1/2) =$

$P(-1/2) =$

Tues 5/4/2021 @ 10:45am

☐ Question: this theorem that we are studying, does it only apply when we divide a polynomial in the form $(x-k)$? would it apply if we divide in the form $(x+k)$?

Answer

Esme's conjecture

$$3x^2 + 1 = 0$$

$$\Rightarrow 3x^2 = -1$$

$$\Rightarrow x^2 = \frac{-1}{3}$$

$$\Rightarrow \sqrt[2]{x^2} = \sqrt[2]{\frac{-1}{3}}$$

$$\Rightarrow |x| = \frac{i}{\sqrt{3}}$$

complex : $\boxed{i = \sqrt{-1}}$

$$\Rightarrow \boxed{x = \pm \frac{i}{\sqrt{3}}}$$

irrational

Point / take-away: Not all zeros are rational