Name: Solutions

How to build deep understanding of math concepts?

VANVS
V : Verbal
A : Abuelita language
N : Nerdy language
V : Visual
S : symbolic

Math 48B, Lesson 5: Dividing Polynomials, Part 1

In Math 48B Lessons 3, 4, 5, and 6, we study techniques to find the zeros of a polynomial by factoring that polynomial into a form:

$$\boxed{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0} = \boxed{a(x - c_1) \cdot (x - c_2) \cdots (x - c_n)}$$

Standard Form of an nth Degree Polynomial (written in descending order)

Complete Zero Factorization Form (Factored completely into linear factors)

In this lesson, we continue our exploration from Lesson 4.

1. FIND ZEROS WITH POLYNOMIAL LONG DIVISION

1A. Please multiply and simplify the following polynomial expression:

$$P(x) = (x + 2)(x - 3)(x - 5)$$

Consider the polynomial:

$$P(x) = \Big[\underbrace{\overset{(A+B)}{(x+2)} \cdot \overset{C}{\boxed{(x-3)}}}_{\text{start here}}\Big] \cdot (x-5)$$

$$= \left[x \cdot (x-3) + 2 \cdot (x-3)\right](x-5)$$

$$= (x^2 - 3x + 2x - 6)(x-5)$$

$$= (x^2 - x - 6) \cdot (x-5)$$

Distributivity
- $A \cdot (B + C) = A \cdot B + AC$
- $(A + B) \cdot C = AC + BC$

Keep going on next page..

$$\Rightarrow P(x) = \overset{A}{\boxed{(x^2 - x - 6)}} \cdot \overset{(B - C)}{(x - 5)}$$

$$= \boxed{(x^2 - x - 6)} \cdot \boxed{x} - (x^2 - x - 6) \cdot 5$$

$$= x \cdot (x^2 - x - 6) - 5x^2 + 5x + 30$$

$$= x^3 - x^2 - 6x - 5x^2 + 5x + 30$$

combine like terms

$$\Rightarrow \boxed{P(x) = x^3 - 6x^2 - x + 30}$$

Policy Issues : Patterns

- ☐ Math teachers dont give enough time on assessments
- ☐ Math teachers don't give enough space to write work

SOLUTION: Use more paper

Long division

denominator

$$\begin{array}{r|l} & \boxed{2\ \ 1} \\ \boxed{2} & \boxed{4}\,2 \\ & -4\ \downarrow \\ \hline & 0\,\boxed{2} \\ & -2 \\ \hline & \boxed{0} \end{array}$$

remainder

$$\begin{aligned} 42 &= 40 + 2 \\ &= \boxed{4} \cdot 10 + \boxed{2} \cdot 1 \\ &= \boxed{2} \cdot 2 \cdot 10 + \boxed{2} \cdot 1 \\ &= 2\,(\boxed{2 \cdot 10} + 1) \\ &= 2\,(20 + 1) \\ &= 2 \cdot 21 \end{aligned}$$

first part of numerator

denominator

$$
\begin{array}{r|rrr}
 & 0 & 4 & \boxed{9} \\
\hline
\boxed{7} & \boxed{3} & 4 & 5 \\
 & -0 & \downarrow & \downarrow \\
\hline
 & \boxed{3} & \boxed{4} & \\
 & -2 & 8 & \downarrow \\
\hline
 & & \boxed{6} & \boxed{5} \\
 & & -6 & 3 \\
\hline
 & & & \boxed{2}
\end{array}
$$

remainder

$$
\begin{aligned}
\Rightarrow \frac{345}{7} &= 49 \text{ R } 2 \\
&= 49 + \frac{2}{7} \\
&= 49\tfrac{2}{7}
\end{aligned}
$$

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1B. Use the polynomial long-division algorithm from Lesson 4 to execute the following division problem:

Numerator quotient

$$\frac{\boxed{N(x)}}{\boxed{D(x)}} = \frac{\boxed{x^3 - 6x^2 - x + 30}}{\boxed{(x+2)}} = \boxed{q(x)}\ \mathrm{R}\ \boxed{r(x)}$$

Denominator remainder

Polynomial long division

first part of numerator

$$\begin{array}{r|l} & \boxed{x^2} - 8x + 15 \\ \hline \boxed{x+2} & \boxed{x^3} - 6x^2 - x + 30 \\ & -x^3 - 2x^2 \\ \hline & 0 \ \boxed{-8x^2 - x} \\ & +8x^2 + 16x \\ \hline & 0 + \boxed{15x} + 30 \\ & -15x - 30 \\ \hline & \boxed{0} \end{array}$$

denominator

remainder

remember: $x^3 = x \cdot x \cdot x$

□ $x^2(x+2) = x^3 + 2x^2$

□ $-8x^2 = -8 \cdot x \cdot x$

□ $-8x(x+2) = -8x^2 - 16x$

□ $15(x+2) = 15x + 30$

Person that needs Thearpy version of signs for mult

	+	−
+	+	−
−	−	+

Numerator

$$\Rightarrow \frac{\boxed{x^3 - 6x^2 - x + 30}}{\boxed{(x+2)}} = \underbrace{\boxed{x^2 - 8x + 15}}_{\text{quotient}}\ \mathrm{R}\ \underbrace{\boxed{0}}_{\text{remainder}} \leftarrow \boxed{\text{remainder notation}}$$

deminator

$$= (x^2 - 8x + 15) + \frac{0}{(x+2)} \leftarrow \boxed{\text{addition notation}}$$

0

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1C. Use Desmos.com to graph the polynomial

$$P(x) = x^3 - 6x^2 - x + 30$$

If our goal is to find the zeros of this polynomial, how can we do that using the graph of our function.

Let's look at our work :

$$\left.P(x) = x^3 - 6x^2 - x + 30\right\} \leftarrow \text{standard form}$$

$$= \underbrace{(x+2)}_{\text{Denominator}} \cdot \underbrace{(x^2 - 8x + 15)}_{\text{quotient}}$$

complete zero factorization

$$= \underbrace{(x+2)}_{\text{zero 1}} \cdot \underbrace{(x-3)}_{\text{zero 2}} \cdot \underbrace{(x-5)}_{\text{zero}}$$

Any time you say find the zeros, we know $y=0$

$$\Rightarrow P(x) = 0 = \boxed{(x+2)} \cdot \boxed{(x-3)} \cdot \boxed{(x-5)}$$

this is the output y-value ($y=0$)

$\Rightarrow$ $x+2=0$ OR $x-3=0$ OR $x-5=0$

$\Rightarrow$ $x=-2$ OR $x=3$ or $x=5$ these are zeros of P(x)

$\Rightarrow$ The x-intercepts happen at points $(-2,0)$, $(3,0)$, $(5,0)$

Notice something crazy:

evaluation bar

$$x = 3 \Rightarrow P(x)\Big|_{x=3} = P(3) = 0$$

P of x evaluated at x equals 3

standard form evaluates

$$\text{Note: } P(3) = x^3 - 6x^2 - x + 30\Big|_{x=3}$$

input value $x = 3$

$$= 3^3 - 6 \cdot 3^2 - 3 + 30$$

$$= 27 - 6 \cdot 9 + 27$$

$$6 \cdot 9 = 6 \cdot (10-1) = 60 - 6 = 54$$

$$= 27 - 54 + 27$$

$$= 0$$

this is why we call $x = 3$ a zero:

for all the zeros of our polynomial the output will be zero

$$\text{Note 2: } P(3) = (x+2)\cdot\boxed{(x-3)}(x-5)\Big|_{x=3}$$

$$= (3+2)\cdot(3-3)\cdot(3-5)$$

$$= (5)\cdot(0)\cdot(-2) = 0 \checkmark$$

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2. REMAINDERS AND POLYNOMIAL LONG DIVISION

Use the polynomial long-division algorithm from Lesson 4 to execute the following division problem:

$$\frac{N(x)}{D(x)} = \frac{3x^2 - 5x - 17}{(3x+1)} = q(x)\,\mathrm{R}\,r(x)$$

Polynomial Long Division

$$\begin{array}{r|llll} & \boxed{x^1} & -2 & \\ \hline (3x+1) & \boxed{3x^2} & -5x & -17 \\ & -3x^2 & -x & \downarrow \\ \hline & 0 & \boxed{-6x} & -17 \\ & & +6x & +2 \\ \hline & & 0 & \boxed{-15} \end{array}$$

remainder

side note:

- $3x^2 = 3 \cdot x \cdot x$
- $x(3x+1) = 3x^2 + x$
- $-2(3x+1) = -6x - 2$

$$\frac{3x^2 - 5x - 17}{(3x+1)} = (x-2)\ \mathrm{R}\ -15$$

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3. DISCOVER THE REMAINDER THEOREM

3A. Consider the following two equations given below. Write your own verbal descriptions of these equations in both formal (nerdy) language and also intuitive (abuelita) language.

Division Equation

$$\frac{N(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}$$

Multiplication Equation

$$N(x) = D(x) \cdot q(x) + r(x)$$

☐ Let's develop notation and words for our division problem

Division Problem

Numerator

remainder

$$\frac{\boxed{N(x)}}{\boxed{D(x)}} = \boxed{q(x)} + \frac{\boxed{r(x)}}{D(x)}$$

Denominator

quotient

Multiplication Problem

quotient

$$\boxed{N(x)} = \boxed{D(x)} \cdot \boxed{q(x)} + \boxed{r(x)}$$

Numerator denominator remainder

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3B. Use the notation and terminology from Lesson 5, Problem 3A (page 5) above to write both division and multiplication equations for all of the polynomial division problems given below

Problem Number	Division Problem
Lesson 4, Problem 2 (page 2)	$\dfrac{6x^2 - 26x + 12}{(x-4)}$
Lesson 5, Problem 1B (page 2) ✓	$\dfrac{x^3 - 6x^2 - x + 30}{(x+2)}$
Lesson 5, Problem 2 (page 4) ✓	$\dfrac{3x^2 - 5x - 17}{(3x+1)}$
Lesson 4, Problem 4B (page 6)	$\dfrac{2x^4 - 6x^3 - 26x^2 + 30x}{(x+3)}$

$$\frac{N(x)}{D(x)} = \frac{x^3 - 6x^2 - x + 30}{(x+2)} = \underbrace{(x^2 - 8x + 15)}_{\text{quotient}} + \frac{\overset{\text{remainder}}{\boxed{0}}}{(x+2)}$$

$$\Rightarrow \quad \underbrace{x^3 - 6x^2 - x + 30}_{N(x)} = \boxed{\underbrace{(x+2)}_{D(x)}} \cdot \underbrace{(x^2 - 8x + 15)}_{q(x)} + \underbrace{0}_{r(x)}$$

Note ☐ the denominator $D(x) = (x+2)$ divides perfectly into the numerator $N(x)$ with no remainder

☐ the denominator $D(x) = (x+2)$ factors into $N(x)$

☐ the denominator $D(x) = (x+2)$ is a perfect factor of $N(x) = x^3 - 6x^2 - x + 30$

$$\Longleftrightarrow \quad N(x) = x^3 - 6x^2 - x + 30 = (x+2) \cdot q(x)$$

$$\frac{N(x)}{D(x)} = \frac{3x^2 - 5x - 17}{(3x+1)} = \underbrace{(x-2)}_{\text{quotient}} + \frac{\overset{\text{remainder}}{-15}}{3x+1}$$

$$\Rightarrow \underbrace{3x^2 - 5x - 17}_{N(x)} = \underbrace{(3x+1)}_{D(x)} \cdot \underbrace{(x-2)}_{q(x)} + \underbrace{-15}_{r(x)}$$

$$= 3x^2 - 6x + x - 2 + -15$$

$$= \boxed{3x^2 - 5x - 2} + \boxed{-15}$$

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3C. In the notation above, look for linear factors $D(x) = (x - c)$ of $N(x)$. What patterns do you notice about the relationship between the remainder and $N(c)$, the value of the function $N(x)$ where $x = c$?

Lesson 5, Problem 1B, page 2

Recall: $N(x) = x^3 - 6x^2 - x + 30$

$D(x) = (x+2) = (x - -2)$ ← here we see the value of $x = -2$ is a zero of $N(x)$ (the x-intercept for this factor is at point $(-2,0)$)

$r(x) = \boxed{0}$

Pattern?

$\Rightarrow N(x)\Big|_{x=-2} = N(-2) = \boxed{0}$

Lesson 5, Problem 2, page 4

Recall: $N(x) = 3x^2 - 5x - 17$

$D(x) = 3x + 1$

$q(x) = x - 2$

$r(x) \neq \boxed{-15}$

hmm...

$\Rightarrow N(x)\Big|_{x=2} = N(2) = 3x^2 - 5x - 17\Big|_{x=2} \neq \boxed{-15}$

$= 3 \cdot 2^2 - 5 \cdot 2 - 17$

$= 3 \cdot 4 - 10 - 17$

$= 12 - 10 - 17 = 12 - 27$

$= 2 - 17$

$$\Rightarrow \quad N(x)\Big|_{x=\frac{-1}{3}} = N\left(-\frac{1}{3}\right)$$

$$= (3x+1)\cdot(x-2) + -15\Big|_{x=\frac{-1}{3}}$$

$$= \left(3\cdot\frac{-1}{3}+1\right)\cdot\left(-\frac{1}{3}-2\right) + -15$$

<u>side note</u>

$$\frac{3}{1}\cdot\frac{-1}{3} = -1$$

$$= (-1+1)\cdot\left(\frac{-7}{3}\right) + -15$$

$$= 0\cdot\frac{-7}{3} - 15$$

$$= 0 - 15$$

$$= -15 \leftarrow \text{remainder...}$$

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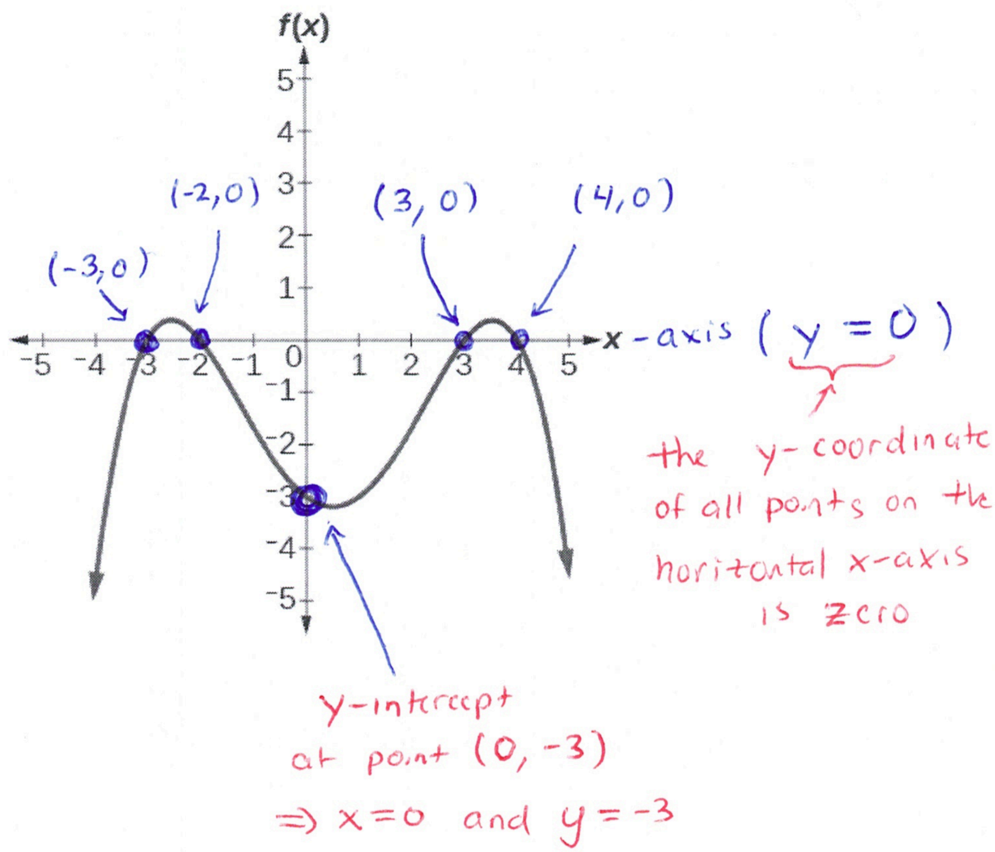
4. CREATE POLYNOMIALS FROM KNOWN ZEROS

4A. Use polynomial long-division to find the quotient and remainder for the following problem

$$\frac{N(x)}{D(x)} = \frac{x^3 - 6x^2 - x + 30}{(x-5)} = q(x)\ \mathrm{R}\ r(x)$$

4B. Use Desmos.com to find the zeros of $x^3 - 6x^2 - x + 30$. Then, using those results, produce the complete zero factorization for this polynomial.

4C. Use the graph below to write a formula for a polynomial function of least degree whose graph looks like the curve given in this figure. Then, discuss how this problem is related to the division problems given above.



This equation is: $P(x) = \frac{-1}{24}(x+3)\cdot(x+2)\cdot(x-3)\cdot(x-4)$

Problem 4c, Solution

verbal

- Based on the graph we see that the zeros are at $-3, -2, 3$, and 4.

- Recall:

remember that when we say that the zeros of the graph are at $-3, -2, 3$, and 4 we can write this in a few different ways:

x-values that make y equal to zero

Zero notation: $x = -3 \Rightarrow x - \boxed{-3} = 0$

$x = -2 \Rightarrow x - \boxed{-2} = 0$

$x = 3 \Rightarrow x - \boxed{3} = 0$

$x = 4 \Rightarrow x - \boxed{4} = 0$

output value of zero

x-intercept notation: $(-3, 0) \Rightarrow x = -3$ and $y = 0$

$(-2, 0) \Rightarrow x = -2$ and $y = 0$

$(3, 0) \Rightarrow x = 3$ and $y = 0$

$(4, 0) \Rightarrow x = 4$ and $y = 0$

locations where curve passes through horizontal axis (write this as a point)

Let's recall our strategy of writing formulas for polynomials... we want to come up with factored form

$\boxed{\text{Draft idea 1: (conjecture)}}$ Put all the zeros together in factored form

equation (has an equals sign)

$$\left[(x+3)\cdot(x+2)\right]\cdot\left[(x-3)\cdot(x-4)\right] = 0$$

$$\Rightarrow \left[x^2 + \underbrace{2x + 3x}_{} + 6\right]\cdot\left[x^2 \underbrace{-4x - 3x}_{} + 12\right] = 0$$

combine like terms

$$\Rightarrow \left[x^2 + 5x + 6\right]\cdot\left[\underbrace{x^2 - 7x + 12}_{}\right] = 0$$

□ Right Distributivity: $(A + B + C)\cdot D = AD + BD + CD$

$$\Rightarrow x^2\left[x^2 - 7x + 12\right] + 5x\cdot\left[x^2 - 7x + 12\right] + 6\left[x^2 - 7x + 12\right] = 0$$

$$\Rightarrow x^4 - \underline{\underline{7x^3}} + \boxed{12x^2} + \underline{\underline{5x^3}} - \boxed{35x^2} + 60x + \boxed{6x^2} - 42x + 72 = 0$$

$$\Rightarrow \boxed{x^4 - 2x^3 - 17x^2 + 18x + 72 = 0}$$

this doesn't work...

the graph of this function is not the same as our given

Notice for our guess, when $x = 0$, we have

$$0^4 - 2 \cdot 0^3 - 17 \cdot 0^2 + 18 \cdot 0 + 72 = 72$$

The y-intercept of our guess is at $(0, 72)$

We want to multiply by a scalar, called a, to reflect about x-axis and vertically compress

$$P(x) = a\,(x+3)\cdot(x+2)\cdot(x-3)\cdot(x-4)$$

$$= a\,(x^4 - 2x^3 - 17x^2 + 18x + 72)$$

$$= a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + \underbrace{a_0 x^0}_{\text{this is the y-value of the y-intercept}}$$

$$\Rightarrow a_0 = \boxed{-3 = 72 \cdot a}$$

$$\Rightarrow \frac{72 \cdot a}{72} = \frac{-3}{72} \Rightarrow a = \frac{-1}{24}$$