Name : _____ © Jeffrey A. Anderson **Class Number**: _____

Sample Exam 2, Version 1A
Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 5 sheets of paper (10 pages front and back) including this cover page.
- There are a total of 25 questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 15 Multiple Choice Questions (60 points)
  - 3 Free-Response Questions (30 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger then 8.5 inches by 11 inches. You may write on both sides of this note card. Your notecard must be handwritten.
- PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T F If $A \in \mathbb{R}^{3\times 3}$ has three pivot columns, then it is possible to find invertible matrices $E_1, E_2, ..., E_p \in \mathbb{R}^{3\times 3}$ such that

$$E_p E_{p-1} \cdots E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. T F Suppose $A \in \mathbb{R}^{m\times n}$ and $B = \text{RREF}(A)$. Then $\text{Col}(A) = \text{Col}(B)$.

3. T F Suppose $A \in \mathbb{R}^{m\times n}$ with $m, n \in \mathbb{N}$ and $m \neq n$. The spaces $\text{Nul}(A)$ and $\text{Col}(A)$ never share an element in common.

4. T F If $A \in \mathbb{R}^{m\times n}$, then $\text{Col}(A) = \mathbb{R}^m$ if and only if $\text{rank}(A) = m$

5. T F Suppose $A, B \in \mathbb{R}^{2\times 2}$. If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form. Problems that are marked "choose all that apply" may have more than one correct answer. In this case, mark all correct answers.

6. Suppose we are given a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & -2 \\ 2 & -2 & 0 \end{bmatrix}$.

   Find the matrix $L \in \mathbb{R}^{3\times 3}$ from the LU factorization of A.

   A. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

7. Suppose that $U \in \mathbb{R}^{3\times 3}$ is the upper triangular matrix from the LU factorization of matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

   in problem 6 above. What do you know about the product of the diagonal elements of U given by $u_{11}u_{22}u_{33}$? Choose all that apply.

   A. $\det(A) = u_{11}u_{22}u_{33}$ B. $u_{11}u_{22}u_{33} = 0$ C. $u_{11}u_{22}u_{33} = 1$

   D. $u_{11}u_{22}u_{33} = -30$ E. $u_{11}u_{22}u_{33} = 30$

8. The LU Factorization of a given 3×3 matrix is $A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4.5 \end{bmatrix}}_{U}$.

   Use the LU factorization of A combined with forward and backward substitution to find the solution to the linear systems problem

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -9 \end{bmatrix}$$

   Which of the following gives $-x_1 + x_2 + x_3$?

   A. -2 B. -1 C. 0 D. 1 E. 2

9. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Which of the following are false? Choose all that apply.

   A. $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. B. $\det(A) = -1$ C. $\text{Nul}\left(A^T\right) \neq \emptyset$ D. $\text{rank}(A) = 2$ E. $\text{Col}\,(A) = \mathbb{R}^2$

10. Let $n \in \mathbb{N}$. Recall that the set of polynomials of degree less than or equal to n is denoted as

$$P_n = \{p(x) : p(x) = a_0 + a_1 x + \cdots + a_n x^n \text{ with } a_i \in \mathbb{R} \text{ for all } i = 0, 1, ..., n\}$$

Which of the following is false?

A. $P_1 \subseteq P_2$

B. P_k is a subspace of P_n for all $0 \leq k \leq n$

C. P_n is a vectors space.

D. The set of constant functions $\{p(x) : p(x) = c \text{ for } c \in \mathbb{R}\}$ is a subspace of P_n

E. The set of linear polynomials with nonzero slope is a subspace of P_n

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11. Let $A = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & -t \end{bmatrix}$. Find the set of values for which $\text{Nul}(A) \neq \{\mathbf{0}\}$:

A. $t = 3$ B. $t = -3$ C. $t = 3$ or $t = -3$ D. $t \neq 3$ E. $t \neq -3$

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12. Suppose $A \in \mathbb{R}^{m \times n}$. Given a nonzero vector $\mathbf{b} \in \mathbb{R}^m$, suppose that you know:

I. Vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$ solve the linear system problem $A\mathbf{x} = \mathbf{0}$

II. Vectors $\mathbf{x}^*, \mathbf{y}^* \in \mathbb{R}^n$ solve the linear system problem $A\mathbf{x} = \mathbf{b}$.

Which of the following is NOT a solution for the linear system problem $A\mathbf{x} = \mathbf{b}$?

A. $\mathbf{x}^* + \mathbf{z}_1$ B. $\mathbf{y}^* + \mathbf{z}_2$ C. $\mathbf{x}^* + \mathbf{y}^*$ D. $3\mathbf{z}_1 + \mathbf{x}^* - 4\mathbf{z}_2$ E. $2\mathbf{x}^* - \mathbf{y}^*$

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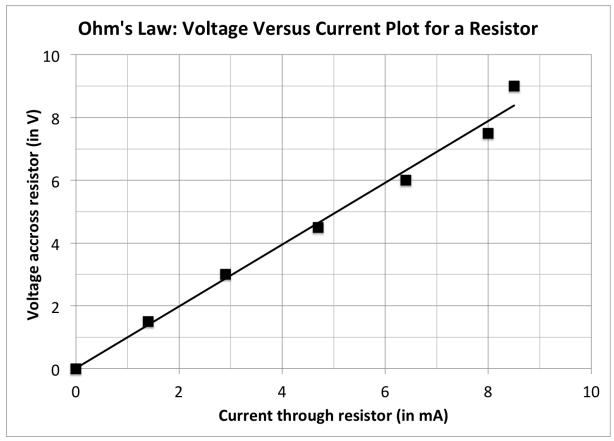
13. Let $B = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 1 \end{bmatrix}}^{E_3} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix}}^{E_2} \cdot \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}^{E_1}$ Find B^{-1}:

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -5 & 7 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -1 & 5 & -7 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$ E. $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

14. Ohm's Law, governing the electrical behavior of resistors, states that the voltage access a resistor depends linearly on the current running through the resistor. Below is some current and voltage data collected from an experiment:

Current through resistor (mA)	Volts across resistor (V)
0.0	0.0
1.4	1.5
2.9	3.0
4.7	4.5
6.4	6.0
8.0	7.5
8.5	9.0



We can model this relationship using the linear function $v(i) = r \cdot i + b$ where the positive constant r measures the resistance value of resistor ($k\Omega$), i represents the current through the resistor (mA), v measures voltage across resistor (V), and b is the intercept of this model with the vertical axis. Solve the least-square problem associated with this model and identify the line of best fit below:

A. $v = 1.0048 \cdot i + 0.0357$ B. $v = 0.0135 \cdot i + 0.9845$ C. $v = 0.9845 \cdot i + 0.0135$

D. $v = 0.0357 \cdot i + 1.0048$ E. $v = 0.9827 \cdot i + 0.0253$

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15. Suppose $\mathbf{y}, \mathbf{b} \in \mathbb{R}^3$ are given by

$$\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $Y = \text{Span}\{\mathbf{y}\}$ and let $Y^\perp = \Big(\text{Span}\{\mathbf{y}\}\Big)^\perp$. Suppose that

$$\alpha\,\mathbf{y} = \text{Proj}_Y(\mathbf{b}) = \text{ the projection of } \mathbf{b} \text{ onto } Y,$$
$$\mathbf{r} = \text{Proj}_{Y^\perp}(\mathbf{b}) = \text{ the projection of } \mathbf{b} \text{ onto } Y^\perp$$

Which of the following statements are true? Choose all that apply.

A. $\alpha\,\mathbf{y} = \begin{bmatrix} 0.8 \\ -0.4 \\ 0 \end{bmatrix}$ B. $\alpha\,\mathbf{y} = \begin{bmatrix} -1.0 \\ 0.0 \\ -1.0 \end{bmatrix}$ C. $\mathbf{r} = \begin{bmatrix} 0.2 \\ 0.4 \\ 1.0 \end{bmatrix}$ D. $\mathbf{r} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ E. $\mathbf{y}^T\mathbf{r} = \mathbf{0}$

For Problems 16 - 20, assume that the matrix $A \in \mathbb{R}^{4\times 6}$ is given by

$$A = \begin{bmatrix} 1 & 2 & -5 & -2 & 6 & 14 \\ 0 & 0 & -2 & -2 & 7 & 12 \\ 2 & 4 & -5 & 1 & -5 & -1 \\ 0 & 0 & 4 & 4 & -14 & -24 \end{bmatrix}$$

16. Find RREF(A):

A. $\begin{bmatrix} 1 & 2 & -5 & -2 & 6 & 14 \\ 2 & 4 & -5 & 1 & -5 & -1 \\ 0 & 0 & -2 & -2 & 7 & 12 \\ 0 & 0 & 4 & 4 & -14 & -24 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 7 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 2 & -2.5 & 0.5 & -2.5 & -0.5 \\ 0 & 0 & 1 & 1 & -3.5 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

17. Which of the following vectors in NOT a solution to $A\mathbf{x} = \mathbf{0}$?

A. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 7 \\ 0 \\ 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ D. $\begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \\ 3 \\ -3 \\ 0 \\ 0 \end{bmatrix}$ E. $\begin{bmatrix} 6 \\ 0 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ -1 \\ 0 \\ -2 \\ -1 \end{bmatrix}$

18. Which of the following sets of vectors are linearly dependent? Choose all that apply.

A. $\{A(:,1), A(:,3), A(:5)\}$ B. $\{A(:,2), A(:,3), A(:6)\}$ C. $\{A(:,1), A(:,3), A(:4)\}$

D. $\{A(:,1), A(:,4), A(:5)\}$ E. $\{A(:,2), A(:,4), A(:6)\}$

19. Find $\dim\big(\text{Nul}(A)\big) + \dim\big(\text{Nul}(A^T)\big)$:

A. 1 B. 2 C. 3 D. 4 E. 5

20. Which of the following must be true? Choose all that apply.

A. $\text{rank}\left(A^T\right) = 3$ B. $\text{Col}\left(A^T\right) \subseteq \mathbb{R}^6$ C. $\left(AA^T\right)^{-1}$ exists

D. $\left(A^TA\right)^{-1}$ exists E. $\text{Col}(A) = \mathbb{R}^3$

Free Response

21. (a) (4 pts) Suppose we are given vectors $\mathbf{y}, \mathbf{b} \in \mathbb{R}^m$. Let $Y = \text{Span}\{\mathbf{y}\}$ and $Y^\perp = \Big(\text{Span}\{\mathbf{y}\}\Big)^\perp$.

Show how to construct the projections

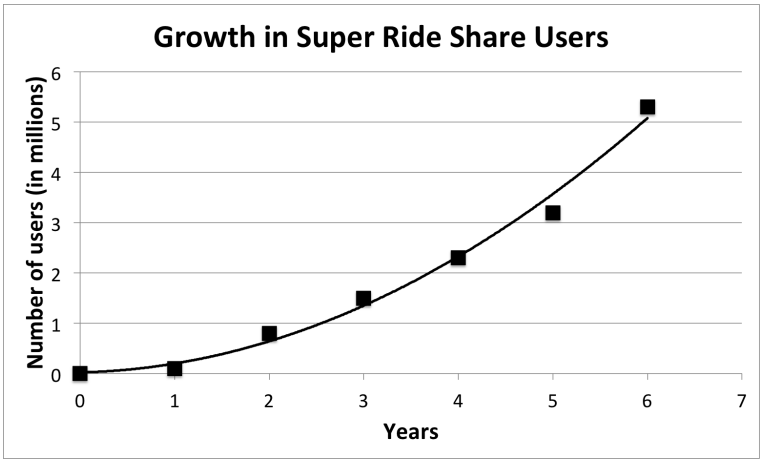
$$\alpha\,\mathbf{y} = \text{Proj}_Y(\mathbf{b}) \qquad \text{and} \qquad \mathbf{r} = \text{Proj}_{Y^\perp}(\mathbf{b})$$

by finding an explicit formula for the scalar α. Explain your assumptions and draw a diagram to support your work.

(b) (6 pts) Use classical Gram-Schmidt to find an orthogonal basis for $\text{Col}(A)$ for $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

22. (10 pts) A ride-sharing app called Super was recently released in 2009. During the first six years of it's business operations, the Super app has seen spectacular growth in its number of users:

Year	Number of Users (in Millions)
0	0.0
1	0.1
2	0.8
3	1.5
4	2.3
5	3.2
6	5.3



We can model this data using a quadratic function $N(t) = a_0 + a_1 t + a_2 t^2$ for unknown positive constant a_0, a_1, a_2. Recall from class that we can set up least-squares problem create this model.

(a) Set up the least-squares model associated with this data set. Explicitly identify the Vandermonde matrix A and the right-hand side vector $\mathbf{b}$.

(b) Solve this least-squares problem to produce the best-fit quadratic function. If you use a calculator, be sure to explain the process you used to find your answer.

23. Recall that for $A \in \mathbb{R}^{3\times 3}$, the determinant of A was given by

$$\det(A) = \sum_{\pi \in S_3} \operatorname{sgn}(\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\, a_{\pi(3),3}$$

(a) (5 pts) List all permutations $\pi \in S_3$. In other words, list all maps $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ that are one-to-one and onto.

(b) (5 pts) Use your work in part (a) and the determinant formula given above to prove that that the determinant of an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ is the product of the main diagonal elements.

Challenge Problem

24. (10 pts) (Optional, Extra Credit, Challenge Problem)

Assume $A \in \mathbb{R}^{n \times n}$ is an invertible matrix. Find the total number of operations required to:

(a) Transform A into A^{-1} using elementary row operations.

(b) Find the LU factorization of A.