Name: Solutions Class #:______

Math 48B, Lesson 16: Exponential Functions

In Math 48B Lessons 14, 15, 16, 17, and 18, we study logarithmic functions:

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$

To begin our exploration, let's recall the rules of powers/exponents.

1. WHAT ARE RULES OF POWERS/EXPONENTS?

Exponent Notation: $N = b^n$ $M = b^m$

product (multiplication)

Product Rule: $b^n \cdot b^m = b^{n+M}$ ← exponent notation

quotient (division)

Quotient Rule: $\frac{b^n}{b^m} = \frac{b^n}{b^m} = b^{n-m}$

repeated exponents

multiplication

Power to a Power: $(b^n)^p = b^{n \cdot p}$

Zero Power: $1 = \frac{b}{b} = \frac{b^1}{b^1}$

Negative Powers: $\frac{1}{b^n}$

Product Rule of logs:

what power do we need to put on b to get N·M

y

$$\log_b(N \cdot M) = y \iff N \cdot M = b^y$$

WTF

y

"log base b of N times M"

$$\iff \underbrace{b^n \cdot b^m} = b^y$$

$$\iff b^{\boxed{n+m}} = b^{\boxed{y}}$$

$$\iff y = n + m$$

$$\Rightarrow \log_b(N \cdot M) = y = n + m$$

But: $\log_b(N) = \underbrace{\cancel{\log_b}}_{\text{log}}(\underbrace{\cancel{b}^n}_{\text{exponent}}) = n$

Recall:

$$\log_b(N) = w$$

$$\Rightarrow b^{\boxed{w}} = N = b^{\boxed{n}}$$

$$\Rightarrow w = n$$

$$\Rightarrow \underbrace{\cancel{\log_b}}_{\text{logs}}(\underbrace{\cancel{b}^n}_{\text{exponent}}) = n$$

"logs cancel exponents"

$$\log_b(M) = \underbrace{\cancel{\log_b}}_{\text{logarithm}}(\underbrace{\cancel{b}^m}_{\text{exponent}}) = m$$

"logs cancel out exponents

$$\Rightarrow \quad \log_b(N \cdot M) = n + m$$

$$\downarrow$$

$$= \log_b(N) + \log_b(M)$$

$\boxed{\text{Product Rule of Logs}}$

$$\Rightarrow \quad \log_b(N \cdot M) = \log_b(N) + \log_b(M)$$

multiplication inside the log

addition outside the log

- "multiplication inside the log transforms into addition outside log"
- the log function turns multiplication problems into addition problem

Product Rule for logs (focuses on superscript)

$$\log_b(N \cdot M) = \log_b(N) + \log_b(M)$$

$$\updownarrow$$

Product Rule for exponents (focus on the whole thing)

$$b^n \cdot b^m = b^{n+m}$$

Major Theme: see Hidden Figures

- ☐ Imagine you had to calculate

Multiplication Pr

$$\begin{array}{r} 532176 = N \\ \times\ 14891 = M \\ \hline \end{array}$$

$$\log_{10}(\underbrace{N \cdot M}_{\text{Yuck!}}) = \underbrace{\log_{10}(532176)}_{10^n = 532176} + \underbrace{\log_{10}(14891)}_{10^m = 14891}$$

$$= \cancel{13.1814772954} + \cancel{9.6081512283}$$

$$= 5.726055 + 4.172924$$

Quotient Rule of Logs

numerator

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

division on the inside of the logs (in between parenthesis)

subtraction outside parenthesis

$$\Rightarrow \underbrace{\log_b\left(\frac{N}{M}\right)} = \overbrace{\log_b(N)}^{n} - \overbrace{\log_b(M)}^{m}$$

what exponent do I need on b to get N divided by M out

$$\frac{N}{M} = \frac{b^n}{b^m} = b^{n-m}$$

Power Rule of logs

multiplication outside the log

$$\log_b(\underbrace{N^p}_{}) = p \cdot \log_b(N)$$

power inside the log

It's almost as if we pull down the power:

Yank this down here

$$\log_b(N^p)$$

used to be here

$$\Rightarrow p \log_b(N)$$

now it's down here!

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2. WHAT ARE RULES OF LOGARITHMS?

Logarithmic Notation: $n = \log_b(N)$ and $m = \log_b(M)$

✓ Product Rule: $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$

✓ Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

✓ Power to a Power: $\log_b(N^p) = p \cdot \log_b(N)$

✓ Inverse Exponential: $\log_b(\cancel{b}^n) = n$

$\cancel{\log_b}(\cancel{b}^{n}) = n$

logs cancel out exponents

✓ Inverse Log: $b^{\log_b(N)} = N$

$\cancel{b}^{\cancel{\log_b}(N)} = N$

exponents cancel out logarithms

○ Change of Base: $\log_b(N) = \frac{\log_a(N)}{\log_a(b)}$

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3. HOW TO USE LOG RULES?

Use the properties of logs we explored in problem 2 above to evaluate the logarithm in each problem:

✓ 3A. $\log_{10}\left(\frac{10000}{100}\right)$

3B. $\log_9(27 \cdot 81)$

3C. $\log_2(16^2)$

✓ 3D. $\log_5\left(\frac{125^2}{\sqrt{625}}\right)$

3E. $\log_2(16 \cdot 32)$

3F. $\log_5(125) - \log_5(25)$

✓ 3G. $\log_6(6^7) - \log_6(6^{10})$

3H. $\log_2\left(\frac{2048}{\sqrt[10]{32}}\right)$

$\boxed{\text{Problem 3A}}$

Method 1: (direct calculation)

$\log_{10}\left(\frac{1000\cancel{0}}{10\cancel{0}}\right) = \log_{10}(100) =$

side note:

$\frac{10000}{100} = \frac{10 \cdot 10 \cdot \cancel{10} \cdot \cancel{10}}{\cancel{10} \cdot \cancel{10}} = 100$

$\Rightarrow \log_{10}(100) = y \Leftrightarrow 10^y = 100$

$\Leftrightarrow y = 2$

☐ How many times do I have to multiply 10 by itself to get 100

Problem 3A, continued ...

$$\log_{10}\left(\underbrace{\frac{10000}{100}}_{\text{division on the inside}}\right) = \underbrace{\log_{10}(10000)}_{\substack{\text{what exponent}\\ \text{do we need on base 10}\\ \text{to get } 10000}} - \underbrace{\log_{10}(100)}_{\substack{\text{what exponent}\\ \text{do we need}\\ \text{on base 10}\\ \text{to get } 100}}$$

$$= 4 - 2$$

$$= 2 \checkmark$$

Problem 3D

Consider $\log_5\left(\frac{125^2}{\sqrt{625}}\right) = 4$

Note

$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$

Heavy on labor
light on management

Method 1:

(Direct calculation)

calculate inside first, then take log

$$125 = 5^3 \Rightarrow (125^2) = (5^3)^2$$

$$\Rightarrow 125^2 = 5^{3 \cdot 2} = 5^6$$

$$625 = 25^2 = (5^2)^2 = 5^{2 \cdot 2} = 5^4$$

$$\Rightarrow \sqrt{625} = \sqrt{\cancel{25^2}} = 25 = 5^2$$

$$\Rightarrow \frac{125^2}{\sqrt{625}} = \frac{5^6}{5^2} = 5^{6-2} = 5^4$$

$$\Rightarrow \log_5\left(\frac{125^2}{\sqrt{625}}\right) = \underbrace{\cancel{\log_5}}_{\text{log}}(\underbrace{\cancel{5}}_{\text{exponent}}{}^{4}) = 4$$

"logs cancel out exponents"

$$4 = \log_5(5^4) = y \Leftrightarrow 5^{\boxed{y}} = 5^{\boxed{4}}$$

$$\Leftrightarrow y = 4$$

$$\log_2(x) = \frac{\log(x)}{\log(2)} =$$

$\boxed{\text{Problem 3D}}$

Method 2:

(Properties of logs)

$$\log_5\left(\underbrace{\frac{125^2}{\sqrt{625}}}\right) = \log_5(125^2) - \log_5(\sqrt{625})$$

division on inside

$\boxed{\text{side note:}}$

- $\sqrt[2]{625} = 625^{\frac{1}{2}}$
- $\sqrt[n]{b} = b^{\frac{1}{n}}$

$$= \log_5(125^{2}) - \log_5(625^{\frac{1}{2}})$$

$$= 2 \cdot \underbrace{\log_5(125)}_{\text{what exponent do we need on base 5 to get 125}} - \frac{1}{2} \underbrace{\log_5(625)}_{\text{What exponent do we on base 5 to get 125}}$$

- $\log_5(125) = y \Rightarrow 5^y = 125$
- $\log_5(625) \Rightarrow 5^y = 625$

$$= 2 \cdot 3 - \frac{1}{2} \cdot 4$$

$$= 6 - 2$$

$$= \underline{4}\ \checkmark$$

Problem 3G

Method 1

$$\cancel{\log_6}(\cancel{6}^{7}) - \cancel{\log_6}(\cancel{6}^{10})$$

log outside

exponent inside

logs cancel out exponents

"logs cancel out exponent"

$$= 7 - 10$$

$$= -3$$

Method 2

$$\log_6(6^{7}) - \log_6(6^{10})$$

exponent on inside

exponent on inside

$$= 7 \cdot \underbrace{\log_6(6)} - 10\underbrace{\log_6(6)}$$

$$= 7 \cdot 1 - 10 \cdot 1$$

$$= 7 - 10 = -3 \checkmark$$

$\boxed{\text{Method 3}}$

$$\log_6(6^7) - \log_6(6^{10})$$

$$= \log_6\left(\frac{6^7}{6^{10}}\right)$$

$$= \cancel{\log_6}(\cancel{6}^{-3})$$

$$= -3$$

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5. HOW TO USE LOG RULES WITH VARIABLES?

Use the properties of logs we explored in problem 2 above to expand each of the following expressions:

5A. $\log_2\left(\frac{36x^2(4x-1)}{(2x+1)}\right)$

5C. $\log(\sqrt[3]{x^2})$

5B. $\ln\left(\frac{x^4 \cdot y^3}{z^5}\right)$

5D. $\ln\left(\frac{\sqrt{(x+1)(2x-1)^2}}{x^2-4}\right)$

Problem 5B

Recall $\ln(x) = \log_e(x)$

natural log

log base e (e is natural exponent)

$$\ln\left(\frac{x^4 \cdot y^3}{\underbrace{z^5}_{\text{division inside}}}\right) = \ln(x^4 \cdot y^3) - \ln(z^5)$$

multiplication inside

$$= \ln(x^4) + \ln(y^3) - \ln(z^5)$$

exponents inside

$$\boxed{= 4 \cdot \ln(x) + 3 \cdot \ln(y) - 5 \cdot \ln(z)}$$

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4. HOW TO USE THE CHANGE OF BASE FORMULA?

Use the change of base formula to calculate each of the following values. Change the base to both the common log and the natural log. Check your work using a calculator.

4A. $\log_2\left(\frac{2}{16}\right)$

4B. $\log_9\left(\frac{9}{243}\right)$

Side Note:

$\log_b(N) = \frac{\log_a(N)}{\log_a(b)}$

4A $y = \log_2\left(\frac{2}{16}\right)$

division inside

Method 1:

(log properties)

$\log_2\left(\frac{2}{16}\right) = \underbrace{\log_2(2)} - \underbrace{\log_2(16)}$

$= 1 - 4$

$= -3$ ✓

Method 2:

(Change of base)

Let's change from $\log_2$ with base $b = 2$ to $\log_{10}$ with base 10

inside $N = \frac{2}{16}$

$\log_2\left(\frac{2}{16}\right) = \frac{\log_{10}\left(\frac{2}{16}\right)}{\log_{10}(2)} = -3$ ✓

base $b = 2$

Change of base formula

side note:

$\log_a(M^P) = p \cdot \log_a(M$

logarithmic form

$$\log_b(N) = y$$

we want to find this one

Exponential Form

$$\underbrace{[b^y]}_{\text{positive number}} = [N]$$

$$\Rightarrow \log_a(b^y) = \log_a(N)$$

for some other base $a > 0$

we hit both sides of the equation with a log operation

$$\Rightarrow y \cdot \log_a(b) = \log_a(N)$$

$$\Rightarrow y = \frac{\log_a(N)}{\log_a(b)} = \log_b(N)$$

this is how we switch from one base to another.

input $N = \frac{9}{243}$

4B. $y = \log_9 \left(\frac{9}{243}\right)$

base $b = 9$

Method 1:

(log properties)

$$\log_9 \Big(\underbrace{\frac{9}{243}}_{\text{division on inside}}\Big) = \overbrace{\log_9}^{f}(9) - \log_9(243)$$

log base 9 of 243

$$= 1 - \log_9(243)$$

$$= 1 - 5/2 = \frac{2}{2} - \frac{5}{2} = \frac{2-5}{2} \stackrel{\checkmark}{=} -\frac{3}{2}$$

Side Note:

$\log_b(N) = y \iff b^y = N$

$y = \log_9(243) \iff 9^y = \underline{\underline{243}}$

- What exponent do we put on 9 to get 243

- $9^1 = 9$
- $9^2 = 9 \cdot 9 = 8$
- $9^3 = 9 \cdot 9 \cdot 9$

$= 81 \cdot 9$

$= (80+1) \cdot 9$

$= 720 + 9$

$= 729$

Side Note : $3^1 = 3$

$3^2 = 9$

$3^3 = 27$

$3^4 = (3^2)^2 = 9^2 = 81$

$3^5 = 3^4 \cdot 3 = 81 \cdot 3$

$= (80 + 1) \cdot 3$

$= 240 + 3$

$= 243$

$243 = 3^5$ & $9 = 3^2$

both numbers can be written in base 3

$\Rightarrow \quad 9^y = 243$

$\Rightarrow \quad (3^2)^y = 3^5 \qquad (b^n)^m = b^{n \cdot m}$

$\Rightarrow \quad \cancel{3}^{\boxed{2y}} = \cancel{3}^{\boxed{5}}$

we have two exponents equal to each other with sam base so exponents are equal

$\Rightarrow \quad 2y = 5$

$\Rightarrow \quad y = 5/2 = \log_9(243)$

Method 2: $y = \log_9 \left(\frac{9}{243}\right)$

(change of base)

Recall: $\log_b (N) = \dfrac{\log_a (N)}{\log_a (b)}$

input $N = \frac{9}{243}$

$$\Rightarrow \log_9 \left(\frac{9}{243}\right) = \frac{\log_{10}\left(\frac{9}{243}\right)}{\log_{10}(9)}$$

base $b = 9$

$$= -1.5 = -\frac{3}{2} \checkmark$$