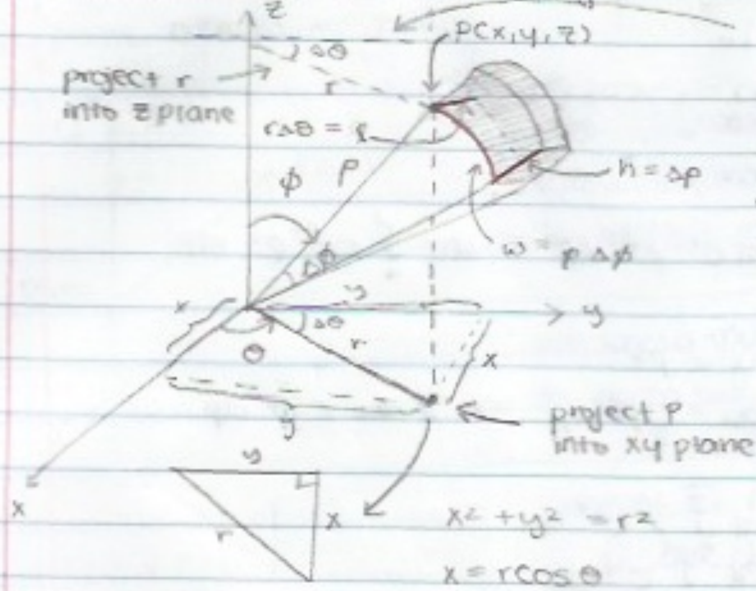
3. (8 points) Evaluate the integral:

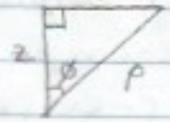
$$\iiint\limits_{D} e^{\left(x^2+y^2+z^2\right)^{3/2}} \, dV$$

where D is the unit ball in $\mathbb{R}^3$ centered at the origin. (Hint: try a change of variables into spherical coordinates).

wow!

3. Evaluate the Integral: $\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\,dV$ where D is the unit ball in $\mathbb{R}^3$ centered at the origin.





$r = \rho\sin\phi$

$z = \rho\cos\phi$

$r^2 + z^2 = \rho^2$

dV = "volume" of spherical box (shaded on diagram)

$= \text{length} \times \text{width} \times \text{height}$

$= \ell \times w \times h$

$= \rho\Delta\phi \times r\Delta\theta \times \Delta\rho$

$= \rho\Delta\phi \times \rho\sin\phi \times \Delta\rho$

$= \rho^2 \sin\phi\,\Delta\phi\,\Delta\theta\,\Delta\rho$

$x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

unit ball, thus unit domain

$D = \{(\rho,\phi,\theta) : 0 \le \rho \le 1,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$x^2 + y^2 = r^2$

$r^2 + z^2 = \rho^2$

$\Rightarrow x^2 + y^2 + z^2 = r^2 + z^2 = \rho^2$

$\Rightarrow I = \iiint_D e^{(x^2+y^2+z^2)^{3/2}}\,dV$

This integral is difficult/impossible to do in cartesian coordinates, so we translate to spherical

$= \iiint_D e^{(\rho^2)^{3/2}}\,dV = \iiint_D e^{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$= \int_0^1\int_0^{\pi}\int_0^{2\pi} e^{\rho^3}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$

Inner integral: $\int_0^{2\pi} e^{\rho^3}\rho^2\sin\phi\,d\theta$

$= e^{\rho^3}\rho^2\sin\phi\,\theta\Big|_0^{2\pi}$

$= 2\pi e^{\rho^3}\rho^2\sin\phi - 0 = 2\pi e^{\rho^3}\rho^2\sin\phi$

Middle integral: $\int_0^{\pi} 2\pi e^{\rho^3}\rho^2\sin\phi\,d\phi$

$= 2\pi e^{\rho^3}\rho^2\int_0^{\pi}\sin\phi\,d\phi$

Middle integral (continued):

$$2\pi e^{\rho^3}\rho^2 \int_0^{\pi} \sin\phi \, d\phi$$
$$= 2\pi e^{\rho^3}\rho^2 (-\cos\phi) \Big|_0^{\pi}$$
$$= 2\pi e^{\rho^3}\rho^2 (-\cos(\pi) - (-\cos(0)))$$
$$= 2\pi e^{\rho^3}\rho^2 (1+1)$$
$$= 4\pi e^{\rho^3}\rho^2$$

outer integral: $\int_0^1 4\pi e^{\rho^3}\rho^2 \, d\rho = 4\pi \int_0^1 e^{\rho^3}\rho^2 \, d\rho$

u-substitution $\quad$ Let $u = \rho^3$

$du = 3\rho^2 d\rho \Rightarrow \frac{1}{3} du = \rho^2 d\rho$

$$\Rightarrow 4\pi \int_0^1 \frac{1}{3} e^u \, du$$
$$= \frac{4}{3}\pi \int_0^1 e^u \, du$$
$$= \frac{4\pi}{3} (e^u \Big|_0^1)$$
$$= \frac{4}{3}\pi (e^{\rho^3} \Big|_0^1)$$
$$= \frac{4}{3}\pi (e-1)$$

Thus,

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV = \frac{4}{3}\pi(e-1)$$

Mistakes:

I practiced the derivation of spherical coordinates several times, so I found this problem fairly straightforward. However, I was overconfident, and confused my ϕ and θ boundaries, resulting in my volumes cancelling out with each other and the integral coming out as zero. I knew this was wrong and didn't make sense as an answer, so I checked my work, but I think a combination of anxiety and sleep deprivation lead me to continuously skip this boundary and focus on finding a flaw in my integration. I ran out of time to check my work, and left what I had. For future integrals, I think taking the time to write out the domain will help make my double checking more efficient, and thus possibly prevent silly mistakes like this.