True/False (10 points: 2 points each) For the problems below, circle $T$ if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. (T) F Let $A \in \mathbb{R}^{m \times n}$. If we define the function $f(\mathbf{x})=A \mathbf{x}$, then the codomain of this function is $\mathbb{R}^{m}$
2. T (F) Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^{n}$. Then $A \mathbf{x}=\sum_{k=1}^{n} x_{k}(A(k,:))^{T}$
3. T F A linear combination of vectors is the same thing as the span of these vectors.
4. (T) F We know $\left(S_{32}(-5)\right)^{T}=I_{4}-5 \mathbf{e}_{2} \mathbf{e}_{3}^{T}$, where $S_{i k}(c)$ is a $4 \times 4$ shear matrix.
5. T F For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$, we know $|\mathbf{x} \cdot \mathbf{y}| \geq\|\mathbf{x}\|_{2}\|\mathbf{y}\|_{2}$

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

Consider the following directed graph. Use this graph to find the correct answer for problem 6.


| Incidence Matrix |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $e_{1}$ | $e_{2}$ | $e_{3}$ | $e_{4}$ | $e_{5}$ | $e_{6}$ |
| $N_{1}$ |  |  |  |  |  |  |
| $N_{2}$ |  |  |  |  |  |  |
| $N_{3}$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $N_{4}$ |  |  |  |  |  |  |

6. Let $A$ represent the $4 \times 6$ incidence matrix. Find $A(:, 2) \cdot A(:, 5)$ :
A. 2
B. 1
C. 0
D. -1
E. -2
7. Let the following $5 \times 7$ matrix be the incidence matrix for a directed graph:

$$
A=\left[\begin{array}{rrrrrrr}
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 & -1 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 1
\end{array}\right]
$$

This matrix corresponds to which of the following directed graphs:

B.

C.

A.
D.

E.

8. Let matrix $P \in \mathbb{R}^{4 \times 5}$ be given as follows:

$$
\left[\begin{array}{lllll}
p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\
p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\
p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\
p_{41} & p_{42} & p_{43} & p_{44} & p_{45}
\end{array}\right]=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{lllll}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45}
\end{array}\right]\left[\begin{array}{lllll}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Using this definition, we see that $a_{12}$ is equal to which of the following:
A. $a_{12}=p_{21}$
B. $a_{12}=p_{24}$
C. $a_{12}=p_{42}$
D. $a_{12}=p_{12}$
E. $a_{12}=p_{25}$
9. Which of the following sets is equivalent to

$$
\operatorname{Span}\left\{\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\}
$$

A. $\mathbb{R}^{2}$
B. $\mathbb{R}^{3}$
C. $\left\{\left[\begin{array}{c}x_{1} \\ x_{2} \\ 0\end{array}\right]: x_{i} \in \mathbb{R}\right.$ for $\left.i=1,2\right\}$
D. $\left\{\left[\begin{array}{l}1 \\ 1 \\ 0\end{array}\right]\right\}$
E. $\left\{\left[\begin{array}{l}x \\ x \\ 0\end{array}\right]: x \in \mathbb{R}\right\}$
10. Let $E \subseteq \mathbb{R} \times \mathbb{R}$ be given by

$$
E=\left\{(x, y): \frac{x^{2}}{1024}+\frac{y^{2}}{729}<1\right\}
$$

Which of the following cannot be true about the relation $E$ ?
A. $\operatorname{Dom}(E)=[-32,32]$
B. $\operatorname{Dom}(E)=(-32,32)$
C. $\operatorname{Rng}(E)=(-27,27)$
D. $E$ is not a function
E. Codomain $(E)=\mathbb{R}$

For problems 11 and 12 , consider the polygons $V$ and $W$.:

BEGIN POLYGON $V$


END POLYGON $W$

11. Which of the following vertex matrices $V$ encodes the begin polygon above? For this model, assume that the $k$ th column of $V$ encodes vertex Vk , for $k \in\{1,2,3,4,5,6\}$ :
A. $\left[\begin{array}{llllll}1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 & 2 & 2 & 1\end{array}\right]$
B. $\left[\begin{array}{llllll}1 & 2 & 2 & 3 & 3 & 1 \\ 4 & 4 & 2 & 2 & 1 & 1\end{array}\right]$
C. $\left[\begin{array}{llllll}4 & 4 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 1\end{array}\right]$
D. $\left[\begin{array}{llllll}1 & 4 & 4 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3\end{array}\right]$
E. $\left[\begin{array}{rrrrrr}-4 & -4 & -2 & -2 & -1 & -1 \\ 1 & 2 & 2 & 3 & 3 & 1\end{array}\right]$
12. As noted above, let $V$ be the vertex matrix that models the begin polygon and $W$ be the vertex matrix that models the end polygon. Which matrix $Q$ below satisfies equation

$$
W=Q V
$$

A. $\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$
B. $\left[\begin{array}{rr}-1 & 0 \\ 0 & -1\end{array}\right]$
C. $\left[\begin{array}{rr}0 & 1 \\ -1 & 0\end{array}\right]$
D. $\left[\begin{array}{rr}0 & -1 \\ -1 & 0\end{array}\right]$
E. $\left[\begin{array}{rr}0 & -1 \\ 1 & 0\end{array}\right]$
13. For sets $A$ and $B$, the statement "If $x \in A$, then $x \in B$ " is written using which of the following?
A. $A \leq B$
B. $B \subseteq A$
C. $A=B$
D. $A \subseteq B$
E. $A \neq B$
14. Consider the following data set that describes the average rent levels for a rental unit with 2 bedrooms and 1 bathroom ( $2 \mathrm{Bd} / 1 \mathrm{Ba}$ ) in Union City, CA.

| Index <br> $i$ | Year <br> $t_{i}$ | Average Rent <br> Level $R$ (in $\$$ ) |
| :---: | :---: | :---: |
| 1 | 2005 | $\$ 1,197$ |
| 2 | 2006 | $\$ 1,252$ |
| 3 | 2007 | $\$ 1,353$ |
| 4 | 2008 | $\$ 1,413$ |
| 5 | 2009 | $\$ 1,275$ |
| 6 | 2010 | $\$ 1,277$ |
| 7 | 2011 | $\$ 1,392$ |
| 8 | 2012 | $\$ 1,490$ |
| 9 | 2013 | $\$ 1,663$ |



From this data, we can model the rent for a $2 \mathrm{Bd} / 1 \mathrm{Ba}$ unit using a linear function in the form

$$
R_{i}=R\left(t_{i}\right)=b+m \cdot t_{i}
$$

where $R_{i}$ is the modeled monthly rent during year $t_{i}$. Choose the correct matrix-vector model for generating vector $\mathbf{R} \in \mathbb{R}^{9}$ given any choice of $b, m \in \mathbb{R}$.
A. $\left[\begin{array}{l}R_{1} \\ R_{2} \\ R_{3} \\ R_{4} \\ R_{5} \\ R_{6} \\ R_{7} \\ R_{8} \\ R_{9}\end{array}\right]=\left[\begin{array}{ll}1 & 2005 \\ 1 & 2006 \\ 1 & 2007 \\ 1 & 2008 \\ 1 & 2009 \\ 1 & 2010 \\ 1 & 2011 \\ 1 & 2012 \\ 1 & 2013\end{array}\right]\left[\begin{array}{c}b \\ m\end{array}\right] \quad$ B. $\left[\begin{array}{l}R_{1} \\ R_{2} \\ R_{3} \\ R_{4} \\ R_{5} \\ R_{6} \\ R_{7} \\ R_{8} \\ R_{9}\end{array}\right]=\left[\begin{array}{ll}2005 & 1197 \\ 2006 & 1252 \\ 2007 & 1353 \\ 2008 & 1413 \\ 2009 & 1275 \\ 2010 & 1277 \\ 2011 & 1392 \\ 2012 & 1490 \\ 2013 & 1663\end{array}\right]\left[\begin{array}{l}b \\ m\end{array}\right] \quad$ C. $\left[\begin{array}{l}R_{1} \\ R_{2} \\ R_{3} \\ R_{4} \\ R_{5} \\ R_{6} \\ R_{7} \\ R_{8} \\ R_{9}\end{array}\right]=\left[\begin{array}{ll}1 & 1197 \\ 1 & 1252 \\ 1 & 1353 \\ 1 & 1413 \\ 1 & 1275 \\ 1 & 1277 \\ 1 & 1392 \\ 1 & 1490 \\ 1 & 1663\end{array}\right]\left[\begin{array}{l}b \\ m\end{array}\right] \quad$ D. $\left[\begin{array}{l}R_{1} \\ R_{2} \\ R_{3} \\ R_{4} \\ R_{5} \\ R_{6} \\ R_{7} \\ R_{8} \\ R_{9}\end{array}\right]=\left[\begin{array}{ll}1 & 2005 \\ 2 & 2006 \\ 3 & 2007 \\ 4 & 2008 \\ 5 & 2000 \\ 6 & 2010 \\ 7 & 2011 \\ 2012 \\ 9 & 2013\end{array}\right]\left[\begin{array}{l}b \\ m\end{array}\right]$
15. Consider the following two column vectors

$$
\mathbf{x}=\left[\begin{array}{r}
-1 \\
1 \\
1 \\
-1
\end{array}\right], \quad \mathbf{y}=\left[\begin{array}{r}
1 \\
-1 \\
1 \\
1
\end{array}\right]
$$

Find the angle $\theta$ between these vectors:
A. $-\frac{1}{2}$
B. $\pi$
C. $\frac{2 \pi}{3}$
D. $\frac{\pi}{3}$
E. $\frac{5 \pi}{6}$

For Problems 16-17, consider the following spring-mass system

16. Consider the mass-spring chain from the diagram above. Recall the model for the mass spring chain is given by $M \ddot{\mathbf{u}}(t)+K \mathbf{u}(t)=\mathbf{F}_{e}(t)$. Identify the stiffness matrix $K$ for the given values of $k_{i}$ ?
A. $\left[\begin{array}{rrr}500 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 500\end{array}\right]$
B. $\left[\begin{array}{rrr}300 & -200 & 0 \\ -200 & 200 & -200 \\ 0 & -200 & 300\end{array}\right]$
C. $\left[\begin{array}{rrr}500 & -200 & -200 \\ -200 & 400 & -200 \\ -200 & -200 & 500\end{array}\right]$
D. $\left[\begin{array}{rrr}-500 & 200 & 0 \\ 200 & -400 & 200 \\ 0 & 200 & -500\end{array}\right]$
E. $\left[\begin{array}{rrr}300 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300\end{array}\right]$
17. Suppose that you are given the displacement vector when $t=T$ at equilibrium to find

$$
\mathbf{u}=\left[\begin{array}{l}
u_{1}(T) \\
u_{2}(T) \\
u_{3}(T)
\end{array}\right]=\left[\begin{array}{l}
0.98 \\
1.96 \\
0.98
\end{array}\right]
$$

measured in meters. Then, which of the following gives the mass vector $\mathbf{m}=\left[\begin{array}{lll}m_{1} & m_{2} & m_{3}\end{array}\right]^{T}$ as measured in kg ? Assume the acceleration due to earth's gravity is $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$. Also assume that the mass of each spring is zero and that these springs satisfy Hooke's law exactly.
A. $\left[\begin{array}{l}10 \\ 40 \\ 10\end{array}\right]$
B. $\left[\begin{array}{r}9.8 \\ 39.2 \\ 9.8\end{array}\right]$
C. $\left[\begin{array}{l}1 \\ 4 \\ 1\end{array}\right]$
D. $\left[\begin{array}{l}0.1 \\ 0.4 \\ 0.1\end{array}\right]$
E. $\left[\begin{array}{l}29.4 \\ 39.2 \\ 29.4\end{array}\right]$
18. Define matrix $A$ by

$$
A=\left[\begin{array}{rrrrr}
2 & 3 & 1 & 4 & 6 \\
1 & -2 & 3 & 2 & 0 \\
-4 & 1 & 0 & 5 & 7 \\
6 & -2 & 8 & 0 & -1 \\
-7 & -2 & -1 & 3 & 1
\end{array}\right]
$$

For which of the following matrices $E$ below will the matrix product

$$
E A=C
$$

not have a zero in the first column?
A. $S_{21}(-0.5)$
B. $S_{31}(2)$
C. $S_{41}(-3)$
D. $S_{51}(3.5)$
E. $S_{41}(3)$
19. Which of the following sets of vectors is linearly dependent?
A. $\left\{\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]\right\}$
B. $\left\{\left[\begin{array}{l}1 \\ 4 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{r}0 \\ -2 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right]\right\}$
C. $\left\{\left[\begin{array}{l}1 \\ 1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{c}1 \\ -1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right]\right\}$
D. $\left\{\left[\begin{array}{l}1 \\ 1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{r}3 \\ 3 \\ -4 \\ -4\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 1\end{array}\right]\right\}$
E. $\left\{\left[\begin{array}{l}1 \\ 0 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}1 \\ 1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}1 \\ 1 \\ 1 \\ 0\end{array}\right]\right\}$
20. Recall that we used a spring in class modeled by the equation $f(e)=k e+b$ where $k=17.57 \mathrm{~N} / \mathrm{m}$ and $b=0.064 \mathrm{~N}$. Which of the following gives an ideal version of vector $\mathbf{e}$ (where entries are measured in $m$ ) if we hang masses encoded in the mass vector

$$
\mathbf{m}=\left[\begin{array}{l}
0.00 \\
0.10 \\
0.20 \\
0.30 \\
0.40
\end{array}\right]
$$

In this case, assume elongation measurements are given in meters ( m ) and are rounded to 4 digits to the right of the decimal place. Each entry of $\mathbf{m}$ is measured in units of kilograms (kg). Remember the unit equation $1 \mathrm{~N}=1 \frac{\mathrm{~kg} \cdot \mathrm{~m}}{\mathrm{~s}^{2}}$. Also, sssume the acceleration due to gravity is $9.8 \mathrm{~m} / \mathrm{s}^{2}$.
A.
$\left[\begin{array}{l}0.064 \\ 1.821 \\ 3.578 \\ 5.335 \\ 7.092\end{array}\right]$
B. $\left[\begin{array}{r}-0.0036 \\ 0.0521 \\ 0.1079 \\ 0.1637 \\ 0.2195\end{array}\right]$
C. $\left[\begin{array}{r}-0.0036 \\ 0.0020 \\ 0.0077 \\ 0.0134 \\ 0.0191\end{array}\right]$
D. $\left[\begin{array}{l}0.00 \\ 0.98 \\ 1.96 \\ 2.94 \\ 3.92\end{array}\right]$
E. $\left[\begin{array}{r}0.064 \\ 17.2826 \\ 34.5012 \\ 51.7198 \\ 68.9384\end{array}\right]$

## Free Response

21. (10 pts) Let $A \in \mathbb{R}^{m \times n}$. Suppose that $\mathbf{x}_{1}, \mathbf{x}_{2} \in \mathbb{R}^{n}$ and $c_{1}, c_{2} \in \mathbb{R}$. The superposition principle of matrix-vector multiplication is given by

$$
A\left(c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}\right)=c_{1} A \mathbf{x}_{1}+c_{2} A \mathbf{x}_{2}
$$

Prove this theorem.

Solution: Option 1: Column-Partition Version- Let $A \in \mathbb{R}^{m \times n}$. Suppose that $\mathbf{x}_{1}, \mathbf{x}_{2} \in \mathbb{R}^{n}$ and $c_{1}, c_{2} \in \mathbb{R}$. Define vector $\mathbf{y} \in \mathbb{R}^{n}$ as the linear combination

$$
\mathbf{y}=c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}
$$

Then, the $k$ th entry of $\mathbf{y}$ is given by

$$
y_{k}=c_{1} \cdot x_{k 1}+c_{2} \cdot x_{k 2}
$$

for $k=1,2, \ldots, n$. In this case, we have that the scalar $x_{k i} \in \mathbb{R}$ is the $k$ th entry of the column vector $\mathbf{x}_{i}$ for $i=1,2$. Now, let's consider the column-partition version of the matrix vector product

$$
\begin{aligned}
A \mathbf{y} & =\sum_{k=1}^{n} y_{k} A(:, k) \\
& =\sum_{k=1}^{n}\left(c_{1} \cdot x_{k 1}+c_{2} \cdot x_{k 2}\right) A(:, k) \\
& =\sum_{k=1}^{n} c_{1} \cdot x_{k 1} A(:, k)+c_{2} \cdot x_{k 2} A(:, k) \\
& \left.=\sum_{k=1}^{n} c_{1} \cdot x_{k 1} A(:, k)\right)+\sum_{k=1}^{n}\left(c_{2} \cdot x_{k 2} A(:, k)\right) \\
& =c_{1}\left(\sum_{k=1}^{n} x_{k 1} A(:, k)\right)+c_{2}\left(\sum_{k=1}^{n} x_{k 2} A(:, k)\right) \\
& =c_{1} A \mathbf{x}_{1}+c_{2} A \mathbf{x}_{2}
\end{aligned}
$$

This is exactly what we wanted to show.

Solution: Option 1: Scalar Version (Using inner products)- Let $A \in \mathbb{R}^{m \times n}$. Suppose that $\mathbf{x}_{1}, \mathbf{x}_{2} \in \mathbb{R}^{n}$ and $c_{1}, c_{2} \in \mathbb{R}$. Define two vectors $\mathbf{y}, \mathbf{b} \in \mathbb{R}^{n}$ as follows

$$
\mathbf{y}=A\left(c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}\right), \quad \mathbf{b}=c_{1} A \mathbf{x}_{1}+c_{2} A \mathbf{x}_{2}
$$

In this problem, we are asked to show that $\mathbf{y}=\mathbf{b}$. We will do this by confirming this equality entry by entry. In particular, we recall that two vectors are equal if and only if they have the same dimensions and their individual coefficients are equal. Letting $y_{i}, b_{i} \in \mathbb{R}$ be the $i$ th entry of vectors $\mathbf{y}$ and $\mathbf{b}$ respectively, we confirm that

$$
y_{i}=b_{i}
$$

for all $i=1,2, \ldots n$.
We begin by considering

$$
\begin{aligned}
y_{i} & =A(i,:) \cdot\left(c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}\right) \\
& =\left[\begin{array}{lll}
a_{i 1} & a_{i 2} & \cdots \\
a_{i n}
\end{array}\right] \cdot\left[\begin{array}{c}
c_{1} x_{11}+c_{2} x_{12} \\
c_{1} x_{21}+c_{2} x_{22} \\
\vdots \\
c_{1} x_{n 1}+c_{2} x_{n 2}
\end{array}\right] \\
& =\sum_{j=1}^{n} a_{i j} \cdot\left(c_{1} x_{j 1}+c_{2} x_{j 2}\right) \\
& =\sum_{j=1}^{n} c_{1}\left(a_{i j} x_{j 1}\right)+c_{2}\left(a_{i j} x_{j 2}\right) \\
& =c_{1} \sum_{j=1}^{n}\left(a_{i j} x_{j 1}\right)+c_{2} \sum_{j=1}^{n}\left(a_{i j} x_{j 2}\right) \\
& =c_{1} A(i,:) \mathbf{x}_{1}+c_{2} A(i,:) \mathbf{x}_{2} \\
& =b_{i}
\end{aligned}
$$

Because this holds true for all $i=1,2, \ldots, n$, we have proved our desired relation.
22. (10 pts) Consider the mass-spring system below:

A. Generate vector models (using appropriate matrices and vectors) to define each of the following:

$$
\mathbf{u}, \mathbf{e}, \mathbf{F}_{s}, \mathbf{y}
$$

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

Solution: Let's set up our model of the 2-mass, 4-spring chain.

## POSITION VECTORS:

Let's define $\mathbf{x}_{0}$ to be the initial position vector. Also, we will let $\mathbf{x}(t)$ store the positions of each mass at any time $t$. In this case, we let

$$
\mathbf{x}_{0}=\left[\begin{array}{l}
x_{1}(0) \\
x_{2}(0)
\end{array}\right], \quad \mathbf{x}(t)=\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]
$$

where $x_{i}(0)$ represents the position of mass $i$ at time $t=0$ as shown in the diagram. Further, $x_{i}(t)$ represents the position of mass $i$ at time $t \in(0, T] \subseteq \mathbb{R}$.

## DISPLACEMENT VECTOR:

With this we can set up our displacement vector $\mathbf{u}(t)$. In this case, we have assumed the zero position of our ruler to be on the ground. Moreover, we orient positive position measurements in the upward direction (toward the ceiling). We want $u_{i}(t)$ to measure the displacement of mass $i$ away from it's initial position. Since $x_{i}(0)>x_{i}(t)$ in our diagram, we see that $u_{i}(t)>0$ if and only if $x_{i}(0)-x_{i}(t)>0$. Thus, we define our displacement vector

$$
\mathbf{u}(t)=\left[\begin{array}{l}
u_{1}(t) \\
u_{2}(t)
\end{array}\right]=\mathbf{x}_{0}-\mathbf{x}(t)=\left[\begin{array}{l}
x_{1}(0)-x_{1}(t) \\
x_{2}(0)-x_{2}(t)
\end{array}\right]
$$

Remark (for students who want to earn above a $90 \%$ ):
In this case, we choose the initial minus final so that positive displacement occurs in the downward position. If we wanted, we could re-orient our model using either of the options:
A. Set $\mathbf{u}(t)=\mathbf{x}(t)-\mathbf{x}_{0}$ and realize that in this case positive displacement occurs when the masses move upward from their initial position.
B. Force the zero position of our ruler to be on the ceiling of our model. Thus, we measure positive position in the downward position. In this case, we could force $\mathbf{u}(t)=\mathbf{x}(t)-\mathbf{x}_{0}$ and ensure that positive displacement occurs in the downward position.

In any case, as mathematicians modeling this system we must state our assumptions CLEARLY and make sure to account for our hypothesis correctly throughout our analysis.

## Solution:

ELONGATION VECTOR:

From this vector, we can define the elongation vector

$$
\mathbf{e}=\left[\begin{array}{c}
e_{1}(t) \\
e_{2}(t) \\
e_{3}(t) \\
e_{4}(t)
\end{array}\right]=\left[\begin{array}{c}
u_{1}(t) \\
u_{2}(t)-u_{1}(t) \\
u_{2}(t)-u_{1}(t) \\
-u_{3}(t)
\end{array}\right]=\left[\begin{array}{rr}
1 & 0 \\
-1 & 1 \\
-1 & 1 \\
0 & -1
\end{array}\right]\left[\begin{array}{c}
u_{1}(t) \\
u_{2}(t)
\end{array}\right]
$$

where $e_{i}(t)$ represents the elongation of spring $i$ at time $t$ for $i \in\{1,2,3,4\}$. As discussed in class, we can write $\mathbf{e}$ as a matrix vector product

$$
\begin{equation*}
\mathbf{e}(t)=A \mathbf{u}(t) \tag{1}
\end{equation*}
$$

$$
\text { where } A=\left[\begin{array}{rr}
1 & 0 \\
-1 & 1 \\
-1 & 1 \\
0 & -1
\end{array}\right]
$$

## FORCE VECTORS FOR SPRINGS:

Now, let's move onto finding the internal forces stored in each spring. To this end, let

$$
\mathbf{F}_{s}(t)=\left[\begin{array}{l}
F_{s_{1}}(t) \\
F_{s_{2}}(t) \\
F_{s_{3}}(t) \\
F_{s_{4}}(t)
\end{array}\right]=\left[\begin{array}{l}
k_{1} e_{1}(t) \\
k_{2} e_{2}(t) \\
k_{3} e_{3}(t) \\
k_{4} e_{4}(t)
\end{array}\right]=\left[\begin{array}{rrrr}
k_{1} & 0 & 0 & 0 \\
0 & k_{2} & 0 & 0 \\
0 & 0 & k_{3} & 0 \\
0 & 0 & 0 & k_{4}
\end{array}\right]\left[\begin{array}{l}
e_{1}(t) \\
e_{2}(t) \\
e_{3}(t) \\
e_{4}(t)
\end{array}\right]
$$

Again, we can interpret our vector $\mathbf{F}_{s}(t)$ using matrix-vector multiplication as

$$
\begin{equation*}
\mathbf{F}_{s}(t)=C \mathbf{e}(t) \tag{2}
\end{equation*}
$$

$$
\text { where } C=\left[\begin{array}{rrrr}
k_{1} & 0 & 0 & 0 \\
0 & k_{2} & 0 & 0 \\
0 & 0 & k_{3} & 0 \\
0 & 0 & 0 & k_{4}
\end{array}\right]
$$

## NET INTERNAL FORCES FOR MASS-SPRING CHAIN:

Let's consider the net internal forces on each mass. To do so, we draw a free-body diagram and focus only on the forces that result from the coupling of the masses and springs.


We now introduce the vector $\mathbf{y}(t)$ to store the net force on each mass. When writing the individual entries of $\mathbf{y}(t)$ we will assume that positive forces result in positive displacements. Since we've oriented positive displacement in the downward direction, we also orient positive force in the downward direction.

$$
\mathbf{y}(t)=\left[\begin{array}{l}
y_{1}(t) \\
y_{2}(t)
\end{array}\right]=\left[\begin{array}{c}
F_{s_{2}}(t)+F_{s_{3}}(t)-F_{s_{1}}(t) \\
-F_{s_{2}}(t)-F_{s_{3}}(t)+F_{s_{4}}(t)
\end{array}\right]=-\left[\begin{array}{rrrr}
1 & -1 & -1 & 0 \\
0 & 1 & 1 & -1
\end{array}\right]\left[\begin{array}{l}
F_{s_{1}}(t) \\
F_{s_{2}}(t) \\
F_{s_{3}}(t) \\
F_{s_{4}}(t)
\end{array}\right]
$$

We transform this into a matrix-vector product

$$
\begin{equation*}
\mathbf{y}(t)=-A^{T} \mathbf{F}_{s}(t) \tag{3}
\end{equation*}
$$

where $A$ was defined in equation (1) for our model of the elongation vector e. Notice that we've factored out a negative sign in order to make this statement.

Remark (for students who want to earn above a $80 \%$ ):

- It is important that we have not consider the external force $F_{e_{i}}(t)$ on mass $i$ when constructing the vector $\mathbf{y}(t)$. We will account for these forces later when considering the vector version of Newton's second law.
- For now, we focus on the internal forces of the system. This enables us to create a description of the mass-spring chain that is independent from the driving forces $F_{e_{i}}(t)$. As we will see, this is particularly useful when we analyze how the system responds to different external forces based on the internal structure.
B. Using your vector models from above, describe $\mathbf{y}$ as a matrix-vector product with stiffness matrix $K$ and vector $\mathbf{u}$. Demonstrate how to calculate $K$ and explicitly calculate it's value in general.

Solution: In this problem, we will using equations (1), (2), and (3) to create stiffness matrix $K$. To this end, note

$$
\begin{aligned}
\mathbf{y}(t) & =-A^{T} \mathbf{F}_{s}(t) & & \text { by equation }(3) \\
& =-A^{T} C \mathbf{e}(t) & & \text { by equation }(2) \\
& =-A^{T} C A \mathbf{u}(t) & & \text { by equation }(1) \\
& =-K \mathbf{u}(t) & &
\end{aligned}
$$

If we let $K=A^{T} C A$, we can then write

$$
\begin{equation*}
\mathbf{y}(t)=-K \mathbf{u}(t) \tag{4}
\end{equation*}
$$

We can form our stiffness matrix $K$ explicitly using matrix-matrix multiplication with

$$
K=\left[\begin{array}{rrrr}
1 & -1 & -1 & 0 \\
0 & 1 & 1 & -1
\end{array}\right]\left[\begin{array}{rrrr}
k_{1} & 0 & 0 & 0 \\
0 & k_{2} & 0 & 0 \\
0 & 0 & k_{3} & 0 \\
0 & 0 & 0 & k_{4}
\end{array}\right]\left[\begin{array}{rr}
1 & 0 \\
-1 & 1 \\
-1 & 1 \\
0 & -1
\end{array}\right]=\left[\begin{array}{lc}
k_{1}+k_{2}+k_{3} & -k_{2}-k_{3} \\
-k_{2}-k_{3} & k_{2}+k_{3}+k_{4}
\end{array}\right]
$$

C. Show how to use Newton's second law leads to an equation of the form

$$
K \mathbf{u}=\mathbf{F}_{e}
$$

where $\mathbf{F}_{e}$ represents the vector of external forces on each mass.

Solution: From Newton's second law, we know that

$$
\text { Net Force }=\text { Mass } \times \text { Acceleration }
$$

We can apply this law to each mass individually to create a differential equation that describes our system, given by

$$
\Sigma \mathbf{F}=\left[\begin{array}{l}
\Sigma F_{1} \\
\Sigma F_{2}
\end{array}\right]=\left[\begin{array}{l}
m_{1} \ddot{u}_{1}(t) \\
m_{2} \ddot{u}_{2}(t)
\end{array}\right]=\left[\begin{array}{cc}
m_{1} & 0 \\
0 & m_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{u}_{1}(t) \\
\ddot{u}_{2}(t)
\end{array}\right]
$$

where $\Sigma F_{i}$ represents the net force on mass $i$ and $\ddot{u}_{i}(t)=\frac{d^{2}}{d t^{2}}\left[u_{i}(t)\right]$ for $i \in\{1,2\}$. We write the matrix-vector multiplication

$$
\Sigma \mathbf{F}=M \ddot{\mathbf{u}}(t) \quad \text { where } M=\left[\begin{array}{cc}
m_{1} & 0  \tag{5}\\
0 & m_{2}
\end{array}\right]
$$

Further, since all forces are assumed to be positive in the downward direction we see

$$
\left[\begin{array}{l}
\Sigma F_{1} \\
\Sigma F_{2}
\end{array}\right]=\left[\begin{array}{l}
F_{e_{1}}(t) \\
F_{e_{2}}(t)
\end{array}\right]+\left[\begin{array}{l}
y_{1}(t) \\
y_{2}(t)
\end{array}\right]
$$

Thus, we can write

$$
\begin{equation*}
\Sigma \mathbf{F}=\mathbf{F}_{e}(t)+\mathbf{y}(t) \tag{6}
\end{equation*}
$$

By combing equations (4), (5), and (6), we see

$$
\begin{aligned}
M \ddot{\mathbf{u}}(t) & =\mathbf{F}_{e}(t)+\mathbf{y}(t) \\
\Longrightarrow \quad M \ddot{\mathbf{u}}(t) & =\mathbf{F}_{e}(t)+-K \mathbf{u}(t)
\end{aligned}
$$

By moving $-K$ onto the other side of the equation, we have

$$
\begin{equation*}
M \ddot{\mathbf{u}}(t)+K \mathbf{u}(t)=\mathbf{F}_{e}(t) \tag{7}
\end{equation*}
$$

Since we have assume that we study the system at equilibrium for $t=T$, we know $\ddot{\mathbf{u}}(T)=\mathbf{0}$ and we have

$$
K \mathbf{u}(T)=\mathbf{F}_{e}(T)
$$

Remark (for students who want to earn above a $100 \%$ ):

- In this derivation, we've used a very general approach to allow $t \in(0, T]$. Only at the very end of our work, did we substitute the value of $t=T$ to represent the case that our masses have settled down to equilibrium. As we will see, this general approach will come in very useful during our discussion of the eigenvalue-eigenvector problem.
- In fact, we have derived a coupled ordinary differential equation in the work above. For those of you that have taken (or will take) Math 2A at Foothill, you may notice that equation (7) is a vector version of the 2 nd order differential equation for a harmonic oscillator with no damping and general forcing function.

23. (10 pts) Let's consider the space of $4 \times 4$ matrices. Let $S_{i k}(c) \in \mathbb{R}^{4 \times 4}$ be a shear matrix, as defined in class. Then, find vector $\tau \in \mathbb{R}^{4}$ such that

$$
S_{41}(-4) \cdot S_{31}(2) \cdot S_{21}(-5)=I_{4}-\boldsymbol{\tau} \mathbf{e}_{1}^{T}
$$

Show your work. Explain how you found vector $\boldsymbol{\tau}$ and demonstrate your mastery of the matrix-matrix multiplication and outer product operations.

Solution: Option 1: Matrix-Matrix Multiplication Version- Let's recall the outer-product definition of shear matrices

$$
S_{i k}(c)=I_{n}+c \mathbf{e}_{i} \mathbf{e}_{k}^{T}
$$

We are told $n=4$ in the problem statement. Using our outer product definition, we see

$$
S_{41}(-4)=\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-4 & 0 & 0 & 1
\end{array}\right], \quad S_{31}(2)=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right], \quad S_{21}(-5)=\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

We can use any version of our definition for matrix multiplication to find

$$
\begin{aligned}
S_{41}(-4) \cdot S_{31}(2) \cdot S_{21}(-5) & =\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-4 & 0 & 0 & 1
\end{array}\right] \cdot\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \cdot\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \\
& =\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
-4 & 0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

Then, we also notice

$$
\begin{aligned}
{\left[\begin{array}{rrrr}
1 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
-4 & 0 & 0 & 1
\end{array}\right] } & =\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]-\left[\begin{array}{rrrr}
0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 \\
4 & 0 & 0 & 0
\end{array}\right] \\
& =\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]-\left[\begin{array}{r}
0 \\
5 \\
-2 \\
4
\end{array}\right]\left[\begin{array}{llll}
1 & 0 & 0 & 0
\end{array}\right]
\end{aligned}
$$

Thus, we see that our desired $\boldsymbol{\tau}$ as described in our problem statement is given by

$$
\boldsymbol{\tau}=\left[\begin{array}{r}
0 \\
5 \\
-2 \\
4
\end{array}\right]
$$

