Name : ____________________ © Jeffrey A. Anderson **Class Number**: ______

Sample Exam 2, Version 2A

Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 6 sheets of paper (12 pages front and back) including this cover page.
- There are a total of 24 questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 15 Multiple Choice Questions (60 points)
  - 3 Free-Response Questions (30 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger then 8.5 inches by 11 inches. You may write on both sides of this note card. Your notecard must be handwritten.
- PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T F If A is a 3×3 matrix with three pivot positions, then for some $p \in \mathbb{N}$ there exist elementary matrices $E_1, E_2, ..., E_p \in \mathbb{R}^3$ such that $E_p \cdots E_2 \cdot E_1 \cdot A = I_3$.

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2. T F Let $m, n \in \mathbb{N}$ with $m > n$ and suppose $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are given. Then, a least-squares solution corresponding to matrix A and vector $\mathbf{b}$ is a vector $\mathbf{x}^* \in \mathbb{R}^n$ such that $A\mathbf{x}^* \in \text{Col}(A)$ and

$$\|\mathbf{b} - A\mathbf{x}^*\|_2 \leq \|\mathbf{b} - A\mathbf{x}\|_2$$

for all $\mathbf{x} \in \mathbb{R}^n$.

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3. T F Let $A \in \mathbb{R}^{m \times n}$ and suppose $\mathbf{x} \in \mathbb{R}^n$. Suppose $\mathbf{b} \in \mathbb{R}^m$ is nonzero. Suppose $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ are solutions to the inhomogeneous system $A\mathbf{x} = \mathbf{b}$. Then any linear combination $c_1\mathbf{x}_1^* + c_2\mathbf{x}_2^*$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

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4. T F Consider the linear systems problem

$$A\mathbf{x} = \mathbf{b}$$

where matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$ are given and vector $\mathbf{x} \in \mathbb{R}^n$ is unknown and desired. If this linear system is inconsistent, there may be an $\mathbf{x} \in \mathbb{R}^n$ such that

$$\|\mathbf{b} - A\mathbf{x}\|_2 = 0.$$

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5. T F If $A \in \mathbb{R}^{3 \times 3}$, then $\det(5\,A) = 5\det(A)$

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

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6. Below we are given the LU Factorization of the matrix A and vector $\mathbf{b}$:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -4 & 2 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Use this LU Factorization to solve the linear system $A\mathbf{x} = \mathbf{b}$ by solving the two linear systems

$$L\mathbf{y} = \mathbf{b} \qquad \text{and} \qquad U\mathbf{x} = \mathbf{y}$$

Then find $\mathbf{y}^T\mathbf{x}$:

A. -22 B. 22 C. -16 D. -18 E. 18

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7. Let $A \in \mathbb{R}^{m \times n}$ and set $f(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$. Which of the following is not equivalent to $\text{Rng}(f)$:

A. $\text{Col}(A)$ B. $\left(\text{Nul}\left(A^T\right)\right)^{\perp}$ C. $\text{span}\{A(:,k)\}_{k=1}^{n}$

D. $\text{span}\{A(i,:)\}_{i=1}^{m}$ E. $\{\mathbf{b} \in \mathbb{R}^m \,:\, \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$

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8. Consider the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Which of the following statements is true:

A. The columns of A are linearly dependent.
B. $\det(A) = -1$
C. $\dim\left(\text{Nul}(A^T)\right) = 1$.
D. The matrix is not invertible.
E. $\left(A^T A\right)^{-1}$ exists

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9. Let $A \in \mathbb{R}^{4 \times 3}$ be defined by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & -4 & 3h \\ 1 & 3 & h+5 \\ 3 & 7 & 4 \end{bmatrix}$$

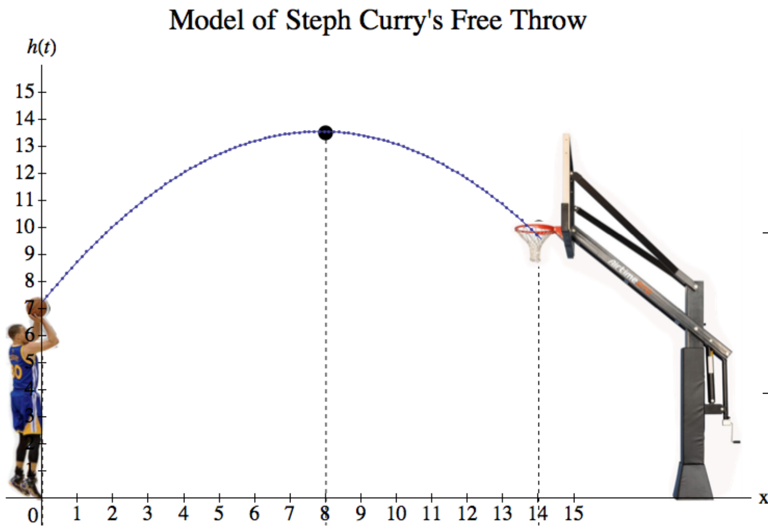
For what value(s) of h is $\dim(\text{Nul}(A)) = 1$:

A. $\text{rank}(A) = 3$ for all h. B. $h = -3$ C. $h = -1$ D. $h = -1$ and $h = 1$ E. $h = 3$

10. Let $M = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$. Then M^{-1} is given by which of the following:

A. $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

11. Stephen Curry plays for the Golden State Warriors as a professional basket ball player in the National Basketball Association (NBA). Suppose we analyze Curry's shooting technique. Below we see a diagram that highlights the typical trajectory of one of Mr. Curry's free-throw shots.



Distance x from release position (ft)	Height h of basketball above ground (ft)
0	7.25
8	13.50
14	10.00

Our data points for this shooting trajectory $\{(x_i, h_i)\}_{i=1}^{3}$ represent the observed height h_i of our basketball when the ball has moved x_i feet in the horizontal direction for $i = 1, 2, 3$. From our study of introductory physics, we choose to model the trajectory of the basket ball using the function

$$h(x) = a_0 + a_1 x + a_2 x^2.$$

Find the corresponding linear-systems problem to produce our desired polynomial model.

A. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 8 & 64 \\ 1 & 14 & 196 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.25 \\ 13.50 \\ 10.00 \end{bmatrix}$ B. $\begin{bmatrix} 0 \\ 8 \\ 14 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.25 \\ 13.50 \\ 10.00 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 7.25 & 52.5625 \\ 1 & 13.50 & 182.2500 \\ 1 & 10.00 & 100.0000 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 14 \end{bmatrix}$

D. $\begin{bmatrix} a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 7.25 \\ 13.50 \\ 10.00 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 7.25 \\ 1 & 8 & 13.50 \\ 1 & 14 & 10.00 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

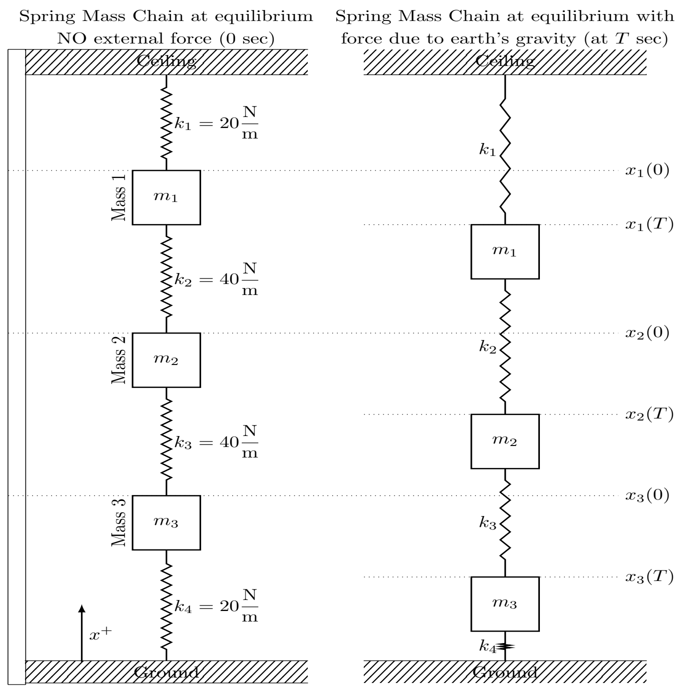
12. Solve the linear systems problem from Problem 11 above using any method. With your solution, determine how high the ball was in the air after traveling $x = 5$ feet in the horizontal direction. In other words, approximate $h(5)$ using the solution of your linear systems problem. Round your answer to the nearest tenth (round to the nearest one digit to the RIGHT of the decimal place).

A. -23.7 ft B. 0 ft C. 12.6 ft D. 12.9 ft E. 11.9 ft

13. Suppose $A \in \mathbb{R}^{n \times n}$. Which of the following guarantees that $\text{Nul}(A) \neq \{\mathbf{0}\}$?

A. $\text{rank}(A) = n$ B. $\det(A) \neq 0$ C. $\text{Col}(A) = \mathbb{R}^n$ D. A^{-1} exists E. $\dim\Big(\text{Nul}\left(A^T\right)\Big) > 0$

For the next problem, consider the following spring-mass system



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14. Suppose that you are apply masses 1, 2, and 3 to the mass-spring chain illustrated above such that

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.400 \\ 0.200 \end{bmatrix}$$

measured in kg. Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law. Then, which of the following gives the displacement vector

$$\mathbf{u} = \begin{bmatrix} u_1(T) \\ u_2(T) \\ u_3(T) \end{bmatrix}$$

measured in meters at $t = T$ when the system is at equilibrium under the force of gravity on earth.

A. $\begin{bmatrix} -0.196 \\ -0.147 \\ -0.196 \end{bmatrix}$ B. $\begin{bmatrix} 0.098 \\ 0.098 \\ 0.049 \end{bmatrix}$ C. $\begin{bmatrix} 0.196 \\ 0.245 \\ 0.196 \end{bmatrix}$ D. $\begin{bmatrix} 0.020 \\ 0.025 \\ 0.020 \end{bmatrix}$ E. $\begin{bmatrix} 0.098 \\ 0.098 \\ 0.098 \end{bmatrix}$

15. Let $A \in \mathbb{R}^{5\times 5}$ with $\det(A) = -4$. Suppose

$$S_{14}(4) \cdot P_{14} \cdot D_3(1/8) \cdot P_{23} \cdot D_3(4) \cdot P_{12} \cdot B = A$$

where we use standard notation for elementary matrices as discussed in class. Then $\det(B)$ is

A. 0 B. 2 C. -2 D. 8 E. -8

For Problems 16 - 18, assume that the matrix $A \in \mathbb{R}^{4\times 7}$ is given by

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 0 & 0 & 1 \\ 2 & 6 & -5 & -2 & -1 & -2 & 1 \\ 0 & 0 & 5 & 10 & 5 & 10 & 5 \\ 2 & 6 & 0 & 8 & 4 & 12 & 8 \end{bmatrix}$$

16. Find RREF(A):

A. $\begin{bmatrix} 1 & 3 & -2.5 & -1 & -.5 & -1 & 0.5 \\ 0 & 0 & 1 & 2 & 1 & 2 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 1.0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0.0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -3 & 0 & 4 & 2 & 0 & -1.0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0.0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 & 3 & 4 & 2 & 1.0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0.0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

17. How many linearly independent solutions are there to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$:

A. 1 B. 3 C. 4 D. 5 E. 7

18. Which of the following is NOT a solution for the linear-systems problem $A\mathbf{x} = \mathbf{0}$?

A. $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} 4 \\ 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ D. $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$ E. $\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.5 \\ 1.0 \end{bmatrix}$

19. Below is a data set for of the gasoline usage from 1/1/2015 at 12am to 3/31/2015 at 11:59pm:

Month i in 2015 (by Number)	Cumulative Gallons g_i of Gas used at end of Month	Odometer reading r_i at end of month (in miles)
0	0.000	86,286.2
1	41.490	87,240.3
2	98.804	88,500.6
3	143.622	89,432.7

We will model this data using a piecewise function

$$R(g) = \begin{cases} R_1(g) & \text{if} \quad 0.000 < \quad g \quad \leq 41.490 \\ R_2(g) & \text{if} \quad 41.490 < \quad g \quad \leq 98.804 \\ R_3(g) & \text{if} \quad 98.804 < \quad g \quad \leq 143.622 \end{cases}$$

where output R gives the odometer reading of the car (in miles) as a function of the number of gallons g used during the year. Each segment of the piecewise linear function given by

$$R_i(g) = m_i(g - g_i) + r_i, \qquad\qquad \text{with } R(g_{i+1}) = r_{i+1}$$

Set up a 3×3 linear-systems problem $G\,\mathbf{m} = \mathbf{r}$ to find the average miles per gallon efficiency of this car during this time period. The diagonal matrix $G \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{r} \in \mathbb{R}^3$ are constructed from our data set. The vector $\mathbf{m}$ stores the unknown slopes m_i of the three line segments connecting these data points. Remember, for all $i = 0, 1, 2$, we have

$$(r_{i+1} - r_i) = m_i \cdot (g_{i+1} - g_i)$$

Which of the following gives the linear-systems problem associated with this model?

A. $\begin{bmatrix} 41.490 & 0 & 0 \\ 0 & 98.804 & 0 \\ 0 & 0 & 143.622 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 87,240.3 \\ 88,500.6 \\ 89,432.7 \end{bmatrix}$

B. $\begin{bmatrix} 87,240.3 & 0 & 0 \\ 0 & 88,500.6 & 0 \\ 0 & 0 & 89,432.7 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 41.490 \\ 98.804 \\ 143.622 \end{bmatrix}$

C. $\begin{bmatrix} 954.1 & 0 & 0 \\ 0 & 1260.3 & 0 \\ 0 & 0 & 932.1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 41.490 \\ 57.314 \\ 44.818 \end{bmatrix}$

D. $\begin{bmatrix} 41.490 & 0 & 0 \\ 0 & 57.314 & 0 \\ 0 & 0 & 44.818 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 954.1 \\ 1260.3 \\ 932.1 \end{bmatrix}$

20. Find average miles per gallon (mpg) efficiency of this car during month 2 of this data set. Round your answer to the integer (one digit to the left of the decimal place).

A. 21 B. 23 C. 896 D. 0 E. 22

Free Response

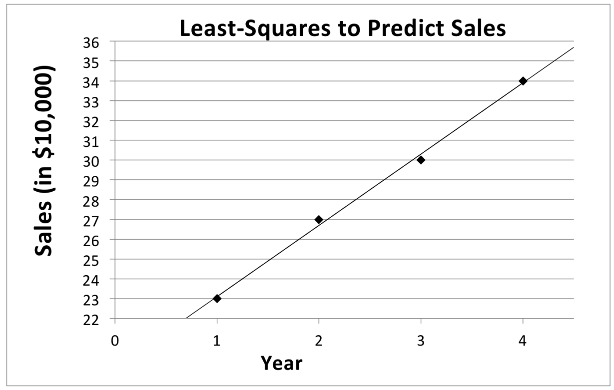
21. Let $n, i, k \in \mathbb{N}$ such that $1 \leq i \leq n$, $1 \leq k \leq n$ and $i \neq k$. Suppose that $c \in \mathbb{R}$.

    (a) Show $\Big(S_{ik}(c)\Big)^{-1} = S_{ik}\left(-c\right)$

    (b) Show $\Big(D_i(c)\Big)^{-1} = D_i\left(\frac{1}{c}\right)$

22. (10 pts) A small bike company selling utility bicycles for daily commuting has been in business for four years. This company has recorded annual sales (in tens of thousands of dollars) as follows:

Year	Sales (in \$10,000)
1	23
2	27
3	30
4	34



This data is plotted in the figure next to the table above. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

a. (4 points) Set up the least squares problem to fit this data to a linear model.

b. (4 points) Solve the least squares problem using the normal equations to find the line of best fit.

c. (2 points) Use your linear model to estimate the sales for year five. Your solution should be include units and be in a full sentence.

23. Recall that Cramer's formula for the inverse of a 2×2 matrix $A \in \mathbb{R}^2$ is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

(a) (5 pts) Using a sequence of elementary matrices, transform A into I_2. Show each matrix you use.

(b) (5 pts) Write A^{-1} as a product of the elementary matrices and confirm Cramer's Rule.

Challenge Problem

24. (10 pts) (Optional, Extra Credit, Challenge Problem)

Assume $A \in \mathbb{R}^{n\times n}$ is an invertible matrix. Suppose we transform A into upper triangle form by multiplying A on the left-hand side by a sequence of unit lower-triangular matrices $L_1, L_2, ..., L_{n-1} \in \mathbb{R}^{n\times n}$. In particular, assume that $L_{n-1} \cdot L_{n-2} \cdots L_2 \cdot L_1 \cdot A = U$ where $U \in \mathbb{R}^n$ is upper-triangular with non-zero diagonal elements. Moreover, assume we can write each unit lower-triangular matrix L_k as a Gauss transformation given by

$$L_k = I_n - \boldsymbol{\ell}_k \, \mathbf{e}_k^T, \qquad \text{where } \boldsymbol{\ell}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \ell_{k+1,k} \\ \vdots \\ \ell_{n,k} \end{bmatrix} \in \mathbb{R}^n$$

Prove that the factor L from the LU factorization of A is given by

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix}$$

Use for Scratch Work