Name: Solutions Class #:______

# Math 48B, Lesson 10: Rational Functions, Part 2

In Math 48B Lessons 8, 9, and 10, we study rational functions in the form:

Numerator:
Standard form of an
nth degree polynomial

$$\boxed{R(x)} = \frac{a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x^1 + a_0\,x^0}{b_m\,x^m + b_{m-1}\,x^{m-1} + \cdots + b_1\,x^1 + a_0\,x^0}$$

Rational Function
(RATIO of polynomials)

Denominator:
Standard form of an
mth degree polynomial

Math Memory training trick:
☐ Test your memory by guess & check. For every "wrong" answer test again

To begin our exploration, we explore some fundamental properties of division.

## 1. WHAT ARE RULES OF FRACTIONS?

Recall each of the following rules for fractions:

$\dfrac{0}{K} = 0$ if $K \neq 0$ (bottom can Never be zero)

$\dfrac{K}{0}$ Error (denominator can't be zero)

$\dfrac{A}{A} = 1$ if $A \neq 0$ (we can't divide by zero)

$\dfrac{A}{1} = A$

$\dfrac{A}{B} \cdot \dfrac{C}{D} = \dfrac{A \cdot C}{B \cdot D}$

$\dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{A}{B} \cdot \dfrac{D}{C} = \dfrac{AD}{BC}$

$\dfrac{1}{\text{HUGE NUMBER}} = \text{tiny number}$

$\dfrac{1}{\text{tiny number}} = \text{HUGE NUMBER}$

Flash card system:

For each formula you want to memorize, create a flash card

Test for 5 - 15 minutes per day

use Green, Yellow, Red techique (memory training)

- [ ] Out of 10 trys, focus

5 red cards

3 yellow cards

2 green cards

- [ ] Can only move one color per day

day 0 : make cards

day 1 : sorting in green / yellow red

day 2-N: start training

Read every word of the problem ←

Before starting a problem, I have read every word to avoid mistakes

2A. Use these instructions to graph the following function:

what ?

$$R(x) = \frac{x^2 - x - 6}{(x-2)(x+1)^2}$$

☐ Identify questions we have from the problem

Use Desmos.com as a tool to run your analysis.

In this problem, we'll use the 8 step instruction box from Math 48B, Lesson 10, page 2 to graph the function

$$R(x) = \frac{x^2 - x - 6}{(x-2)\cdot(x+1)^2}$$

We'll run through each step, one-by-one

$\boxed{\text{Step 1:}}$ Evaluate the function at 0 to find y-intercep

Translation (abuelita dice): We equal the function to $y = R(x)$ and replace all x-values with zero

Let $y = R(x)\Big|_{x=0}$ ← recall: y-intercept is point on y-axis where $x = 0$

$$\Rightarrow R(0) = \frac{x^2 - x - 6}{(x-2)\cdot(x+1)^2}\Bigg|_{x=0}$$

$$= \frac{0^2 - 0 - 6}{(0-2)\cdot(0+1)^2}$$

$$= \frac{0 - 6}{-2\cdot 1^2}$$

$$= \frac{-6}{-2} = +3$$

y-intercept at $(0, +3)$

Simplify, ~~solve~~,

**Step 2:** ✓ $\boxed{\text{Factor}}$ the numerator and denominator

Translation:
(abuelita language)

Let's look at

$$R(x) = \frac{\boxed{x^2 - x - 6}}{\boxed{(x-2)\cdot(x+1)^2}}$$

Numerator

Denominator

Let's get the whole thing in the form of factored form.

For numerator (quadratic/second degree) lets use AC method.

Let's use that AC method:

$$\square x^2 - x - 6 = ax^2 + bx + c$$

$a = 1, \quad b = -1, \quad c = -6$

$$\Rightarrow \quad x^2 - x - 6 = (x-3)\cdot(x+2)$$

Multiply

$-6$

$-3 \qquad +2$

$-1$

Add

Step 2: Factor denominator

Notice: for

$$R(x) = \frac{x^2 - x - 6}{(x-2)\cdot(x+1)^2}$$

$$\Rightarrow R(x) = \frac{(x+2)\cdot(x-3)}{(x-2)\cdot(x+1)\cdot(x+1)}$$

Notice, denominator is already factored ✓

$\boxed{\text{Step 3:}}$ Look at factors on top:

$$R(x) = \frac{\boxed{(x-3)\cdot(x+2)}}{(x+1)^2\cdot(x-2)}$$

- ☐ Do any factors show up on top and bottom

IN this case, NO ✓

- ☐ Set factors in top equal to zero

$\Rightarrow (x-3)\cdot(x+2) = 0$

$\Rightarrow x-3=0$ OR $(x+2)=0$

$\Rightarrow x=3$ or $x=-2$

These give us x-intercepts where $y=R(x)=0$

x-intercepts are at points

$\boxed{(3,0)}$ OR $\boxed{(-2,0)}$

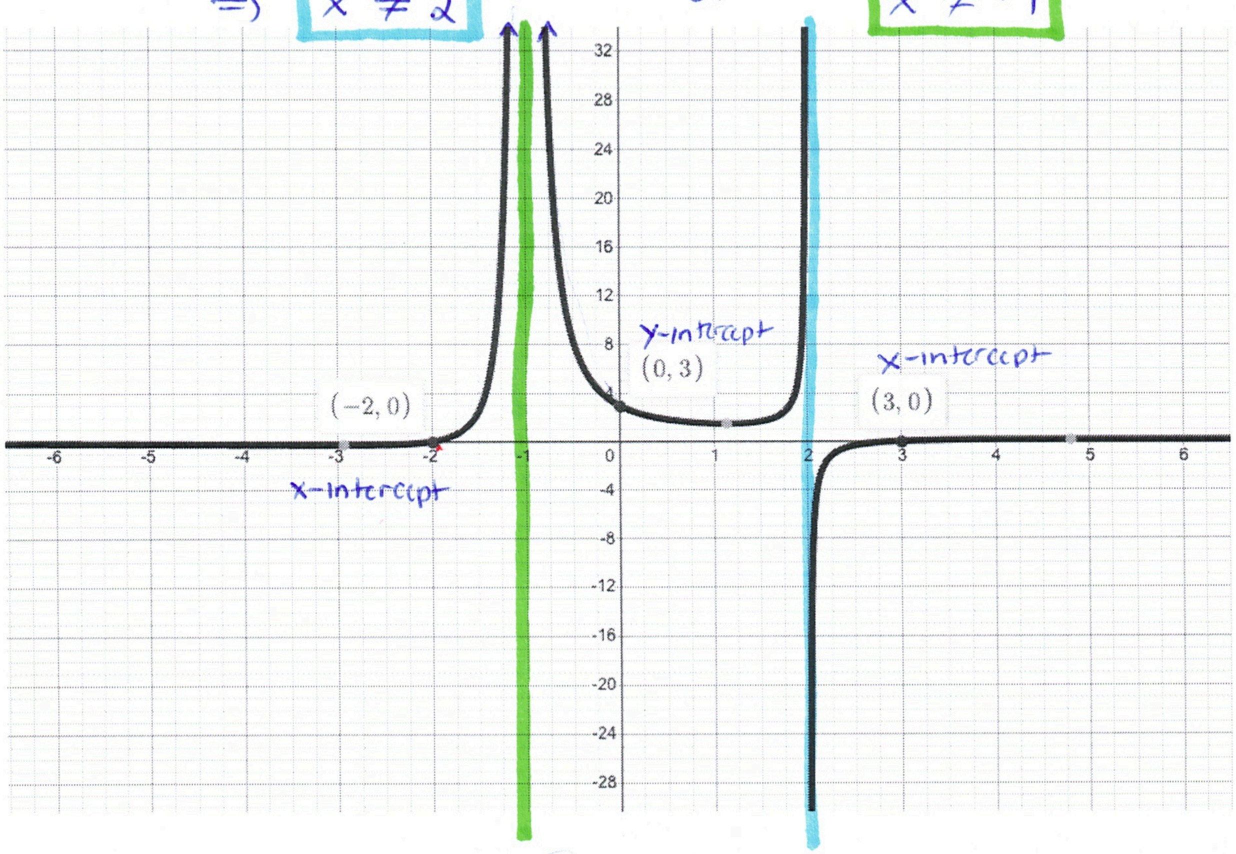
Notice: $R(x) = \dfrac{\boxed{(x+2)\cdot(x-3)}}{(x+1)^2\cdot(x-2)}$

The bottom can't be zero: so

$\Rightarrow (x+1)^2 \cdot (x-2) \neq 0$

$\Rightarrow x-2 \neq 0$ or $x+1 \neq 0$

$\Rightarrow \boxed{x \neq 2}$ or $\boxed{x \neq -1}$




$x=-1$ vertical asymptote

$x=2$ vertical asymptote

Behavior at vertical asymptote $x=-1$

$\lim_{x\to -1^-} R(x) = +\infty$

$\lim_{x\to -1^+} R(x) = +\infty$

as we approach $-1$ from the left, graph goes up & up & up

Behavior at vertical asymptote $x=2$

☐ $\lim_{x \to 2^-} R(x) = +\infty$

the limit as x approaches 2 from the negative side of R of x equals positive infinity

☐ $\lim_{x \to 2^+} R(x) = -\infty$

the limit of R(x) as x approach 2 from above is negative infinity

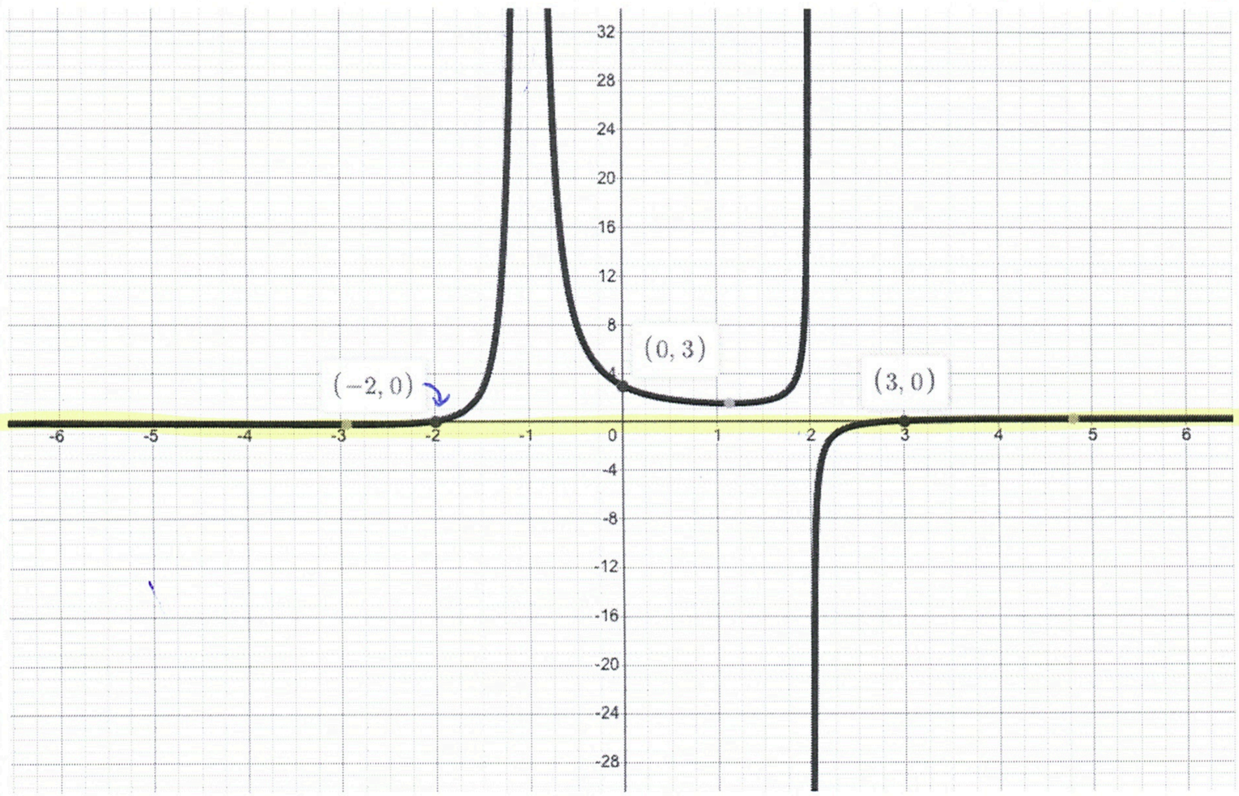
Notice: $R(x) = \dfrac{x^2 - x - 6}{(x+1)^2 (x-2)}$

the horizontal asymptote is at x-axis given by $\boxed{y = 0}$

Horizontal asymptote

- a line that defines long-term behavior (it can be crossed)

Horizontal asymptote




x-axis ($y = 0$)

$\boxed{\text{Behavior at horizontal asymptote } y = 0}$

$$\lim_{x \to -\infty} R(x) = 0^-$$

$$\lim_{x \to +\infty} R(x) = 0^+$$

the limit as x goes to negative infinity of R of x is zero

Note: $(x+1)^2 = (x+1)(x+1)$

$= x^2 + 2x + 1$

$\Rightarrow (x+1)^2 \cdot (x-2) = (x-2) \cdot (x^2 + 2x + 1)$

$= x^3 + 2x^2 + x - 2x^2 - 4x - 2$

$= x^3 - 3x - 2$

$\Rightarrow R(x) = \dfrac{x^2 - x - 6}{x^3 - 3x - 2}$ ← 2nd degree ← third degree

$= \dfrac{a_2 x^2 + a_1 x^1 + a_0 x}{b_3 x^3 + b_2 x^2 + b_1 x^1 + b_0 x^0}$

To find horizontal asymptote, we're looking for long-term behavior

$$\Rightarrow \quad R(x) = \frac{\boxed{x^2} - x - 6}{\boxed{x^3} - 3x - 2} \left[\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right]$$

$$\Rightarrow \quad R(x) = \frac{\cancel{\frac{1}{x}}^{\,0} - \cancel{\frac{1}{x^2}}^{\,0} - \cancel{\frac{6}{x^3}}^{\,0}}{1 - \underbrace{\cancel{3/x^2}}_{0} - \underbrace{\cancel{2/x^3}}_{0}}$$

as $x \to \infty$

## Sign analysis

$-1 > x > -2 \quad : \quad \dfrac{(x+2)\cdot(x-3)}{(x+1)^2\,(x-2)} = \dfrac{+\cdot-}{+\cdot-} = +$

$x < -2 \quad : \quad \dfrac{-\;\cdot\;-}{+\;\cdot\;-} = -$

# Review:

## Lesson 1-7: Polynomial functions

We studied the following ideas

- How to find $\boxed{\text{zeros}}$ of a polynomial?
  1. AC method for quadratic
  2. Long division
  3. Graphical method

- Two forms of a polynomial

these are the zeros

- Factored form: $a(x - c_1)\cdot(x - c_2)\cdots(x - c_n)$

nth degree polynomial

Standard / Nonfactored form:

$$\underbrace{a_n x^n}_{\text{nth degree}} + \cdots + a_2 x^2 + a_1 x^1 + a_0 x^0$$

- Long division is a way to find zero factors out of standard form: we want a zero remainder for a zero factor

# Lesson 8 - 10 Rational functions

Ration function : ratio of two polynomial

degree n polynomial

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0 x^0}$$

degree m polynomial

Assuming no common factors zeros give us x-intercepts since $\frac{0}{K} = 0$

$$\Leftrightarrow \quad R(x) = \frac{a(x-c_1)(x-c_2)\cdots(x-c_n)}{b(x-d_1)(x-d_2)\cdots(x-d_n)}$$

these are locations of either:

1. vertical asymptotes
2. holes in the functions (a factor ontop looks the identical to factor on bottom)

To find horizontal asymptotes

1) we can graph the function

2) analyze long term behavior by looking at degree on top/bottom