# Overview of PreCalculus into Calculus

Math 48A: Function graphs & Function Transformations

- ☐ Linear Function : $f(x) = a_1 x + a_0$
- ☐ Absolute value Functions : $f(x) = |x|$
- ☐ Quadratic Functions : $f(x) = a_2 x^2 + a_1 x + a_0$
- ☐ Root Functions : $f(x) = \sqrt[n]{x}$
- ☐ Function Transformations

$$g(x) = a \cdot f(x-h) + k$$

Math 48B: Function Graphs

- ☐ Polynomial Functions

$$f(x) = a_n x^n + \cdots + a_1 x^1 + a_0 x^0$$

- ☐ Rational Functions

$$f(x) = \frac{a_n x^n + \cdots + a_1 x^1 + a_0 x^0}{b_m x^m + \cdots + b_1 x^1 + b_0 x^0}$$

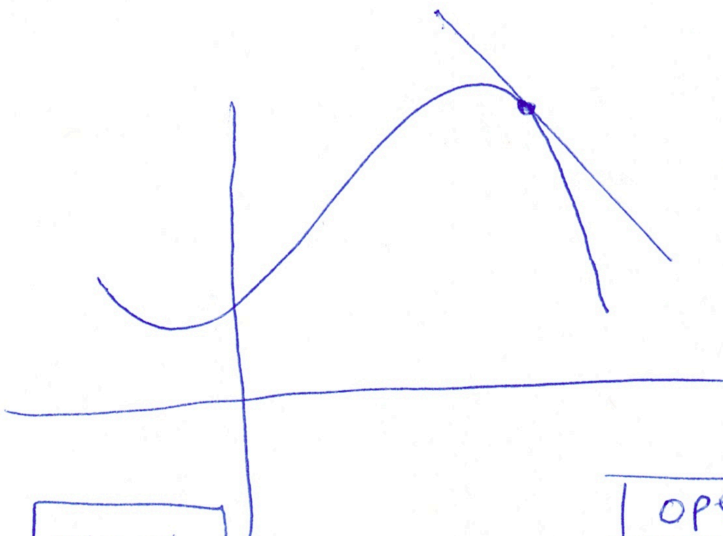
- ☐ Exponential Functions : $f(x) = b^x$
- ☐ Logarithmic Functions : $f(x) = \log_b(x)$

Math 48C: More functions: Trigonometric functions

| | |
|---|---|
| sine | $f(x) = \sin(x)$ |
| cosine | $f(x) = \cos(x)$ |
| tan | $f(x) = \tan(x)$ |

Math 1A: Single-variable derivatives

→ ☐ Finding tangent lines on functions



| Objects | | operation | | inverse operation |
|---|---|---|---|---|
| Numbers | → | addition | ⟷ | subtraction |
| Functions Math48ABC | → | derivitives Math 1A | ⟷ | integrals Math 1B |

Name: Solutions Class #:______

# Math 48B, Lesson 12: Exponential Functions

In Math 48B Lessons 11, 12, and 13, we study exponential functions:

$$y = b^x$$

To begin our exploration, let's recall the rules of powers/exponents.

## 1. WHAT ARE RULES OF POWERS/EXPONENTS?

constant in superscript

variable in superscript

Powers vs exponents: $y = x^n$ $y = b^x$

variable in base

constant base

Product Rule: $\boxed{b^n \cdot b^m = b^{n+m}}$

Quotient Rule: $\boxed{\frac{b^n}{b^m} = b^{n-m}}$

Zero Power: $1 = \frac{b}{b} = \frac{b^1}{b^1} = b^{1-1} = b^0 \Rightarrow \boxed{b^0 = 1}$

Negative Powers: $\frac{1}{b^n} = \frac{b^0}{b^n} = b^{0-n} = b^{-n} \Rightarrow \boxed{b^{-n} = \frac{1}{b^n}}$

Power to a Power: $\boxed{(b^n)^p = b^{n \cdot p}}$

Name:______________________________ Class #:______

## 2. WHAT IS EXPONENTIAL GROWTH?

2A. Fill in the table below. To the best of your ability, fill this table out by hand.

| $x$ | $f(x) = 2^x$ (base $b=2$) | $g(x) = 4^x$ (base $b=4$) | $h(x) = 5^x$ | $j(x) = 10^x$ |
|---|---|---|---|---|
| $-4$ | $2^{-4} = \frac{1}{16}$ | $4^{-4} = \frac{1}{256}$ | $\frac{1}{625}$ | $\frac{1}{10000}$ |
| $-3$ | $2^{-3} = \frac{1}{8}$ | $4^{-3} = \frac{1}{64}$ | $\frac{1}{125}$ | $\frac{1}{1000}$ |
| $-2$ | $2^{-2} = \frac{1}{4}$ | $4^{-2} = \frac{1}{16}$ | $\frac{1}{25}$ | $10^{-2} = \frac{1}{100}$ |
| $-1$ | $2^{-1} = \frac{1}{2}$ | $4^{-1} = \frac{1}{4}$ | $\frac{1}{5}$ | $10^{-1} = \frac{1}{10}$ |
| 0 | $2^0 = 1$ | $4^0 = 1$ | $5^0 = 1$ | $10^0 = 1$ |
| 1 | $2^1 = 2$ | $4^1 = 4$ | $5^1 = 5$ | $10^1 = 10$ |
| 2 | 4 | 16 | 25 | $10^2 = 100$ |
| 3 | 8 | 64 | 125 | $10^3 = 1000$ |
| 4 | 16 | 256 | 625 | $10^4 = 10000$ |

here

2B. Graph the functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ from problem 2A above.

Let $h(x) = 5^x$

$x=1 \Rightarrow h(x) = 5^x \Big|_{x=1} = 5^1 =$

$x=2 \Rightarrow 5^x = 5^2 = 25$

$x=3 \Rightarrow 5^x = 5^3 = 5^2 \cdot 5 = 25 \cdot 5$

$= (20+5)\,5$

$= 100 + 25$

$= 125$

$x=4 \Rightarrow 5^x = 5^4 = 5^3 \cdot 5 = 125 \cdot 5$

$= (100 + 25) \cdot 5$

$= 500 + 125$

$= 625$

Let $g(x) = 4^x$

Grocery Store Challenge

* Add up bill in your head before you pay

Let $x=0 \Rightarrow 4^0 = 1$

Let $x = 1 \Rightarrow 4^1 = 4$

$x = 2 \Rightarrow 4^2 = 4^1 \cdot 4^1 = 4 \cdot 4 = 16$

$x=3 \Rightarrow 4^3 = 4^{2+1} = 4^2 \cdot 4^1 = 16 \cdot 4$

$= (10+6)4$

$= 40 + 24$

$= 64$

$$\begin{array}{r} {}^{2}\\ 16 \\ 4 \\ \hline 64 \end{array}$$

$x=4 \Rightarrow 4^4 = 4^3 \cdot 4 = 64 \cdot 4$

$= (60+4)4$

$= 240 + 16$

$= 256$

Let $f(x) = 2^x$

$x = -4 \Rightarrow 2^x = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

Note:

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

$x = -3 \Rightarrow 2^x = 2^{-3} = 2^{-4+1} = 2^{-4} \cdot 2^1$

$= \frac{1}{16} \cdot \frac{2}{1}$

$= \frac{2}{16} = \frac{2}{2 \cdot 8} = \frac{1}{8}$

$x = -2 \Rightarrow 2^x = 2^{-2} = \frac{1}{4}$

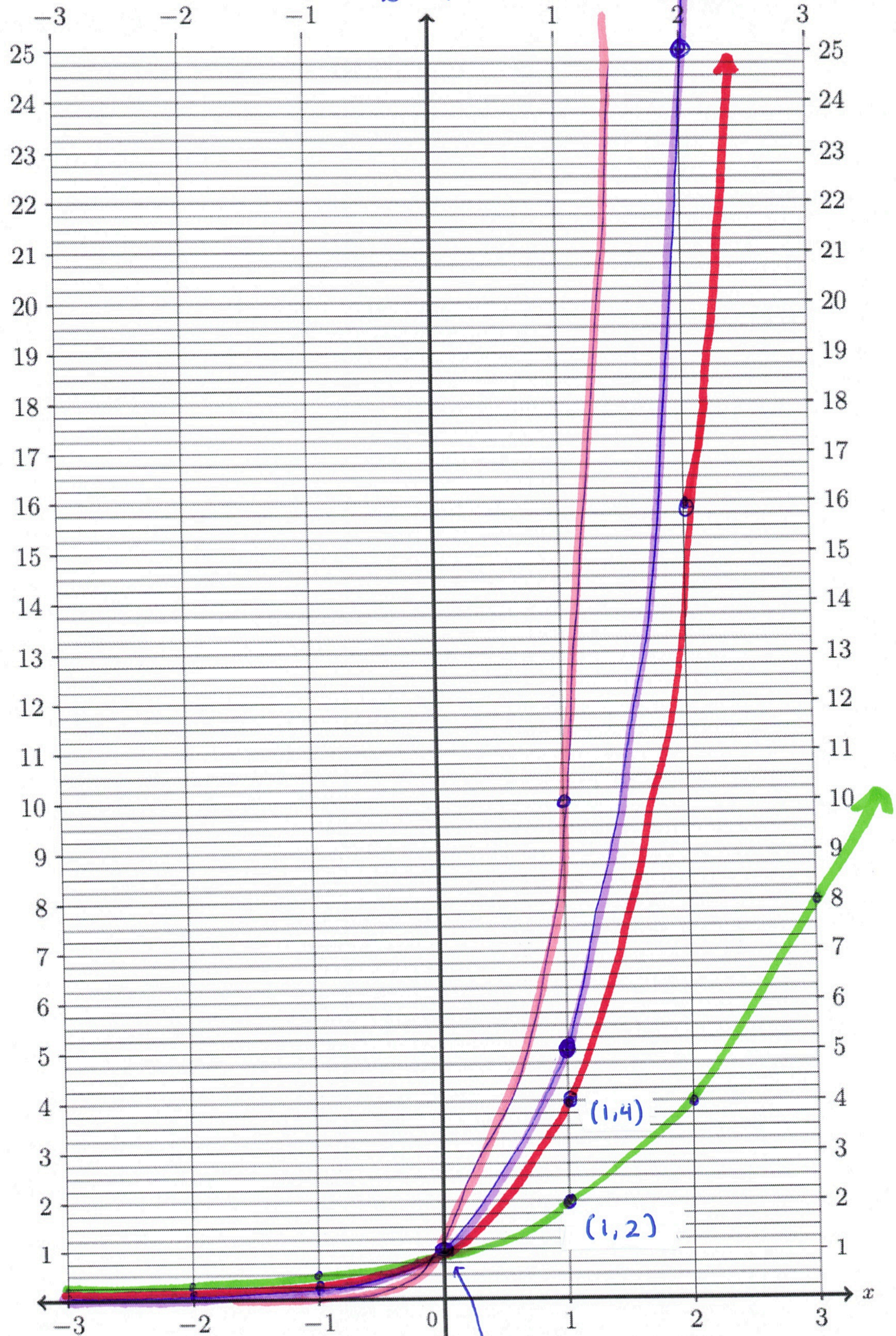
$x = -1 \Rightarrow 2^x = 2^{-1} = \frac{1}{2}$

Name:________________________

Exponential Growth

$b > 1$

$h(x) = 5^x$




Horizontal asymptote

y-intercept : $b^0 = 1$
(0, 1)

Note: for $x = 1$, we note that $b^x = b^1 = b$ at point $(1, b)$

Capital R: R (has one leg)

(Real number) Blackboard R: $\mathbb{R}$ (has two legs)

2C. Identify patterns in the graphs of the functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ from problems 2AB above. Make a conjecture about the general behavior of the graph of the function

$$y = b^x \qquad \text{for} \qquad 1 < b$$

In your conjecture, identify the domain, range, y-intercept, and the end behavior as $x \to -\infty$ as well as $x \to +\infty$.

Let $B(x) = b^x$.

Domain: $\text{Dom}(B(x)) = \{\text{all real number}\} = (-\infty, \infty) = \mathbb{R}$

Range: $\text{Rng}(B(x)) = (0, +\infty) = \mathbb{R}^+$ ← positive real numbers

never touches zero: horizontal asymptote

y-intercept: This will alway be at point $(0,1)$

when we take $b^0 = 1$

(anything to the power zero is one)

No x-intercepts since we have a horizontal asymptote

$0 \neq \frac{1}{b^n}$ for any value of $n$

n much larger than 1

even though $\frac{1}{b^n} \approx 0$ for $n >> 1$

approximately equal (really close)

End Behavior

$x \to -\infty$

(behavior towards the left)

$$\lim_{x \to -\infty} b^x = 0^+ \quad \text{when} \quad b > 1$$

"the limit of b to the x as x approaches negative infinity equals zero from positive direction"

End Behavior

$x \longrightarrow +\infty$

$$\lim_{x \to +\infty} B(x) = +\infty \quad \text{when} \quad b > 1$$

Let $f(x) = 2^x$

Domain: $Dom(f) = (-\infty, \infty) = \mathbb{R}$

Range: $Rng(f) = (0, \infty)$

never touchs zero (open parenthesis)

y-intercept will be at point $(0, 1)$ since $\underbrace{2^0}$

any thing to power of zero is one

End Behavior: $x \to -\infty$ : $\square \lim_{x \to -\infty} f(x) = 0^+$ ← this is a horizontal asymptote...

As x-values get more and more negative the fraction gets tiny

$b^{-n}$ with $n \gg 1$

$\Rightarrow b^{-n} = \frac{1}{b^n}$

HUGE Denom

End Behavior $x \to +\infty$ : $\square \lim_{x \to +\infty} f(x) = +\infty$

$$\boxed{\frac{1}{\text{HUGE NUMBER}} = \text{tiny number}}$$

Notice: $\left(\frac{1}{b}\right)^x = \left(\frac{b^0}{b^1}\right)^x$

$$= \left(b^{0-1}\right)^x$$

$$= \left(b^{-1}\right)^x$$

$$= b^{-1\cdot x}$$

$$= b^{-x}$$

$$\Rightarrow \left(\frac{1}{b}\right)^x = b^{-x}$$

Note: If $f(x) = b^x$, then $g(x) = \left(\frac{1}{b}\right)^x$

with $g(x) = f(\underbrace{-x}_{\text{flipped/rotated about y-axis}})$

$x = -4 \Rightarrow \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-4} = \frac{1}{(1/2)^4}$

$= 1 \div \left(\frac{1}{2}\right)^4$

Note:

$\left(\frac{1}{2}\right)^4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$= \frac{1\cdot1\cdot1\cdot1}{2\cdot2\cdot2\cdot2}$

$= \frac{1}{1} \div \frac{1}{16}$

$= \frac{1}{1} \cdot \frac{16}{1}$

$= 16$

Note: $\left(\frac{1}{b}\right)^x = b^{-x} \Rightarrow \left(\frac{1}{2}\right)^{-4} = 2^{-(-4)} = 2^4 = 16$

$x = -3 \Rightarrow \left(\frac{1}{2}\right)^{-3} = 2^{-(-3)} = 2^3 = 8$

$b = 2$

Name: Solution Class #:______

## 3. WHAT IS EXPONENTIAL DECAY?

3A. Fill in the table below. To the best of your ability, fill this table out by hand.

| $x$ | base $b=1/2$ <br> $F(x)=\left(\frac{1}{2}\right)^x$ | base $b=1/4$ <br> $G(x)=\left(\frac{1}{4}\right)^x$ | base $b=1/5$ <br> $H(x)=\left(\frac{1}{5}\right)^x$ | base $b=1/10$ <br> $J(x)=\left(\frac{1}{10}\right)^x$ |
|---|---|---|---|---|
| $-4$ | 16 | 256 | 625 | 10 000 |
| $-3$ | 8 | 64 | 125 | 1000 |
| $-2$ | 4 | 16 | 25 | 100 |
| $-1$ | 2 | 4 | 5 | 10 |
| 0 | $\left(\frac{1}{2}\right)^0=1$ | 1 | 1 | 1 |
| 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{10}$ |
| 2 | $\frac{1}{4}$ | $\frac{1}{16}$ | $\frac{1}{25}$ | $\frac{1}{100}$ |
| 3 | $\frac{1}{8}$ | $\frac{1}{64}$ | $\frac{1}{125}$ | $\frac{1}{1000}$ |
| 4 | $\frac{1}{16}$ | $\frac{1}{256}$ | $\frac{1}{625}$ | $\frac{1}{10000}$ |
| | $\boxed{\left(\frac{1}{2}\right)^x=2^{-x}}$ | $\boxed{\left(\frac{1}{4}\right)^x=4^{-x}}$ | $\boxed{\left(\frac{1}{5}\right)^x=5^{-x}}$ | $\boxed{\left(\frac{1}{10}\right)^x=10^{-x}}$ |

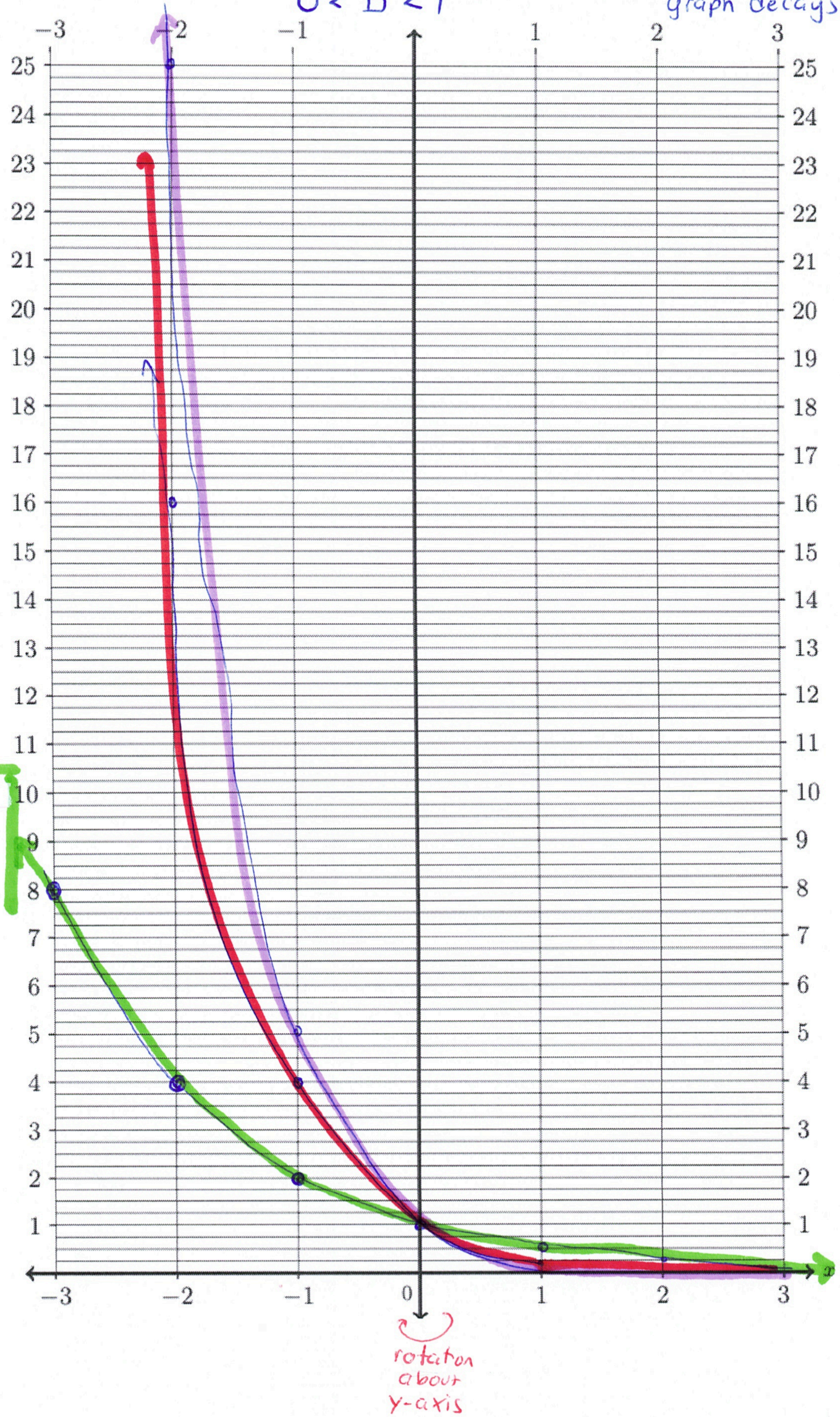
3B. Graph the functions $F(x)$, $G(x)$, $H(x)$, and $J(x)$ from problem 3A above.

Name: Solution

$\boxed{\text{Exponential decay}}$

we have exponential function $b^x$, this graph decays

$0 < b < 1$




$F(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

$F(x) = f(-x)$

14

Homework ← hint: see problem 2C

3C. Identify patterns in the graphs of the functions $F(x)$, $G(x)$, $H(x)$, and $J(x)$ from problems 3AB above. Make a conjecture about the general behavior of the graph of the function

$$y = b^x \qquad \text{for} \qquad 0 < b < 1$$

In your conjecture, identify the domain, range, y-intercept, and the end behavior as $x \to -\infty$ as well as $x \to +\infty$.

Name:______________________________ Class #:______

## 4. TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS?

4A. For exponential function $y = a \cdot b^{x-h} + k$, what do parameters $a$, $h$, and $k$ do to the graph of $y = b^x$?

Remember from Math 48A, we studied function transformations

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Horizontal Shift

h : ☐ moves graph left and right
☐ horizontal shift
☐ if $h > 0$, we move to the right (positive)
☐ if $h < 0$, we move to left (negative)

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Vertical shift

k : ☐ moves graph up and down
☐ vertical shift
☐ if $K > 0$, we move upwards
☐ if $K < 0$, we move downward

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Reflection or stretch/compress

a : ☐ shrink / compress graph
☐ stretch / elongate graph
☐ reflect around x-axis when $a < 0$

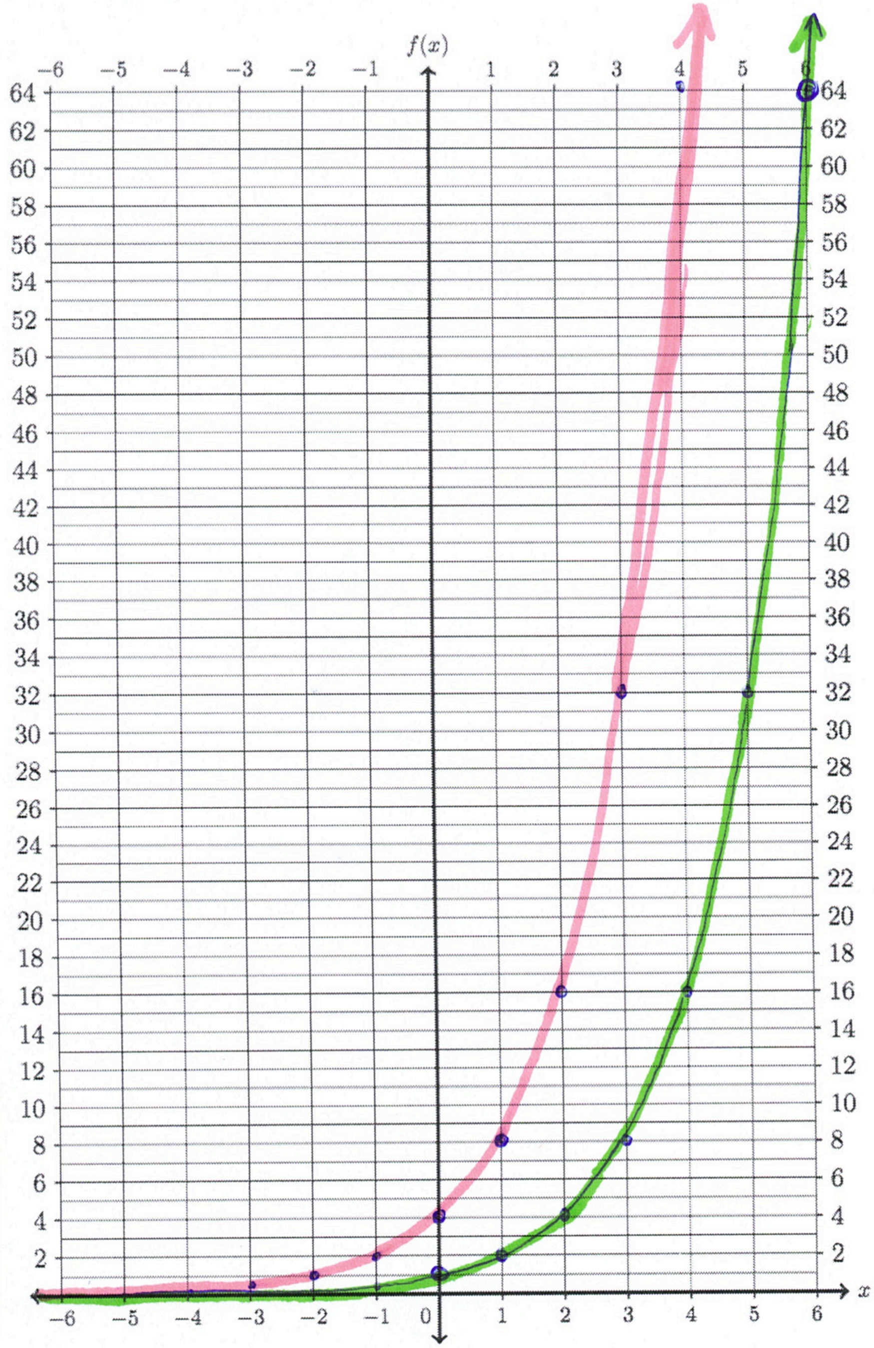
Name:______________________________ Class #:______

4B. Test your hypothesis from Problem 4A above by graphing the function below.

| $x$ | $f(x) = 2^x$ | $g(x) = 2^{\boxed{x+2}}$ |
|---|---|---|
| -4 | $\frac{1}{16}$ | $\frac{1}{4}$ |
| -3 | $\frac{1}{8}$ | $\frac{1}{2}$ |
| -2 | $\frac{1}{4}$ | 1 |
| -1 | $\frac{1}{2}$ | 2 |
| 0 | 1 | 4 |
| 1 | 2 | 8 |
| 2 | 4 | 16 |
| 3 | 8 | 32 |
| 4 | 16 | 64 |
| 5 | 32 | 128 |
| 6 | 64 | 256 |
| 7 | 128 | 512 |

$x+2=0$

shift two up



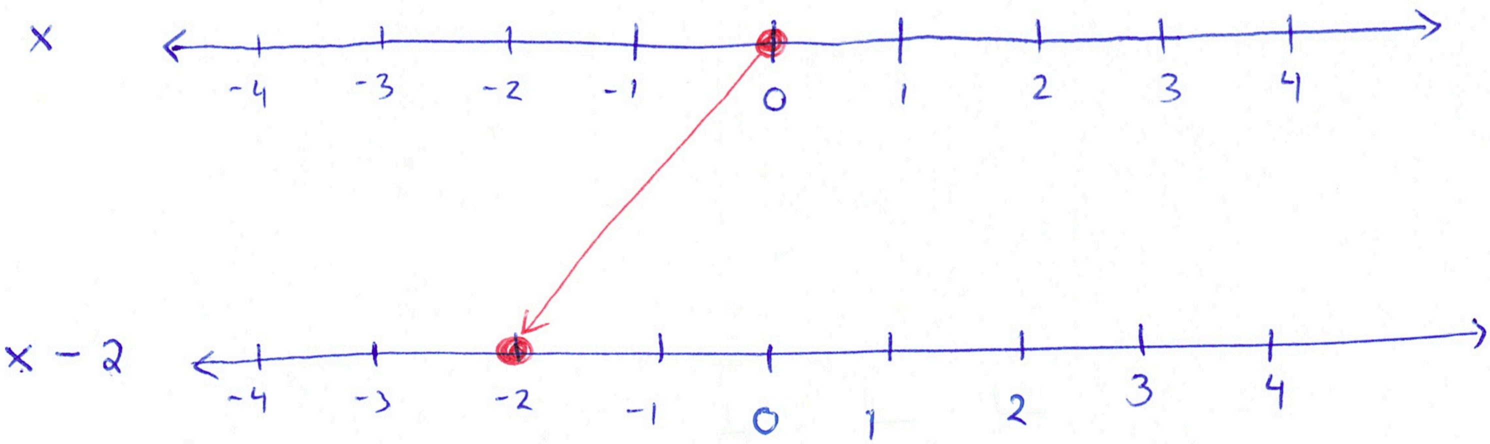

Note: $g(x) = 2^{x+2} = 2^{x-(-2)}$

$= 2^{x-h}$ with $\underbrace{h = -2}$

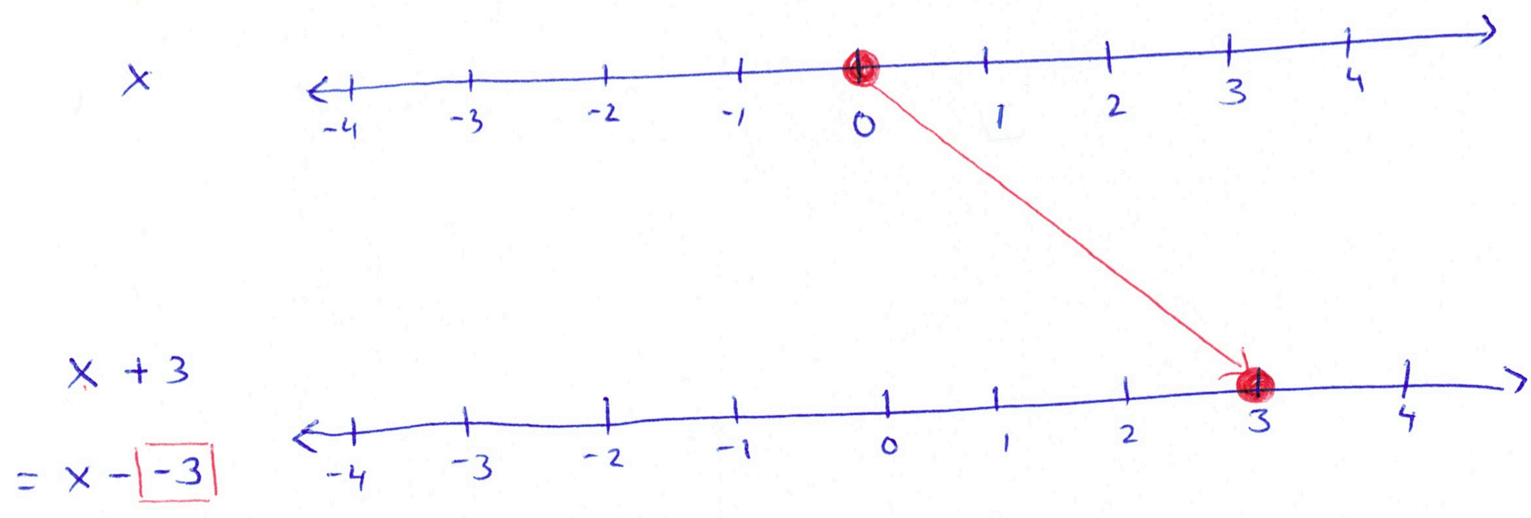
move left (negative)

$= f(x - -2)$

16




when we try to align these,
the bottom shifts to the right




when we try to align
these, the bottom shifts
to the left