Overview of Pre calculus

- Pre calculus is all about the study of functions. We study all of the following

families of functions

absolute function : $f(x) = |x|$

Root function : $f(x) = \sqrt[n]{x}$

Linear functions : $f(x) = mx + b$

Quadratic functions : $f(x) = ax^2 + bx + c$

Polynomial functions : $f(x) = a_n x^n + \cdots + a_1 x + a_0$

Rational function : $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Exponential functions : $f(x) = b^x$

$\boxed{\text{Logarithmic functions : } f(x) = \log_b(x)}$

functions transformation: $g(x) = a f(x-h) + k$

Name:______________________________ Class #:______

Math 48B, Lesson 14: Exponential Functions

In Math 48B Lessons 14, 15, 16, 17, and 18, we study logarithmic functions:

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$
"y equals log base b of x"	"x equals base b to exponent y"

To begin our exploration, let's recall the rules of powers/exponents.

1. WHAT IS AN INVERSE FUNCTION?

1A. What does it mean for a function to be one-to-one?
1B. What is the horizontal line test?
1C. What is an inverse function?
1D. When do inverse functions exist?
1E. Assume we start with function $y = f(x)$ where every point on the graph of this function is given as (x, y). How do we create the graph of the inverse function?
1F. Explain inverse function notation $f^{-1}(x) = y$.

Horizontal line test:
- draw the graph of a function
- at any point along the y-axis draw a horizontal line and count how many times that line touches the graph
- we say that we "pass" the horizontal line test if only one point touches each horizontal line.
- used to determine if a function is one-to-one

One-to-one function:

☐ In a function, remember that every x-value has exactly one y-value (vertical line test)

☐ In a one-to-one function, every output y-value comes from only (exactly) one x-value horizontal (horizontal line test)

☐ Vertical line test is a test to see if we have a function

☐ Horizontal line test tests to see is we have a one-to-one function

☐ One-to-one function: ☐ if $y = f(x_1)$ and $y = f(x_2)$, then $x_1 = x$

☐ if (x_1, y) and (x_2, y) are on te graph of a 1-to-1 function, then $x_1 = x_2$

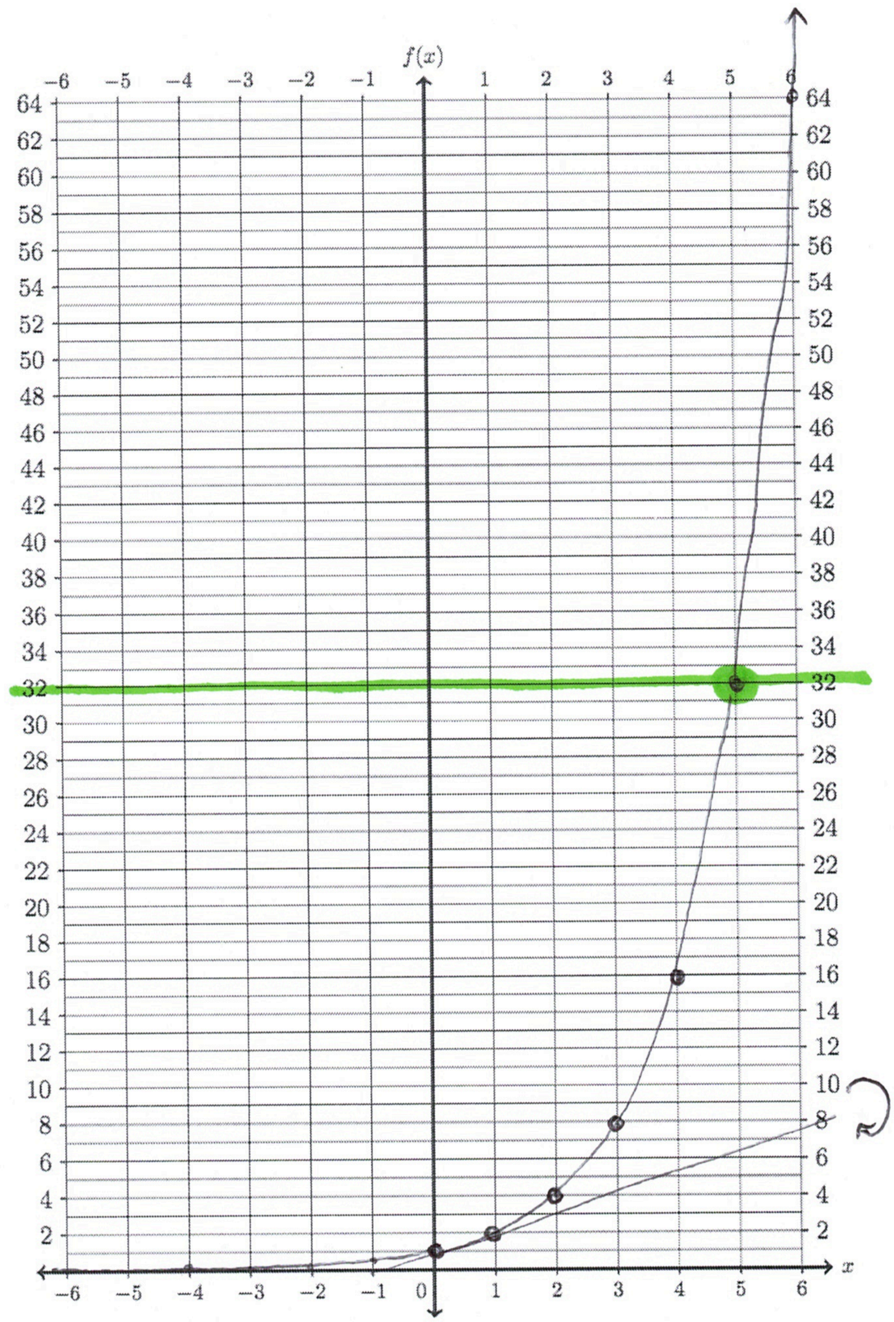
Name:____________________________

$$b^{-n} = \frac{1}{b^n}$$

2. WHAT IS THE INVERSE OF AN EXPONENTIAL FUNCTION?

2A. Graph of the exponential function $y = 2^x$ below and then apply the horizontal line test to this function. What can you say about the existence of an inverse?

x	y	(x, y)
-4	$\frac{1}{16}$	$(-4, \frac{1}{16})$
-3	$\frac{1}{8}$	$(-3, \frac{1}{8})$
-2	$\frac{1}{4}$	$(-2, \frac{1}{4})$
-1	$\frac{1}{2}$	$(-1, \frac{1}{2})$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	4	$(2, 4)$
3	8	$(3, 8)$
4	16	$(4, 16)$
5	32	$(5, 32)$
6	64	$(6, 64)$



If $x = -4$, $y = 2^x = 2^{-4} = \frac{1}{16}$

This passes the horizontal line test because for each y-value (horizontal line) we have only one x-value

Remember:

- ☐ an inverse function is designed to put the function evaluation problem backwards

- ☐ this is like rewinding a function evaluation

Function Diagram

Input ... Output

Forward Direction

$$x \longrightarrow \boxed{f(x)} \longrightarrow y \qquad y = f(x)$$

Backward Direction

$$x \longleftarrow \boxed{f(x)} \longleftarrow y \qquad x = \underbrace{f^{-1}}_{\text{inverse function}}(y)$$

given y

tell me which x produces that y

- ☐ What we start with in function evaluation is what we end with in function inverses

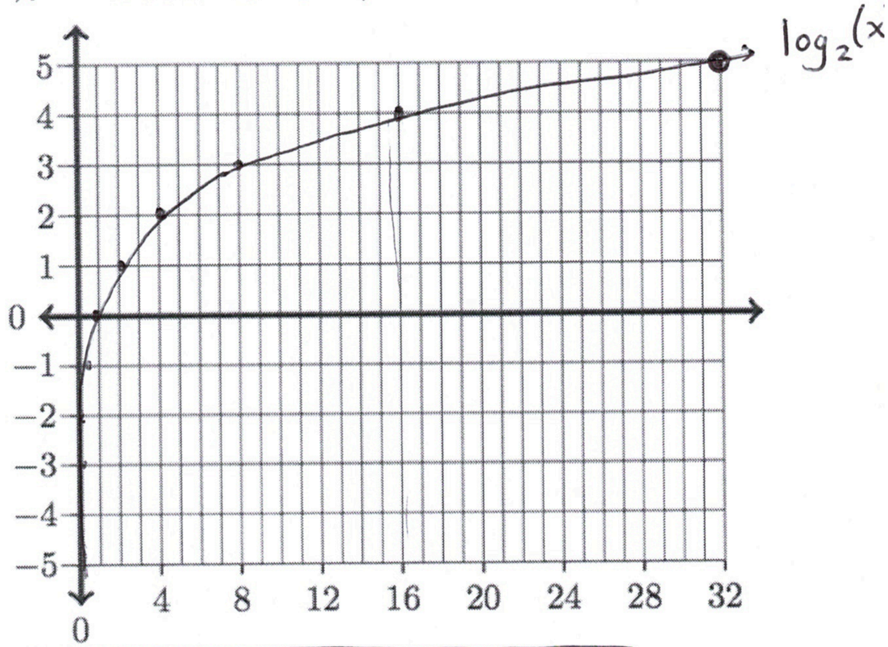
$2^x = \frac{1}{8} \Rightarrow x = f^{-1}(\frac{1}{8})$

Class #:______

2B. Graph the inverse of the exponential function $y = 2^x$ below and then apply the horizontal line test to this function. What can you say about the existence of an inverse?

In this case, the roles of x and y switch: y - becomes input (we start there) and x - becomes output

y	x	(y, x)
$\frac{1}{16}$	-4	$(\frac{1}{16}, -4)$
$\frac{1}{8}$	-3	$(\frac{1}{8}, -3)$
$\frac{1}{4}$	-2	$(\frac{1}{4}, -2)$
$\frac{1}{2}$	-1	$(\frac{1}{2}, -1)$
1	0	$(1, 0)$
2	1	$(2, 1)$
4	2	$(4, 2)$
8	3	$(8, 3)$
16	4	$(16, 4)$
32	5	$(32, 5)$
64	6	$(64, 6)$



$\boxed{\text{Reflected about the } y = x \text{ line}}$

(swapping x and y)

$\longrightarrow \quad 2^x = y$ with $y = 64$

$\Rightarrow \quad 2^x = 64$

what exponent do I hit 2 with to get 64

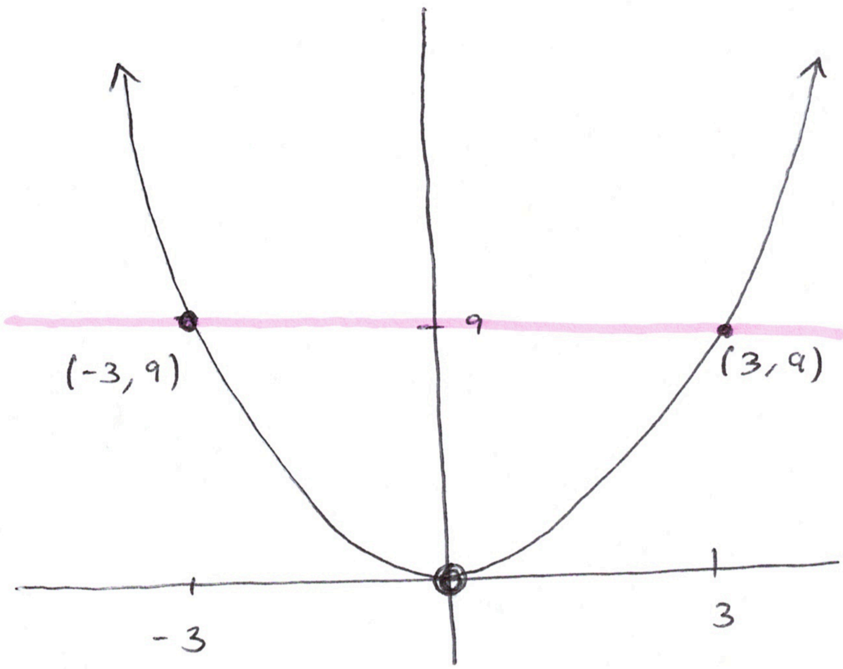
$\Rightarrow \quad x = 6$

$\Rightarrow \quad x = \log_2(64) = 6$

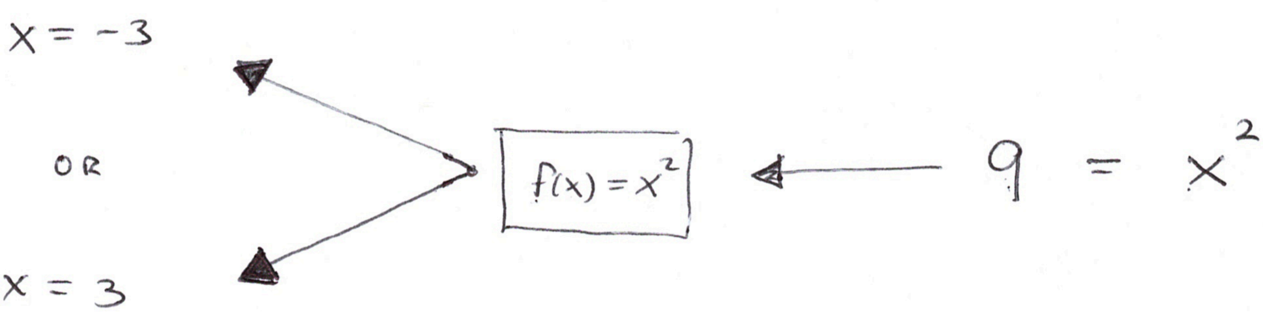
- Inverse functions exist when the original graph passes the horizontal line test

- If a graph passes the horizontal line test, the inverse function exists.

- Notice for $y = x^2$:



fails horizontal line test
(this line touches twice)



Inverse is <u>not a function</u>
(one y-value produces more than one x-value)

log form | exponential form

3H. $y = \log_b(x) \iff b^y = x$

Consider $5^{\log_5(y)} \Rightarrow m = \log_5(y)$

$\Rightarrow 5^m = y$

$\Rightarrow 5^{\log_5(y)} = y$ ✓

these are inverses
(they undo each other)

□ $x + 5 = 7$

to get rid of addition, we use subtraction

□ $5 \cdot x = 25$

to get rid of multiplication we use division

Name:____________________________ Class #:______

3. HOW TO MOVE BACK AND FORTH FROM LOGS INTO EXPONENTS?

Consider the two equivalent forms for logarithmic functions:

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$

Write each of the following problems in both forms to find the desired output value.

3A. $y = \log_{10}(1000)$

3B. $y = \log_3(81)$

3C. $y = \log_2(\frac{1}{32})$

3D. $y = \log_5(\frac{1}{125})$

3E. $y = \log_{16}(64)$

3F. $y = \log_{27}(9)$

3G. $y = \log_9(9^5)$

3H. $y = 5^{\log_5(y)}$

The hardest part of logs is notation. Jeff recommends writing both log and exponet form to start with.

$$\boxed{\text{Forward problem}} \longrightarrow \text{swap } x \text{ and } y \longrightarrow \boxed{\text{Inverse Problem}}$$

$y = 2^x$

$x = 2^y$

$\Leftrightarrow \log_2(x) = y$

"y equals log base 2 of x"

log form | Exponent form

3A. $y = \log_{10}(1000) \iff 10^y = 1000$

"y equals log base 10 of 1000"

output y | input x = 1000

$y = \log_{10}(1000) \iff 1000 = 10^y$

base b = 10

How many times do I have to multiply 10 by itself to get 1000

$3 = \log_{10}(1000) \iff y = 3$

base 10 to the 3 equals 1000

---

y equals log b of x is equivalent to x equals b to the y

3B. $y = \log_b(x) \iff x = b^y$

x = 81

$y = \log_3(81) \iff \boxed{81 = 3^y}$

base b = 3

what exponent goes on base 3 to get 81 out

multiply 3 by itself y times to get 81

$\log_3(81) = 4 \iff y = 4$

$3^2 = 3 \cdot 3 = 9$

$9 \cdot 3 = (10-1)\,3$

$= 30 - 3$

$= 27$

$3 \cdot 3 \cdot 3 = 9 \cdot 3 = 27 = 3^3$

$\underline{3 \cdot 3 \cdot 3 \cdot 3} = 27 \cdot 3 = 81 = 3^4 = 9 \cdot 9$

$27 \cdot 3 = (30-3) \cdot 3$

$= 90 - 9$

$= 81$

3C.

$$\boxed{y = \log_b(x) \iff b^y = x}$$

input $x = \frac{1}{32}$

$$y = \log_2\left(\frac{1}{32}\right) \iff 2^y = \frac{1}{32} = \boxed{\frac{1}{2^5}}$$

base $b = 2$

- ☐ What exponent do I put on base $b=2$ to get $\frac{1}{32}$ out?

$$\Rightarrow 2^y = \frac{1}{2^5} = 2^{-5}$$

$$\Rightarrow y = -5 = \log_2\left(\frac{1}{32}\right)$$

$$\boxed{b^{-n} = \frac{1}{b^n}}$$

Note

$2 \cdot 2 = 4$

$2 \cdot 2 \cdot 2 = 8$

$2 \cdot 2 \cdot 2 \cdot 2 = 16$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

3F: $y = \log_b(x) \iff b^y = x$

input $x = 9$

$y = \log_{27}(9) \iff \boxed{27^y = 9}$ WTF

base $b = 27$

$\iff (3^3)^y = 3^2$

$(b^n)^m = b^{n \cdot m}$

$\frac{b^m}{b^n} = b^{m-n}$

$\boxed{3^x = 3^w \iff x = w}$ $\iff \boxed{3}^{\boxed{3y}} = \boxed{3}^{\boxed{2}}$

whatever exponent is here must equal

$\iff \frac{3y}{3} = \frac{2}{3}$

$\iff y = \frac{2}{3}$

$\iff \log_{27}(9) = \frac{2}{3}$

$\iff 27^{2/3} = 9$

$$27^{2/3} = 27^{\frac{1}{3}\cdot\frac{2}{1}} = \left(27^{1/3}\right)^2 \qquad \sqrt[n]{x} = x^{\frac{1}{n}}$$

$$= \left(\sqrt[3]{27}\right)^2$$

$$= (3)^2$$

$$= \underline{9} \checkmark$$

Log Form		Exponent Form

3E. $y = \log_b(x) \iff x = b^y$

Input $x = 64$

$y = \log_{16}(64) \iff 64 = 16^y$

base $b = 16$

$\iff 4^3 = (4^2)^y$

$\iff 4^{\boxed{3}} = 4^{\boxed{2y}}$

$\iff \frac{3}{2} = \frac{2y}{2}$

$\iff \boxed{y = 3/2} = 1.5$

General Principle: If $b^x = b^w$, then $x = w$

$$\log_2(x) = 5$$

$$\Rightarrow \cancel{2}^{\cancel{\log_2}(x)} = 2^5$$

exponents cancel out logarithm $\longleftrightarrow$ logarithms cancel out exponents

$$\Rightarrow x = 32$$

3G. $\cancel{\log_9}(\cancel{9}^5) = 5$

$$\log_9(9^5) = y \Rightarrow 9^{\boxed{y}} = 9^{\boxed{5}}$$

$$\Rightarrow y = 5$$

$$\cancel{\log_9}(\cancel{9}^{\boxed{5}}) = 5$$

Name:_______________________________ Class #:_______

4. WHAT ARE PROPERTIES OF LOGARITHMS?

Consider the two equivalent forms for logarithmic functions:

★

Logarithmic Form	Exponential Form
$y = \log_b(x)$	$x = b^y$

Using these two equivalent forms, come up with formula to describe the properties of logarithmic functions in each of the following cases.

4A. $\log_b(1) = 0$ $\iff$ $b^0 = 1$

4B. $\log_b(b) = 1$ $\iff$ $b^1 = b$

4C. $\log_b(b^x) = x$ ← logs cancel out exponents (logs are inverses of exponents)

4D. $b^{\log_b(x)} = x$ ← exponents cancel out logs (exponents are inverses of logs)

Name:___________________________

$$b^{0-n} = \frac{b^0}{b^n} = b^{-n} = \frac{1}{b^n}$$

5. WHAT DOES THE GRAPH OF A LOGARITHM LOOK LIKE?

Fill out the table for the logarithmic function $y = \log_3(x)$ below. The, use Desmos.com to create a graph and describe the relevant features of that graph including the domain, range, x-intercept, and the end behavior as $x \to +\infty$.

x	y
$\frac{1}{81}$	-4
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3
81	4
243	5
729	6

$$y = \log_3(x) \iff 3^y = x$$

$$y = \log_3\left(\tfrac{1}{81}\right) \Rightarrow 3^{\boxed{y}} = \frac{1}{81} = \frac{1}{3^4} = 3^{\boxed{-4}}$$

$$y = \log_3\left(\tfrac{1}{27}\right) \Rightarrow 3^y = \frac{1}{27} = \frac{1}{3^3} = 3^{-3}$$

Behavior or shape of log functions