Name: Solutions Class #:

Math 48B, Lesson 15: Graphs of Logarithmic Functions

In Math 48B Lessons 14, 15, 16, 17, and 18, we study logarithmic functions:

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$

To begin our exploration, let's recall the rules of powers/exponents.

1. HOW TO EVALUATE LOGARITHMS?

Consider the two equivalent forms for logarithmic functions:

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$

Use these two equivalent forms to evaluate the following logarithm problems.

✓ 1A. $4 + \log_{10}(0.001)$ ✓ 1B. $\log_4(\frac{1}{32})$ ✓ 1C. $\log_e(e^{2/5})$

Problem 1A) Consider

✓ $1 = 4 + \underbrace{\log_{10}(0.001)}_{\text{focus on hard stuff first, then come back}} = 4 + -3$

input $x = 0.001$

Let $y = \log_{10}(0.001)$ ← logarithmic form

$b = 10$

"log base 10 of 0.001"

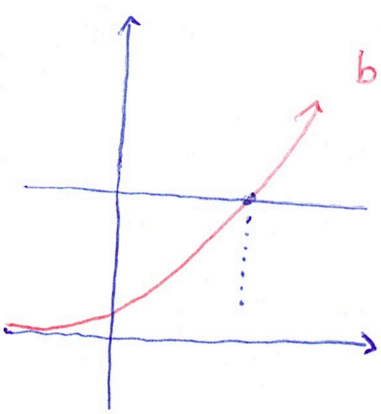
Notice we can put this in exponential form

$$y = \log_b(x) \iff b^y = x$$

input/output swap from exponential

$$\Rightarrow \quad y = \log_{10}(0.001) \iff 10^y = 0.001$$

$$\iff 10^y = \frac{1}{1000} = \frac{1}{10^3}$$



$$\iff 10^{\boxed{y}} = 10^{\boxed{-3}}$$

$$\iff y = -3$$

Remember: $\frac{1}{10^3} = 10^{-3}$

If $b^{\boxed{x}} = b^{\boxed{w}}$, then $x = w$

output1 = output2 (b^x is one-to-one $\Rightarrow$ inputs have to be same)

$x = \frac{1}{32}$

Problem 1B

Consider $\log_4\left(\frac{1}{32}\right) = y$

base $b = 4$

Recall $y = \log_4\left(\frac{1}{32}\right) \iff \boxed{4}^{y} = \boxed{\frac{1}{32}}$

$$\iff \left(2^2\right)^y = \left(\frac{1}{2^5}\right) = 2^{-5}$$

side note:

$4^1 = 4$

$4^2 = 16$

$4^3 = 64$

$32 = 2^5$

$4 = 2^2$

$\frac{1}{b^n} = \frac{b^0}{b^n} = b^{0-n} = b^{-n}$

$(b^n)^m = b^{n \cdot m}$

$$\iff 2^{\boxed{2y}} = 2^{\boxed{-5}}$$

$$\iff 2y = -5$$

$$\iff y = -\frac{5}{2} = -2.5$$

we see $\log_4\left(\frac{1}{32}\right) = -\frac{5}{2} = -2.5$

since

$$4^{\boxed{-\frac{5}{2}}} = 4^{\boxed{\frac{1}{2}\cdot\frac{-5}{1}}}$$

$$= \left(4^{1/2}\right)^{-5}$$ $\square\ b^{n\cdot m} = (b^n)^m$

$$= \left(\sqrt[2]{4}\right)^{-5}$$ $\square\ x^{\frac{1}{n}} = \sqrt[n]{x}$

$$= 2^{-5}$$

$$= \frac{1}{2^5}$$

$$= \frac{1}{32}$$ ✓

1C. Consider

$$y = \log_e\big(\overbrace{e^{2/5}}^{\text{input } x = e^{2/5}}\big) \Rightarrow e^y = e^{2/5}$$

base $b = e$

$$\Rightarrow y = 2/5$$

$$\Rightarrow \cancel{\log_e}(\cancel{e}^{2/5}) = \frac{2}{5}$$

logs cancel out exponents

$\boxed{\text{Natural log}}$

Note: $\log_e(x) = \ln(x)$ "L N of e"

log base e of x

the natural log function

$\boxed{\text{Common Log}}$

$\sqrt{x}$

$\log_{10}(x) = \log(x)$

if no base is written, then $b = 10$

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$b^n \cdot b^m = b^{n+m}$

2. WHAT DOES THE GRAPH OF A LOGARITHM LOOK LIKE?

2A. Fill out the table for the logarithmic function $y = \log_2(x)$ below. The, use Desmos.com to create a graph and describe the relevant features of that graph including the domain, range, x-intercept, and the end behavior as $x \to +\infty$.

$\frac{1}{64}$

x	y
$\frac{1}{32}$	-5
$\frac{1}{16}$	-4
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3
16	4
32	5
64	6

$\lim_{x \to +\infty} f(x) = +\infty$

$\leftarrow \log_2\left(\frac{1}{2}\right)$

$\leftarrow \log_2(1)$

$\leftarrow \log_2(2)$ what exponent do we put on base 2 to get 2 out

$\leftarrow \log_2(64) \leftarrow$ base 2 raised to what power is 64

$$\boxed{y = \log_b(x) \iff b^y = x}$$

eg₁: $x = 4 \Rightarrow \log_2(4) = y$

$\Rightarrow 2^y = 4$

what exponent do we put on 2 to get 4 out

eg₂: $x = 16 \Rightarrow y = \log_2(16)$

$\Rightarrow 2^y = 16$

$\Rightarrow \log_2(16) = y = 4$

eg 3: $x = 1 \Rightarrow y = \log_2(1)$

what exponent do we put on base 2 to get 1

$\Rightarrow 2^y = 1$

$\Rightarrow y = 0 = \log_2(1)$

Recall:

- $1 = \frac{2}{2} = \frac{2^1}{2^1} = 2^{1-1} = 2^0$
- $\frac{b^m}{b^n} = b^{m-n}$
- $b^0 = 1$
- $b^{-n} = \frac{1}{b^n}$

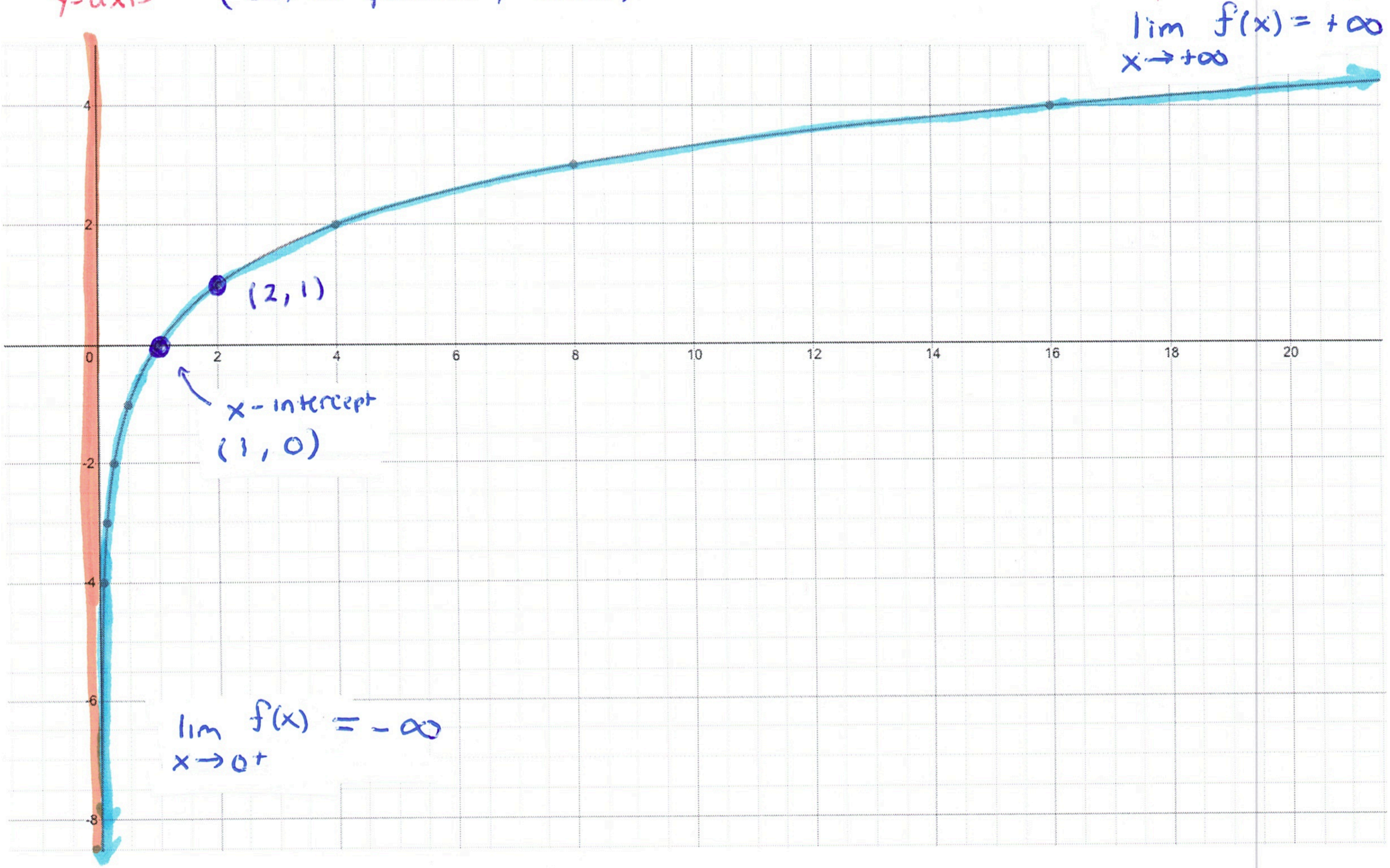
Look at $\boxed{f(x) = \log_2(x)}$

✓ The domain of $f(x)$: $\text{Dom}(f) = (0, +\infty)$

open parenthesis (does touch)

✓ The range of $f(x)$ (set of possible y-values) : $\text{Rng}(f) = (-\infty, +\infty)$

y-axis



vertical asymptote
$(x = 0)$

Never touches the y-axis

No y-intercept

☐ x-intercept at point : $(1, 0)$

$$\log_2(1) = 0 \iff 2^0 = 1$$

☐ Point $(2, 1)$ happens since

$$\log_2(2) = 1 \iff 2^1 = 2$$

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2B. Fill out the table for the common logarithmic function

$$y = \log_{10}(x) = \log(x)$$

The, use Desmos.com to create a graph and describe the relevant features of that graph including the domain, range, x-intercept, and the end behavior as $x \to +\infty$.

x	y
0.00001	-5
0.0001	-4
0.001	-3
$\frac{1}{10^2}$ = 0.01	-2
$\frac{1}{10^1}$ · 0.1	-1
1	0
10	1
100	2
1000	3
10000	4
100000	5
1,000,000	6

$y = \log_b(x) \Leftrightarrow b^y = x$

eg.: $x = 100 \Rightarrow \log_{10}(100) = y$

$\Rightarrow 10^{\boxed{y}} = 100 = 10^2$

what is exponent on 10 to get 100?

$\Rightarrow y = \log_{10}(100) = 2$

$\leftarrow \log_{10}(1000) \leftarrow$ what exponent do we need on 10 to get 1000

$x = 1000 \Rightarrow \log_{10}(1000) = y$

How many times do we multply 10 by itself to get 10[00]

$\leftarrow \log_{10}(1000000)$

what exponent on 10 to get 1000000?

$\Rightarrow 10^y = 1000 = 10\cdot10\cdot10$

$\Rightarrow y = \log_{10}(1000) = 3$

Note: $10^y \neq 0$ for all y ; $10^0 = 1$

Define $g(x) = \log_{10}(x)$

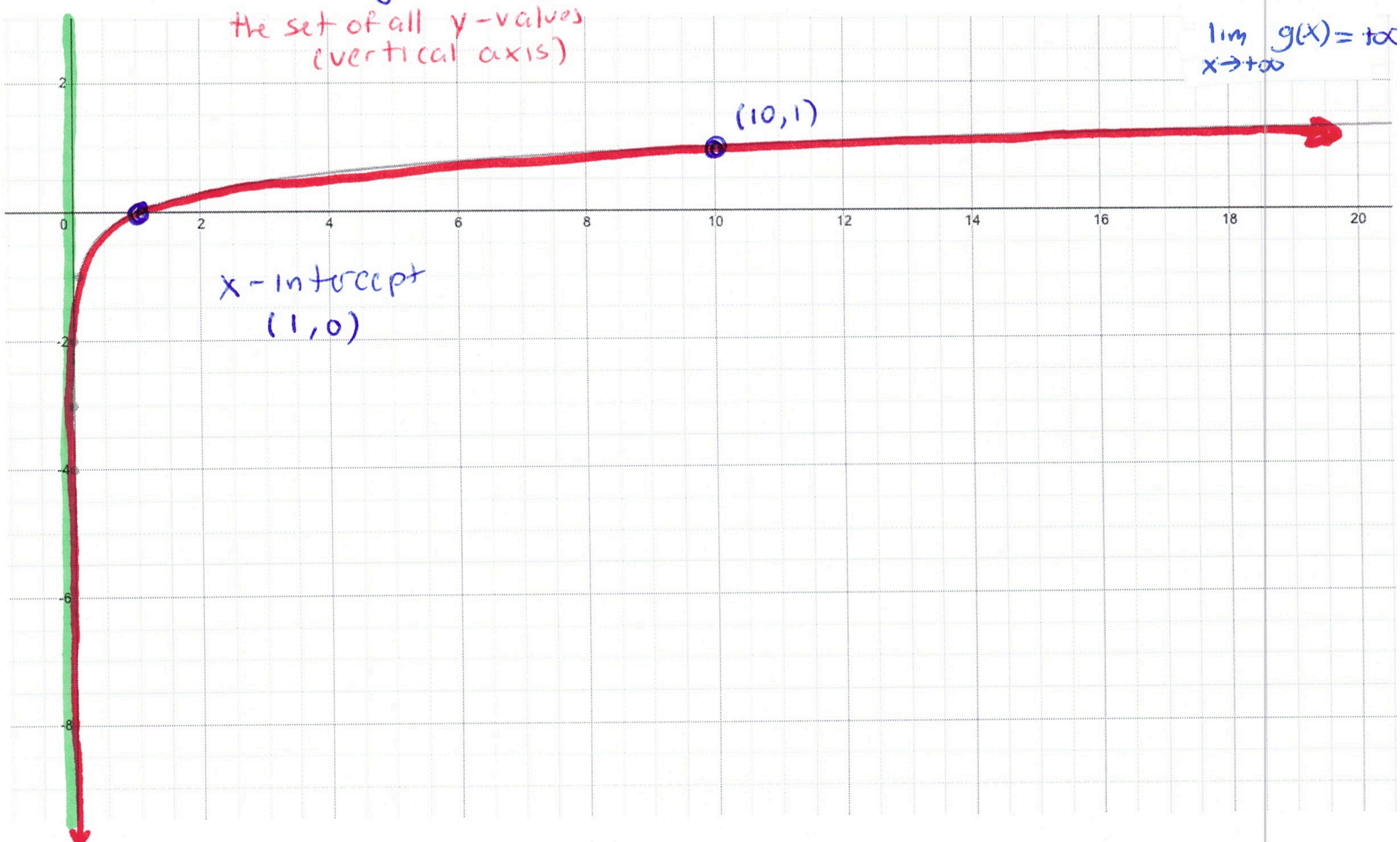
Domain of $g(x)$: $\quad Dom(g) = (0, +\infty)$

the set of all possible x-values (horizontal axis)

open parenthess

Range of $g(x)$: $\quad Rng(g) = (-\infty, +\infty)$

the set of all y-values (vertical axis)



y-axis
$x = 0$

$\boxed{\text{vertical asymptote} \quad \lim_{x \to 0^+} g(x) = -\infty}$

No y-intercept

- [ ] x-intercept at point $(1, 0)$

$\log_{10}(1) = 0 \iff 10^0 = 1$

- [ ] $\log_{10}(10) = 1 \iff 10^1 = 10$

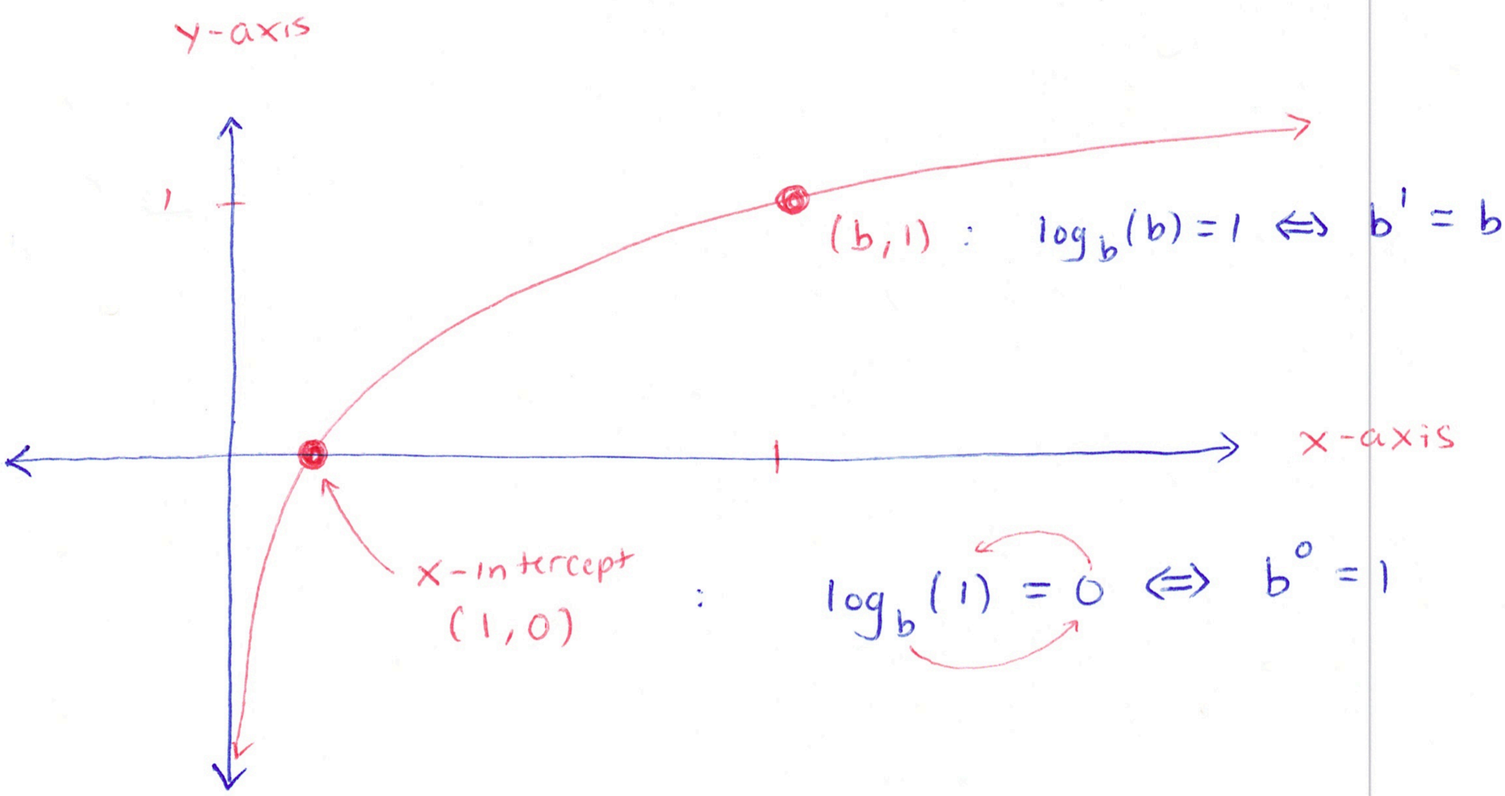
what

Let's graph the function $h(x) = \underbrace{\log_b(x)}_{\text{parent log function}}$

for any $\boxed{b > 1}$

Domain of $h(x)$: $\text{Dom}(\log_b(x)) = (0, +\infty)$

Range of $h(x)$: $\text{Rng}(\log_b(x)) = (-\infty, \infty)$



vertical asymptote at $x = 0$

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4. TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS?

4A. For logarithmic function $y = a \cdot \log_b(x - h) + k$, what do parameters a ,h, and k do to the graph of $y = \log_b(x)$? Develop graphs on Desmos.com to highlight each parameter and demonstrate the effect on your graph. Capture

Let $b > 1$ and consider $y = a \cdot \log_b (x - h) + k$

Scalar a : this does two function

- stretch or compress the graph
  - if $|a| > 1$, we stretch graph

    (if $a > 1$ or $a < -1$, we have a stretch
  - if $0 < |a| < 1$, we compress graph

    (if $0 < a < 1$ or $-1 > a > 0$, we compress
- Reflect about the x-axis
  - if $a < 0$, we reflect about x-axis

scalar h : horizontal shift

- if $h>0$, we shift to the left
- if $h<0$, we shift to right

scalar k : vertical shift

- if $k>0$, we shift upward
- if $k<0$, we shift down

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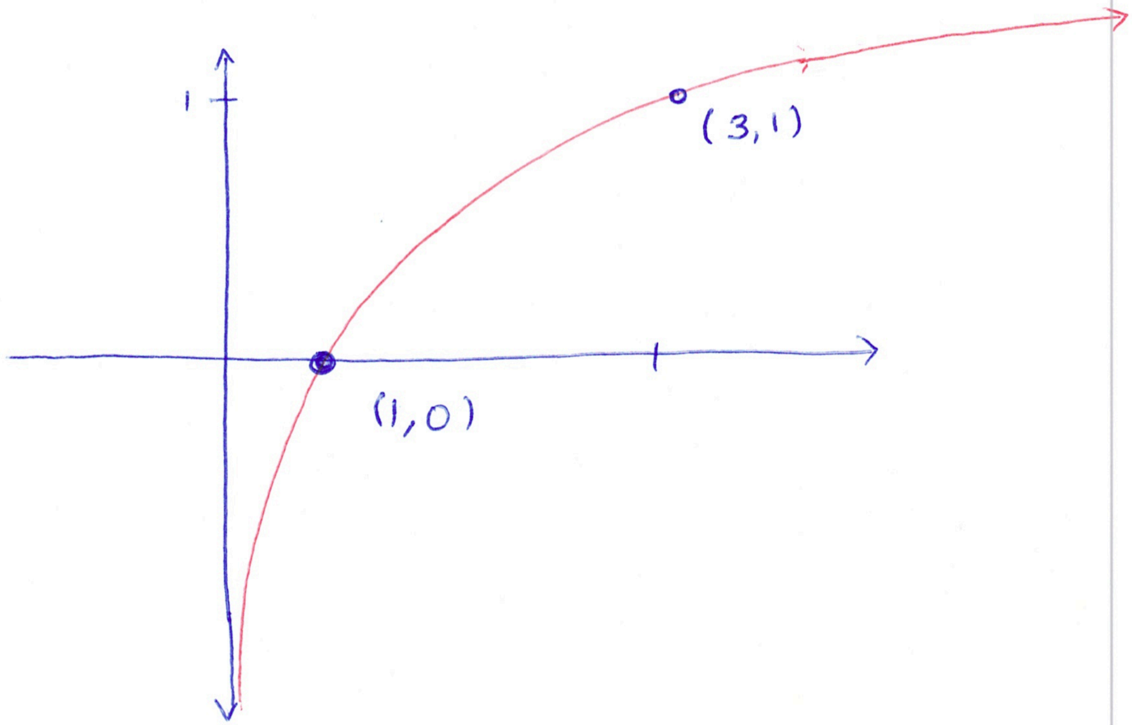
4B. Test your hypothesis from Problem 4A above by graphing the function

$$f(x) = -2\log_3(x-4) + 5$$

Let's recall $y = \log_3(x)$ has graph



Notice

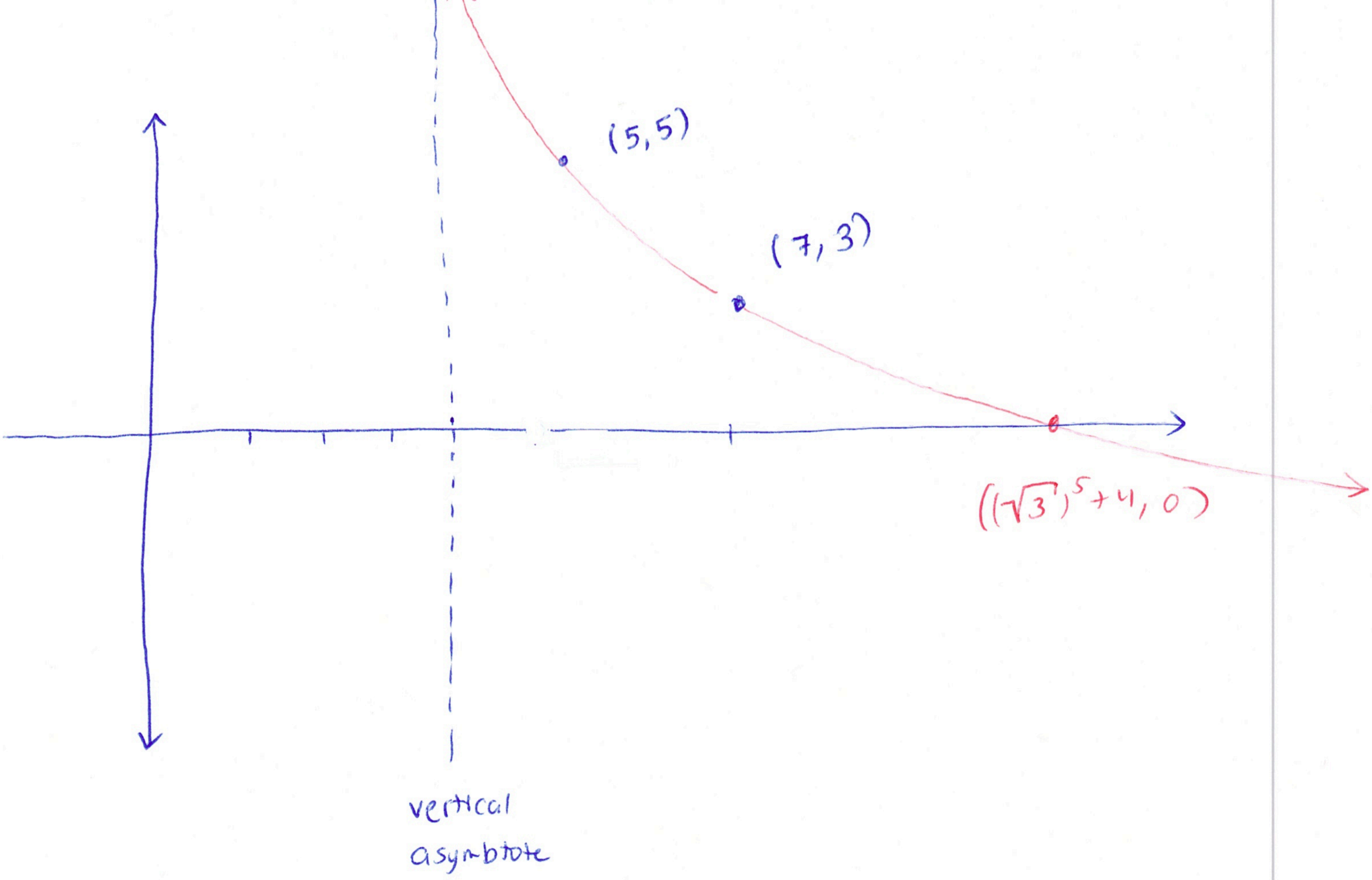
reflection about x-axis

shift up by 5 units

$$f(x) = -2\log_3(x-4) + 5$$

stretch

right shift by 4 units



$$\mathrm{Dom}(f) = (4, \infty)$$

$$\mathrm{Rng}(f) = (-\infty, \infty)$$

x - intercept :

$$0 = -2 \log_3 (x-4) + 5$$

$$\Rightarrow -5 = -2 \log_3 (x-4)$$

$$\Rightarrow \frac{5}{2} = \log_3 (x-4)$$

$$\Rightarrow 3^{5/2} = x-4$$

$$\Rightarrow x = 3^{5/2} + 4$$