For problems 1 - 4, let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a two-variable function with explicit representation $z = f(x, y)$. Let $A\big(a, b, f(a, b)\big)$ be a point on the surface

$$S_f = \{(x, y, z) \,:\, (x, y) \in D \text{ and } z = f(x, y)\}.$$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the domain of function f.

1. (6 points) Please derive the limit definition of the directional derivative from first principles. If you're confused where to start, please follow the 5 steps process to constructing a derivative that we discussed in our Lesson 11 videos.

1)

Let $z=f(x,y)$ define a surface where $f(x,y)$ is differentiable. Let point $(a, b, f(a,b))$ be a point on the surface

$$S_f = \{(x,y,z) : (x,y) \in D \text{ and } z=f(x,y)\}$$

Let $\vec{u} = \langle u_1, u_2 \rangle \in \mathbb{R}^2$ be a unit vector in the xy-plane, the domain of f

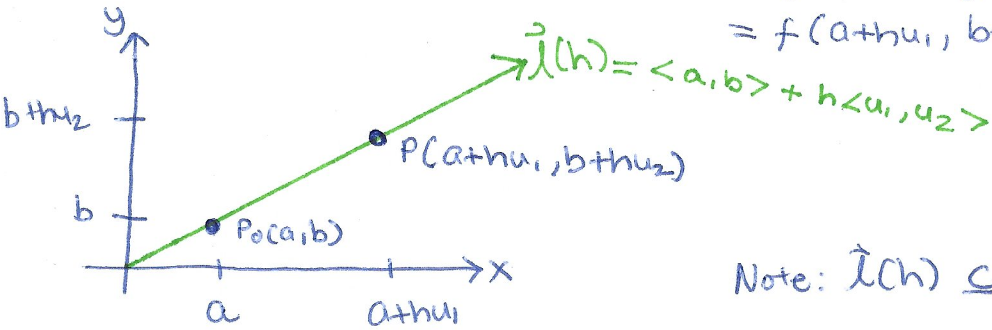
* We have to find the slope of a tangent line to our given surface $z=f(x,y)$ at an input in a general direction defined by the unit vector.
Using the 5 steps from Lesson 11 we will derive the limit definition of the directional derivative from first principles.

(1) Graph a curve & related to the given function

- we can create a vector-valued equation with a single parameter h

$$\Rightarrow \vec{l}(h) = \vec{P_0} + h \cdot \vec{u} \qquad \text{from : } \vec{P_0} = \langle a,b \rangle$$
$$= \langle a,b \rangle + h \cdot \langle u_1, u_2 \rangle \qquad h = \text{parameter}$$
$$= \langle a+hu_1, b+hu_2 \rangle \qquad \vec{u} = \langle u_1, u_2 \rangle$$
$$= \langle x(h), y(h) \rangle$$

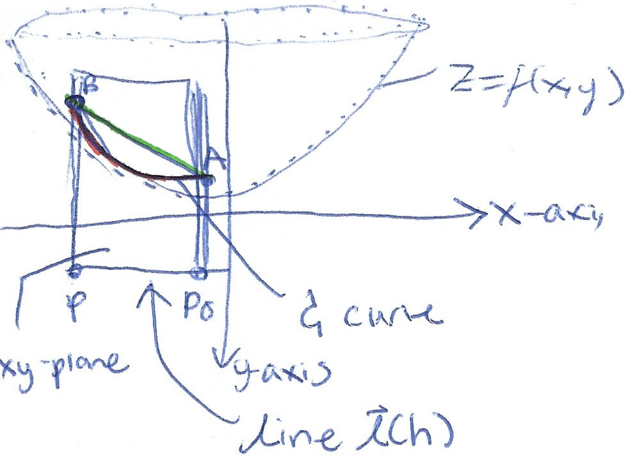
- Notice where $z=f(x,y)$, then
$$z(h) = f(\vec{l}(h))$$
$$= f(x(h), y(h))$$
$$= f\langle a+hu_1, b+hu_2 \rangle$$



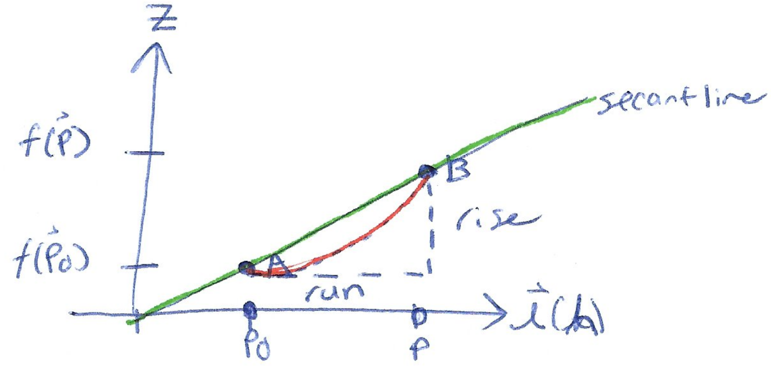
Note: $\vec{l}(h) \subseteq \mathbb{R}^2$

2) Find two pts & draw a secant line : $\vec{l}(h) = \langle a,b \rangle + h\langle u_1, u_2 \rangle$ (parameterized secant line)
$$= \vec{P_0} + h \cdot \vec{u}$$
$$= \langle a+hu_1, b+hu_2 \rangle$$
$$= \langle x(h), y(h) \rangle$$

$$\vec{P_0} = (a,b)$$
$$\vec{P} = (a+hu_1, b+hu_2)$$



where



(3) Measure the slope of the secant line through points A & B

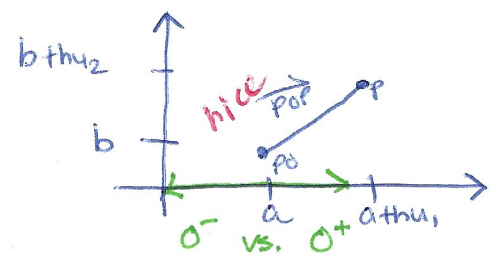
$\Rightarrow$ Slope of the secant line algebraically is $m_{AB} = \frac{\text{rise}}{\text{run}}$.

$$m_{AB} = \frac{\text{rise}}{\text{run}} \Rightarrow \frac{f(\vec{P}) - f(\vec{P_0})}{\|\vec{P_0P}\|_2}$$

$$= \frac{f(a+hu_1, b+hu_2) - f(a,b)}{\|\vec{P_0P}\|_2}$$

$$= \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

where h can be $\pm$; $h \to 0^-$, $h \to 0^+$!!



Note: $\|\vec{P_0P}\|_2 \geq 0$
but a^- or a^+

Sidework:

$$\Rightarrow \vec{P_0P} = \langle a+hu_1, b+hu_2 \rangle - \langle a,b \rangle$$
$$= \langle a+hu_1 - a, b+hu_2 - b \rangle$$
$$= \langle h \cdot u_1, h \cdot u_2 \rangle$$
$$= h \cdot \langle u_1, u_2 \rangle$$
$$= h \cdot \vec{u}$$

$\vec{u} = 1$ b.c its a unit vector

$$= h$$

$\therefore$ The slope of $m_{AB} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

Note about $\vec{u}$: Since $\|\hat{u}\| = 1$, by assumption we travel exactly h units to get from P_0 to P. The run is simply h, where h is a signed distance.

$$\Rightarrow \|\vec{P_0P}\|_2 = \|h \cdot \vec{u}\|_2$$
$$= |h| \cdot \|\hat{u}\|_2 \quad \text{unit vector magnitude} = 1$$
$$\equiv |h| \quad \text{(homogeneity)}$$

where

$$= \sqrt{(hu_1)^2 + (hu_2)^2}$$
$$= \sqrt{h^2(u_1^2 + u_2^2)}$$
$$= \sqrt{h^2} \cdot \sqrt{u_1^2 + u_2^2}$$
$$= |h| \cdot \vec{u} \equiv |h|$$

h can be $\pm$; $h \to 0^-$, $h \to 0^+$!!

P1.99.2

2. (4 points) Using the limit definition for the directional derivative of f in the direction of $\mathbf{u}$ at the point (a, b) that you derived in problem 1 above, show how to construct a composite function $g(t)$. This single variable function should have the property that the derivative $g'(t)$ is the same value as the limit we constructed to compute the directional derivative in problem 1.

(2) We draw the secant line through points A & B on the curve C given by $Z=f(x,y)$. We know if we move along $\vec{l}(h)$ in the domain of $Z=f(x,y)$ we can trace a curve C along the surface where the outputs on C are given by

$$Z(h) = f(\vec{l}(h)) = f(x(h), y(h)) = f(a+hu_1, b+hu_2)$$

The plane $\perp$ to xy-plane contains the parameterized line $\vec{l}(h)$.
The curve C is given by:

$$C = \{\langle x(h), y(h), f(x(h), y(h))\rangle : h \in I \subset \mathbb{R}\}$$

where $0 \in I$ for $x(0)=a$, $y(0)=b$, $Z(0)=f(a,b)$ and $I=(-\varepsilon, \varepsilon)$

Recall from the ordinary derivative in Math 1A:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{(difference quotient)}$$

h is a sign sensitive parameter! We said in Math 1A that the limit of a function is also sign sensitive. Where the limit of $f(x)$ as x approaches a is L

$$\lim_{x\to a} f(x) = L$$

wow! very nice!

for $F'(a) = \lim_{x\to a} \frac{F(x)-F(a)}{x-a}$

if for any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ s.t $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$. The value of δ in the precise definition of a limit depends only on ε which is why when we graph the intersection of a plane through the point $(a,b,f(a,b))$ with the normal vector $\vec{n} = \langle -u_2, u_1, 0\rangle$ and the surface $Z=f(x,y)$, we must understand "h" can approach $\pm$ values from the lefthand side and righthand side when we evaluate the slope of the secant line. The curve $C = \{\langle x(h), y(h), z(h)\rangle : h \in I \in \mathbb{R}\}$ where $h \in I \subseteq \mathbb{R}$ for $I = (-\varepsilon, \varepsilon)$ to indicate the signed distance

(4) Now we transform the secant line into a tangent line using a limiting process

$$\Rightarrow m_{AB} = \frac{f(\vec{P}) - f(\vec{P_0})}{\|\overrightarrow{P_0P}\|_2} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

// apply the limit definition where:

$$\lim_{\vec{P} \to \vec{P_0}} \frac{f(\vec{P}) - f(\vec{P_0})}{\|\overrightarrow{P_0P}\|_2}$$

(5) Construct the "derivative" as the slope of the tangent line

$$\Rightarrow \lim_{\vec{P} \to \vec{P_0}} \frac{f(\vec{P}) - f(\vec{P_0})}{\|\overrightarrow{P_0P}\|_2} = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

When this limit exists, it's called the directional derivative of f at point (a,b) in the direction of $\vec{u}$, where $\vec{u}$ is a unit vector in a general direction.

$$\therefore D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

// where $h \in I \subseteq \mathbb{R}$,
$I = (-\varepsilon, \varepsilon)$
meaning $h \to 0^-$ & $h \to 0^+$

Problem #2

In problem 1, $D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$.

We can construct a composite function $g(t)$ using the direction of $\vec{u}$ and the point (a,b).

We said previously using the limit definition with parameter h reserved for secant lines, that the parameterized line $\vec{l}(h)$ can be crafted by $\vec{l}(h) = \vec{P_0} + h \cdot \vec{u}$.

From Problem # 1

- uses parameter "h"

$\Rightarrow \vec{l}(h) = \vec{P_0} + h \cdot \vec{u}$
$= \langle a,b \rangle + h \cdot \langle u_1, u_2 \rangle$

$\Rightarrow$ where $z = f(x,y)$, then
$z(h) = f(\vec{l}(h))$.
Or, $g(h) = f(\vec{l}(h))$
$= \underbrace{f(a+hu_1, b+hu_2)}$

// From $\vec{l}(h) = \vec{P_0} + h \cdot \vec{u}$
$= \langle a,b \rangle + h \cdot \langle u_1, u_2 \rangle$
$= \langle a+hu_1, b+hu_2 \rangle$
$= \langle x(h), y(h) \rangle$

vs.

Now Problem #2

- uses parameter "t"

$\Rightarrow \vec{l}(t) = \vec{P_0} + t \cdot \vec{u}$
$= \langle a,b \rangle + t \cdot \langle u_1, u_2 \rangle$

$\Rightarrow$ Where $z = f(x,y)$, then
$z(t) = f(\vec{l}(t))$
Or, $g(t) = f(\vec{l}(t))$
$= \underbrace{f(a+tu_1, b+tu_2)}$

// From $\vec{l}(t) = \vec{P_0} + t\vec{u}$
$= \langle a,b \rangle + t \cdot \langle u_1, u_2 \rangle$
$= \langle a+tu_1, b+tu_2 \rangle$
$= \langle x(t), y(t) \rangle$

We can continue to transform $g(t) = f(\vec{l}(t))$ using $g(h)$ & $g(0)$ like we used $z(h) = f(\vec{l}(h))$ where $f(\vec{P})$ & $f(\vec{P_0})$.

From Problem #1

$\Rightarrow M_{AB} = \frac{f(\vec{P}) - f(\vec{P_0})}{\|\vec{P_0P}\|_2}$

Note: $M_{AB} = \frac{g(h) - g(0)}{\|\vec{P_0P}\|_2}$

where

$g(h) = f(\vec{l}(h)) = f(a+hu_1, b+hu_2)$

vs

Now for Problem #2

$\Rightarrow M_{AB} = \frac{g(t) - g(0)}{\|\vec{P_0P}\|_2}$

Note: $g(t) = f(\vec{l}(t)) = f(a+tu_1, b+tu_2)$

(Problem 1) "h"

$\Rightarrow g(h) = f(\vec{l}(h)) = f(a+hu_1, b+hu_2)$

$$\Rightarrow m_{AB} = \frac{g(h)-g(0)}{\|\overrightarrow{P_0P}\|_2}$$

$$= \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

see work from problem 1

Notice for $f(a+hu_1, b+hu_2)$ we can rewrite in terms of $x(h)$ & $y(h)$ from

$g(h) = f(\vec{l}(h)) = f(x(h), y(h))$
$= f(a+hu_1, b+hu_2)$

we can apply the limit definition again

$$\Rightarrow \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

Note: For $f(a,b)$ when $h \to 0$,

$\vec{l}(0) = \langle a,b\rangle + 0\cdot\langle u_1, u_2\rangle$
$\vec{l}(0) = \langle a,b\rangle$
$\vec{l}(0) = f(x(0), y(0))$

$$\Rightarrow \lim_{h\to 0} \frac{f(x(h),y(h)) - f(x(0),y(0))}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(\vec{l}(h)) - f(\vec{l}(0))}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{g(h) - g(0)}{h}$$ ✓

(Problem #2) "t"

$\Rightarrow g(t) = f(\vec{l}(t)) = f(a+tu_1, b+tu_2)$

$$\Rightarrow m_{AB} = \frac{g(t)-g(0)}{\|\overrightarrow{P_0P}\|_2}$$

$$= \frac{f(a+tu_1, b+tu_2) - f(a,b)}{\|\overrightarrow{P_0P}\|_2}$$

see work from problem 1 and apply steps to parameter "t"

$$= \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

We can apply the limit definition

$$\Rightarrow \lim_{t\to 0} \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

Note: $\vec{l}(t) = \langle a,b\rangle + 0\cdot\langle u_1, u_2\rangle$
$\vec{l}(0) = \langle a,b\rangle$
$\vec{l}(0) = f(x(0), y(0))$

and

$g(t) = f(\vec{l}(t)) = f(x(t), y(t))$

$$\Rightarrow \lim_{t\to 0} \frac{f(x(t),y(t)) - f(x(0),y(0))}{t}$$

$$\Rightarrow \lim_{t\to 0} \frac{f(\vec{l}(t)) - f(\vec{l}(0))}{t}$$

$$\Rightarrow \lim_{t\to 0} \frac{g(t) - g(0)}{t}$$ ✓

we can apply the limit definition for $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Problem #1

vs.

Problem #2 Proof of relation

$\Rightarrow g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$

$\Rightarrow g'(0) = \lim_{t \to 0} \frac{g(0+t) - g(0)}{t}$

$\therefore D_{\vec{u}}f(a,b) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = g'(0)$ & $D_{\vec{u}}f(a,b) = \lim_{t \to 0} \frac{g(0+t) - g(0)}{t}$

$\therefore$ For $g(t) = f(l(t))$, then $g'(0) = g'(t)\Big|_{t=0} = D_{\vec{u}}f(a,b)$ ✓

Proving $g'(t)$ is the same value as the limit we constructed to compute the directional derivative in Problem #1, but in problem #1 "h" is used and in problem #2 "t" is used.

3. (4 points) Derive the dot product formula for the directional derivative. Be sure to specifically refer to the the function $g(t)$ from problem 2 above along with the multivariable chain rule with two intermediate variables and one independent variables. When appropriate, please explicitly state and use the multivariable chain rule in your work. Also, make sure to explain the value of t that you use to take the ordinary derivative in this derivation.

Problem # 3

From the previous problem, we created the composite function $g(t) = f(x,y) = f(x(t), y(t))$ and proved the composite function has the same value as the limit constructed in problem #4

Now we can continue to evaluate the directional derivative of f at (a,b) to form the dot product formula for the directional derivative.

From the previous problem,

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = D_{\vec{u}}f(a,b)$$

for $g(t) = f(\vec{l}(t))$, then $g'(0) = g'(t)\big|_{t=0} = D_{\vec{u}}f(a,b)$

$$\Rightarrow D_{\vec{u}}f(a,b) = \frac{d}{dt}\left[g(t)\right]\Big|_{t=0}$$

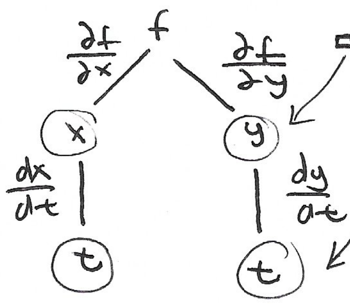
(Note: $g(t) = f(\vec{l}(t)) = f(x(t), y(t))$)

$$= \frac{d}{dt}\left[f(\vec{l}(t))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[f(x(t), y(t))\right]\Big|_{t=0}$$

(Note: use the multivariable chain rule to continue

$$= \left[\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right]\Bigg|_{t=0}$$



Notice:
- Intermediate variables are x & y
- independent variable is t

$$= \left[(f_x(x,y)\cdot x'(t)) + (f_y(x,y)\cdot y'(t))\right]\Big|_{t=0}$$

(where $x(t) = \overset{0}{\cancel{a}} + tu_1$)

$\Rightarrow x'(t) = u_1$ ~~from~~

$$= \left[(f_x(x(t),y(t))\cdot x'(t)) + (f_y(x(t),y(t))\cdot y'(t))\right]\Big|_{t=0}$$

$y(t) = \overset{0}{\cancel{b}} + tu_2$

$\Rightarrow y'(t) = u_2$

$$= f_x(x(0),y(0))\cdot u_1 + f_y(x(0),y(0))\cdot u_2$$

$$= f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$

(group like terms)

$$= \underbrace{\langle f_x(a,b), f_y(a,b)\rangle}\cdot\underbrace{\langle u_1, u_2\rangle}$$

(notice the unit vector)

$$= \vec{\nabla}f(a,b) \cdot \vec{u}$$

$$\therefore\ \underbrace{D_{\vec{u}}f(a,b)}_{\text{Directional Derivat}} = \underbrace{\vec{\nabla}f(a,b)}_{\text{gradient}} \cdot \underbrace{\vec{u}}_{\text{unit vector}}$$

4. (6 points) Using your work in problem 3, explain which unit vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ in the domain D give

   A. the direction of steepest ascent on the surface.
   B. the direction of no change on the surface.
   C. the direction of steepest descent on the surface.

   Please provide evidence that your concept images associated with these directions incorporate multiple categories of knowledge including verbal, graphical, and symbolic representations of these ideas. To earn top scores, your solution should combine the work you did in problem 3 with the cosine formula for the dot product. Also, please make specific connections to between your explanations of each direction and your knowledge of the extreme values of the cosine function.

Problem #4 –

From problem #3, we defined the directional derivative of f at point (a,b) to be equal to the dot product of the gradient f and the unit vector $\vec{u} = \langle u_1, u_2 \rangle$ in the domain D.

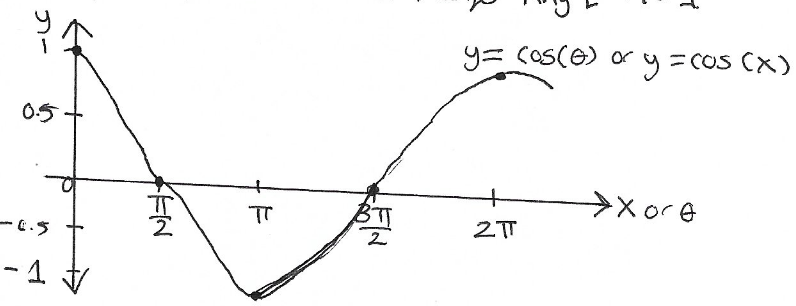
We apply the cosine formula for the dot product, $\vec{x} \cdot \vec{y} = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta$ to relate the gradient and the unit vector through a common angle

$$\Rightarrow D_{\vec{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}$$

$$= \|\vec{\nabla} f(a,b)\|_2 \cdot \underbrace{\|\vec{u}\|_2}_{1} \cdot \cos(\theta)$$

$$= \|\vec{\nabla} f(a,b)\|_2 \cdot \cos(\theta)$$, where θ is the angle between $\vec{\nabla} f$ & $\vec{u}$.

We know cosine is restricted from $-1 \leq \cos\theta \leq 1$ for $\theta \in [0, 2\pi]$. Cosine has a domain $D\ [0, 2\pi]$ and a Range $Rng\ [-1, 1]$



We know from the directional derivative that the idea of a directional derivative conveys the slope. The directional derivative is a scalar object, a number. It's the rate of change when the point (a,b) in $\mathbb{R}^3$ moves in that direction of the scalar object. The slope at $P(a,b)$ exists at that unique point to create an instantaneous tangent line we can measure. To figure out the direction of ascent, descent, or no change we look to see when $D_{\vec{u}} f(a,b)$ is the largest, the smallest, or doesn't change.

Note: $\nabla f(a,b)$ is perpendicular to the contour through point (a,b) and points where f is increasing. ∇f shows the slope in the direction of steepest ascent or descent when then influences $D_{\vec{u}} f(a,b)$ to be large or small when $\vec{\nabla} f \parallel \vec{u}$ or $\vec{\nabla} f \perp \vec{u}$ or $-\vec{\nabla} f \parallel \vec{u}$.

A. the direction of steepest ascent on the surface

Notice when $\cos\theta = 1$, $\theta = 0$ & 2π. We pick the smaller angle and see how the angle describes the behavior of $\vec{\nabla} f$ & $\vec{u}$. As $\vec{u}$ varies, the maximum value of the directional derivative is (+) $\vec{\nabla} f$. We know two vectors are $\perp$ when the angle between them is 0. This indicates $\vec{u}$ is in the same direction of the $\vec{\nabla} f$ perpendicular to the level curve. The unit vector and the gradient of f point in the same direction at the point (a,b) when the angle θ between them is 0. This produces a directional derivative that has the steepest slope when the gradient of f points in the direction of steepest ascent of the surface and is the slope in that direction. This indicates the direction of steepest ascent or the fastest, or greatest increase of f when ∇f is alligned with $\vec{u}$.

Notice: $\vec{\nabla} f(a,b) \perp$ through $P(a,b)$ & points where f is increasing

Algebraically

$$\Rightarrow D_{\vec{u}} f(a,b) = \|\vec{\nabla} f(a,b)\|_2 \cdot \|\vec{u}\|_2 \cdot \cos(0)$$

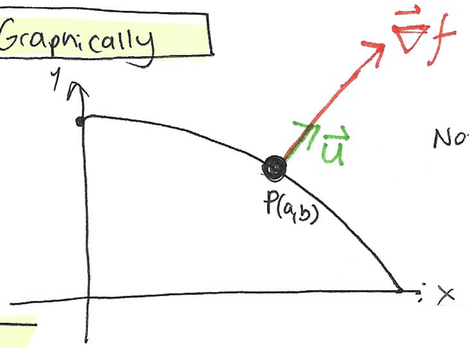
$$= \|\vec{\nabla} f(a,b)\|_2 \cdot 1 \cdot 1$$

$$\Rightarrow \underbrace{D_{\vec{u}} f(a,b) = \|\vec{\nabla} f(a,b)\|_2}$$

The largest value of $D_{\vec{u}} f(a,b)$ is produced by $\vec{\nabla} f$.

!!

Graphically



Notice: $\vec{u} \perp$ to level curve

B) the direction of no change

When $\theta = \frac{\pi}{2}$, there is no change on of direction on the surface. The directional derivative will be zero, meaning the slope of the line at $P(a,b)$ will be zero & a horizontal tangent line.

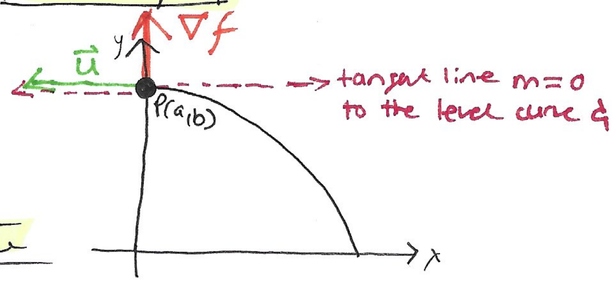
Algebraically

$$\Rightarrow D_{\vec{u}} f(a,b) = \|\vec{\nabla} f(a,b)\|_2 \cdot \underbrace{\|\vec{u}\|_2}_{1} \cdot \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0}$$

$$\Rightarrow D_{\vec{u}} f(a,b) = 0$$

the rate of change is zero!

Graphically



(C) the direction of steepest descent on the surface

Here, the directional derivative portrays the greatest rate of decrease of f or the smallest value of $D_{\vec{u}} f(a,b)$ when $\vec{\nabla} f$ & $\vec{u}$ are antiparallel when $\theta = \pi$. When $\cos(\pi) = -1$, the unit vector points in the opposite direction of the gradient of f, since the gradient always points towards the direction of greatest increase. The $D_{\vec{u}} f(a,b)$ is minimum and this is the direction of steepest descent or fastest decrease of f when $\vec{\nabla} f$ & $\vec{u}$ oppose directions.

Algebraically

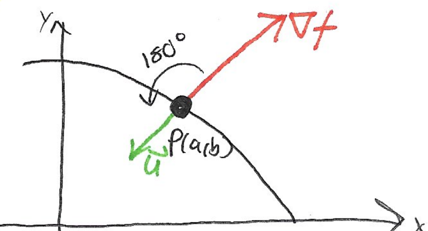
$$\Rightarrow D_{\vec{u}} f(a,b) = \|\vec{\nabla} f(a,b)\|_2 \cdot \underbrace{\|\vec{u}\|_2} \cdot \cos(\pi)$$

$$= \|\nabla f(a,b)\|_2 \cdot 1 \cdot -1$$

$$D_{\vec{u}} f(a,b) = -\|\vec{\nabla} f(a,b)\|_2$$

Graphically

Notice: $\vec{u} \perp$ to level curve



For problems 5 - 6, let $f(x, y) = 15 - x^2 - 4y^2 + 2x - 40y$.

5. (8 points) Find a vector-valued equation for the tangent line to the level curve

$$L_{100}(f) = \{(x, y) \,:\, f(x, y) = 100\}$$

at the point $(-3, -5)$.

Problem #5

Given $L_{100}(f)=\{(x,y): f(x,y)=100\}$ at the point $(-3,-5)$, we will create a vector-valued eqn for the tangent line to this level curve.

Notice $f(x,y)=15-x^2-4y^2+2x-40y$ is not in a easily graphable format. We can format this function into an ellipse by completing the square

$$\Rightarrow 15-x^2-4y^2+2x-40y=100$$
$$\underline{-15 \qquad\qquad\qquad\qquad -15}$$

$$\Rightarrow -x^2-4y^2+2x-40y=85$$

$$\Rightarrow \frac{-(x^2-2x)}{4}-\frac{4(y^2-40y)}{4}=\frac{85}{4}$$

$$\Rightarrow -\frac{1}{4}(x^2-2x+1)-1(y^2+10y)=\frac{85}{4}-\frac{1}{4}$$

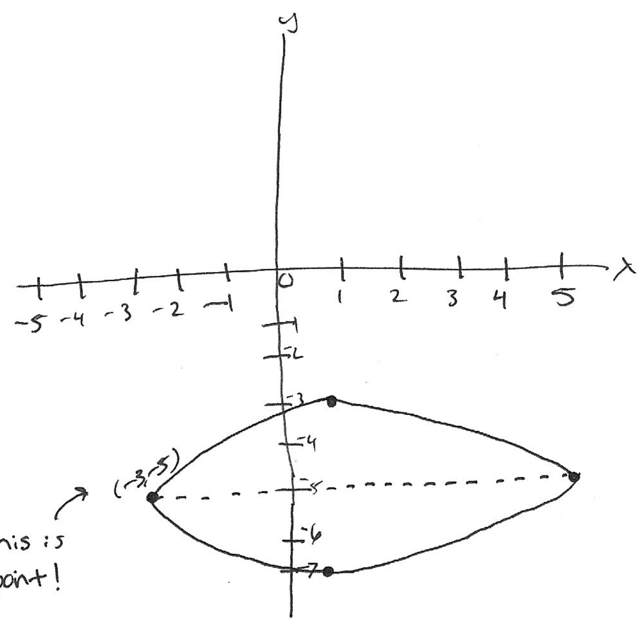
$$\Rightarrow -\frac{1}{4}(x-1)^2-1(y^2+10y)=\frac{85}{4}-\frac{1}{4}$$

$$\Rightarrow \frac{-\frac{1}{4}(x-1)^2}{4}-\frac{1(y^2+10y+25)}{4}=\frac{\frac{85}{4}-\frac{1}{4}-25}{4}$$

$$\Rightarrow \frac{-1(x-1)^2}{-16}+\frac{(y+5)^2}{4}=1$$

$$\Rightarrow \frac{(x-1)^2}{16}+\frac{(y+5)^2}{4}=4$$

$$\Rightarrow \frac{(x-1)^2}{4^2}+\frac{(y+5)^2}{2^2}=1$$



Notice, this is our given point!

// This is an equation for an ellipse where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a = length of x-semi axis

b = length of y-semi axis

where $P(h,k)$ is $(1,-5)$, $a=4$, $b=2$.

Next, we can create the vector-valued equation of the tangent line using $\vec{r}(t)=\vec{r}_0+t\cdot\vec{v}$ where $\vec{r}_0=\langle -3,-5\rangle$, t is our given parameter for the tangent line, and we use implicit differentiation to solve for $\vec{v}$, the direction of the line.

$$\Rightarrow 15-x^2-4y^2+2x-40y=100$$

$$\Rightarrow \frac{d}{dx}[15-x^2-4y^2+2x-40y]=\cancel{\frac{d}{dx}[100]} \nearrow 0$$

$$\Rightarrow 0-2x-8y\cdot y'+2-40\cdot y'=0$$

$$\Rightarrow -2x-8yy'+2-40y'=0$$
$$\underline{+2x \qquad\qquad -2 \qquad\qquad}$$

$$\Rightarrow -8yy'-40y'=2x-2$$

$$\Rightarrow y'(-8y-40)=2x-2$$

$$\Rightarrow y'=\frac{2x-2}{-8y-40} \quad \text{or} \quad \frac{dy}{dx}=\frac{2x-2}{-8y-40}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(-3,-5)}=\langle 0,-8\rangle$$

Sidework:

$$\left.\frac{dy}{dx}=\frac{2x-2}{-8y-40}\right|_{(-3,-5)}$$

$$=\frac{2(-3)-2}{-8(-5)-40}$$

$$=\frac{-8}{0} \quad \begin{array}{l}\text{for "y"}\\ \text{for "x"}\end{array}$$

$$\left.\frac{dy}{dx}\right|_{(-3,-5)}=\langle 0,-8\rangle$$

Using this calculation, he construct the vector-valued equation.

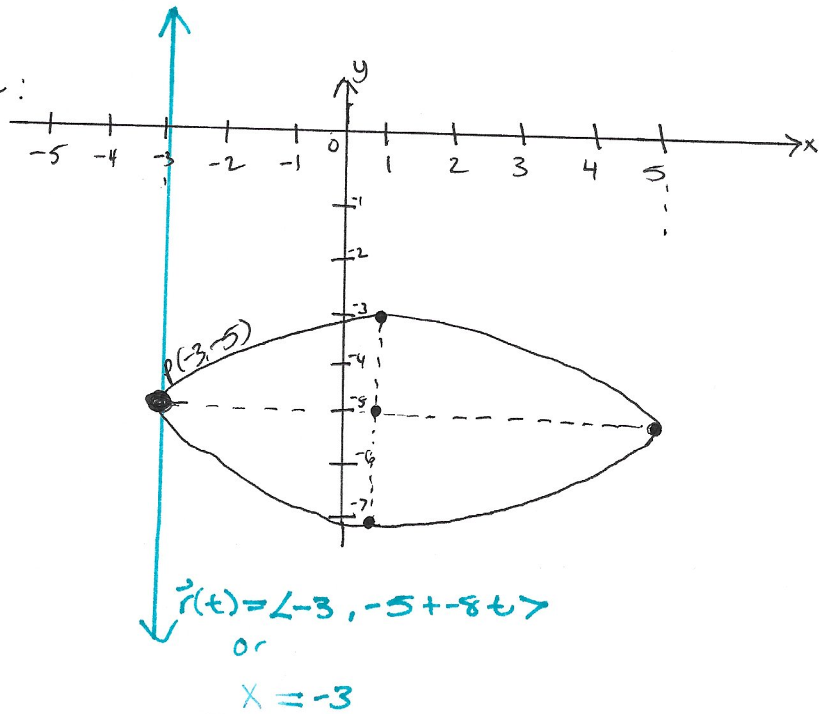
$\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}$

$\vec{r}(t) = \langle -3, -5 \rangle + t \cdot \langle 0, -8 \rangle$

$= \langle -3 + 0t, -5 + -8t \rangle$

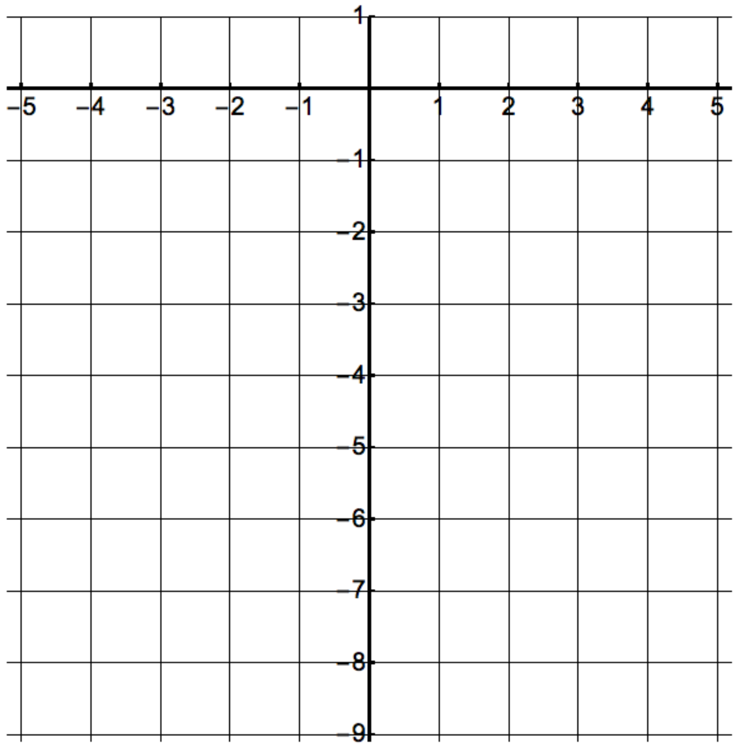
$= \langle -3, -5 + -8t \rangle$

Graph of Level curve with tangent line:



6. (6 points) On the axes below, sketch the level curve $L_{100}(f)$ and it's the tangent line from problem 5 above. Also, sketch the vector $\mathbf{u} \in \mathbb{R}^2$ with tail at point $(-3, -5)$ where $\mathbf{u}$ is the unit vector in the direction of the gradient vector $\nabla f(-3, -5)$ given by

$$\mathbf{u} = \frac{\nabla f(-3,-5)}{\|\nabla f(-3,-5)\|_2}$$



Now, use full sentences to explain how your graph above relates your knowledge about the shape of the surface $f(x, y)$ and your solution to problem 6 above.

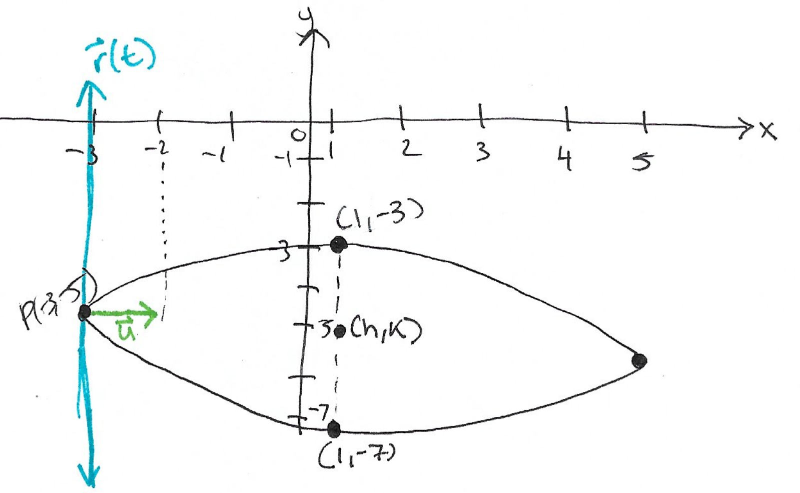
Problem # 6 -

$\vec{u} = \dfrac{\nabla f(-3,-5)}{\|\vec{\nabla} f(-3,-5)\|}$ where $\langle 2x-2, -8y-40 \rangle$

$\Rightarrow \nabla f(-3,-5)$
$= \langle 2x-2, -8y-40 \rangle |_{(-3,-5)}$
$= \langle 2(-3)-2, -8(-5)-40 \rangle$
$= \langle -8, 0 \rangle$

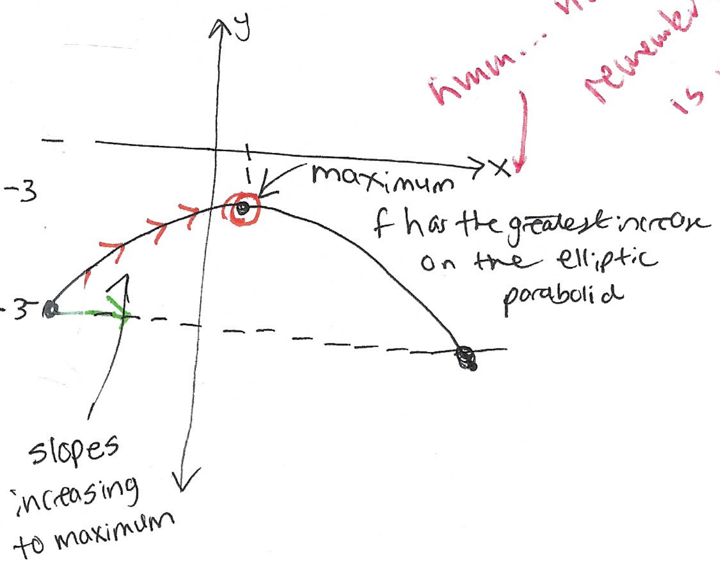
where $\|\vec{\nabla} f(-3,-5)\|_2 = \sqrt{(-8)^2 + (0)^2}$
$= \sqrt{64}$
$= 8$

$\Rightarrow \vec{u} = \dfrac{\vec{\nabla} f(-3,-5)}{\|\vec{\nabla} f(-3,-5)\|_2} = \dfrac{\langle -8, 0 \rangle}{\langle -8, 0 \rangle} = \langle 1, 0 \rangle$



$\vec{u} = \langle 1, 0 \rangle$
$\vec{r}(t) = \langle -3, -5 + -8t \rangle$

hmm... not quite. remember image is in $\mathbb{R}^3$.



Explaination of Surface $f(x,y)$

// The surface is an elliptic parabolid. The gradient, $\vec{\nabla} f$, points in the direction of steepest ascent. The unit vector points inward in the positive direction with a magnitude of 1.

The gradient is a vector in a certain direction on the surface $f(x,y)$ and the unit vector is in any direction, generally. From the unique given point $(-3,-5)$, we can further understand how the graph of f changes. From the gradient and the unit vector, we can understand the change of f as (x,y) changes in the direction of $\vec{u}$ from the given point $(-3,-5)$. Then we can infer the following from $\vec{u} = \dfrac{\nabla f(-3,-5)}{\|\nabla f(-3,-5)\|_2}$, when $\theta = 0$,

$\cos\theta = 1$, so the directional derivative of f is maximized and its value is $\|\vec{\nabla} f(-3,-5)\|$ and that is the direction of steepest ascent and where f has the greatest increase when $\vec{\nabla} f(-3,-5)$ and the unit vector point in the same direction. The directional derivative ~~hit~~ has the steepest slope and f has the greatest increase.