Name : ______________________ © Jeffrey A. Anderson **Class Number**: _____

Sample Exam 1, Version 2A

Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 5 sheets of paper (10 pages front and back) including this cover page.
- There are a total of 25 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 15 Multiple Choice Questions (60 points)
  - 3 Free-Response Questions (30 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger than 8.5 inches by 11 inches. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T F Let $A \in \mathbb{R}^{m \times n}$. If we define the function $f(\mathbf{x}) = A\mathbf{x}$, then the codomain of this function is $\mathbb{R}^m$

2. T F Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $A\mathbf{x} = \sum_{k=1}^{n} x_k \Big(A(k,:)\Big)^T$

3. T F A linear combination of vectors is the same thing as the span of these vectors.

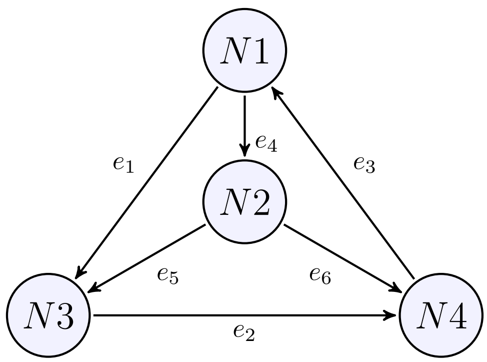
4. T F We know $\Big(S_{32}(-5)\Big)^T = I_4 - 5\,\mathbf{e}_2\,\mathbf{e}_3^T$, where $S_{ik}(c)$ is a 4×4 shear matrix.

5. T F For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we know $|\mathbf{x} \cdot \mathbf{y}| \geq \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2$

Multiple Choice

(60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

Consider the following directed graph. Use this graph to find the correct answer for problem 6.



Incidence Matrix

	e_1	e_2	e_3	e_4	e_5	e_6
N_1						
N_2						
N_3						
N_4						

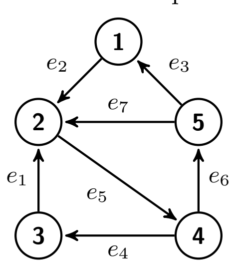
6. Let A represent the 4×6 incidence matrix. Find $A(:,2) \cdot A(:,5)$:

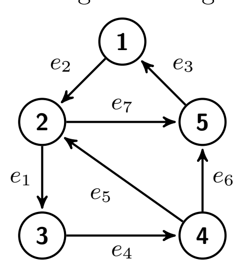
A. 2 B. 1 C. 0 D. -1 E. -2

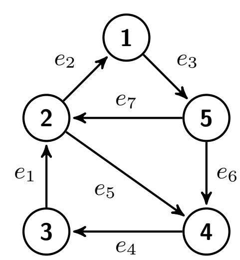
7. Let the following 5×7 matrix be the incidence matrix for a directed graph:

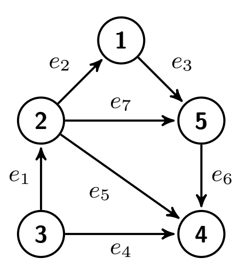
$$A = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

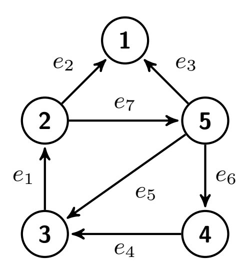
This matrix corresponds to which of the following directed graphs:

A. 

B. 

C. 

D. 

E. 

8. Let matrix $P \in \mathbb{R}^{4\times 5}$ be given as follows:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using this definition, we see that a_{12} is equal to which of the following:

A. $a_{12} = p_{21}$ B. $a_{12} = p_{24}$ C. $a_{12} = p_{42}$ D. $a_{12} = p_{12}$ E. $a_{12} = p_{25}$

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9. Which of the following sets is equivalent to

$$\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

A. $\mathbb{R}^2$ B. $\mathbb{R}^3$ C. $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2 \right\}$ D. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ E. $\left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$

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10. Let $E \subseteq \mathbb{R} \times \mathbb{R}$ be given by

$$E = \left\{ (x, y) : \frac{x^2}{1024} + \frac{y^2}{729} < 1 \right\}$$

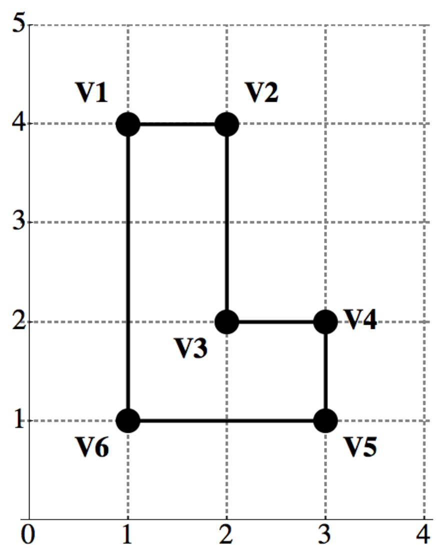
Which of the following cannot be true about the relation E?

A. $\operatorname{Dom}(E) = [-32, 32]$ B. $\operatorname{Dom}(E) = (-32, 32)$ C. $\operatorname{Rng}(E) = (-27, 27)$

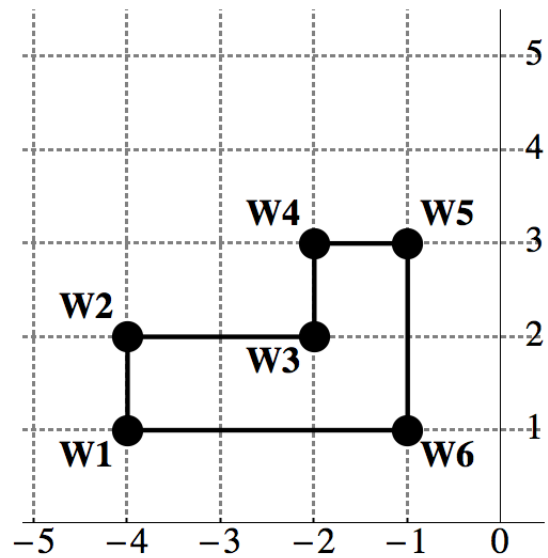
D. E is not a function E. $\operatorname{Codomain}(E) = \mathbb{R}$

For problems 11 and 12, consider the polygons V and W.:

BEGIN POLYGON V



END POLYGON W



11. Which of the following vertex matrices V encodes the begin polygon above? For this model, assume that the kth column of V encodes vertex Vk, for $k \in \{1, 2, 3, 4, 5, 6\}$:

A. $\begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 & 2 & 2 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 1 \\ 4 & 4 & 2 & 2 & 1 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 4 & 4 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 4 & 4 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{bmatrix}$ E. $\begin{bmatrix} -4 & -4 & -2 & -2 & -1 & -1 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$

12. As noted above, let V be the vertex matrix that models the begin polygon and W be the vertex matrix that models the end polygon. Which matrix Q below satisfies equation

$$W = QV$$

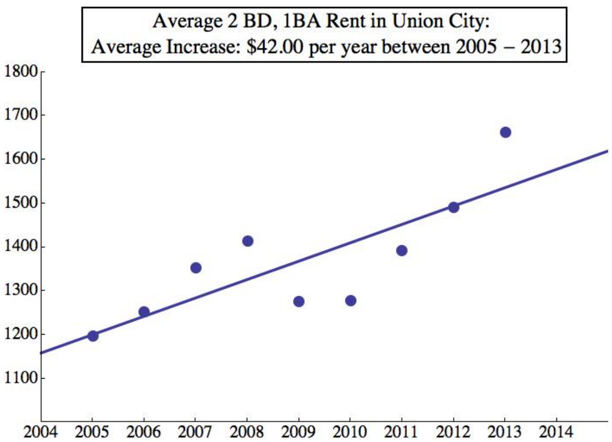
A. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ B. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ D. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ E. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

13. For sets A and B, the statement "If $x \in A$, then $x \in B$" is written using which of the following?

A. $A \leq B$ B. $B \subseteq A$ C. $A = B$ D. $A \subseteq B$ E. $A \neq B$

14. Consider the following data set that describes the average rent levels for a rental unit with 2 bedrooms and 1 bathroom (2Bd/1Ba) in Union City, CA.

Index i	Year t_i	Average Rent Level R (in \$)
1	2005	\$1,197
2	2006	\$1,252
3	2007	\$1,353
4	2008	\$1,413
5	2009	\$1,275
6	2010	\$1,277
7	2011	\$1,392
8	2012	\$1,490
9	2013	\$1,663



From this data, we can model the rent for a 2Bd/1Ba unit using a linear function in the form

$$R_i = R(t_i) = b + m \cdot t_i$$

where R_i is the modeled monthly rent during year t_i. Choose the correct matrix-vector model for generating vector $\mathbf{R} \in \mathbb{R}^9$ given any choice of $b, m \in \mathbb{R}$.

A. $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix} = \begin{bmatrix} 1 & 2005 \\ 1 & 2006 \\ 1 & 2007 \\ 1 & 2008 \\ 1 & 2009 \\ 1 & 2010 \\ 1 & 2011 \\ 1 & 2012 \\ 1 & 2013 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$
B. $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix} = \begin{bmatrix} 2005 & 1197 \\ 2006 & 1252 \\ 2007 & 1353 \\ 2008 & 1413 \\ 2009 & 1275 \\ 2010 & 1277 \\ 2011 & 1392 \\ 2012 & 1490 \\ 2013 & 1663 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$
C. $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix} = \begin{bmatrix} 1 & 1197 \\ 1 & 1252 \\ 1 & 1353 \\ 1 & 1413 \\ 1 & 1275 \\ 1 & 1277 \\ 1 & 1392 \\ 1 & 1490 \\ 1 & 1663 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$
D. $\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix} = \begin{bmatrix} 1 & 2005 \\ 2 & 2006 \\ 3 & 2007 \\ 4 & 2008 \\ 5 & 2000 \\ 6 & 2010 \\ 7 & 2011 \\ 8 & 2012 \\ 9 & 2013 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$

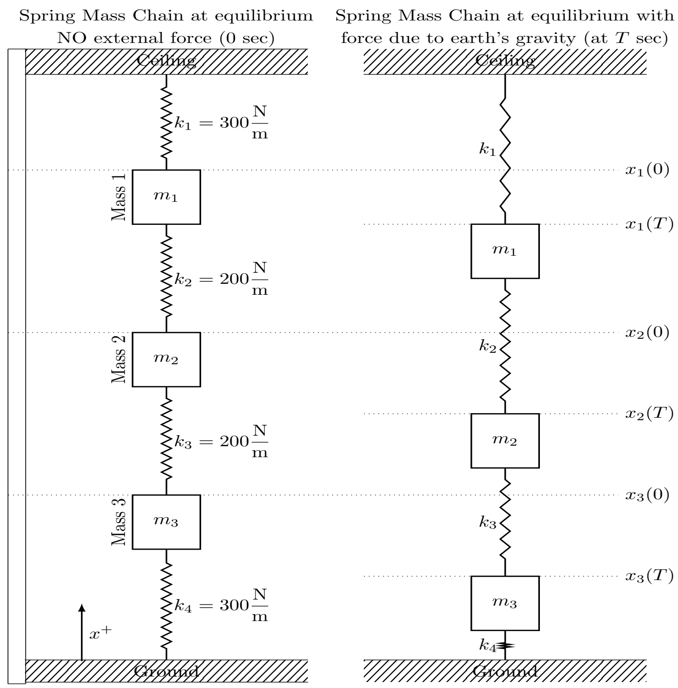
15. Consider the following two column vectors

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Find the angle θ between these vectors:

A. $-\frac{1}{2}$ B. π C. $\frac{2\pi}{3}$ D. $\frac{\pi}{3}$ E. $\frac{5\pi}{6}$

For Problems 16 - 17, consider the following spring-mass system



16. Consider the mass-spring chain from the diagram above. Recall the model for the mass spring chain is given by $M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{F}_e(t)$. Identify the stiffness matrix K for the given values of k_i?

A. $\begin{bmatrix} 500 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 500 \end{bmatrix}$ B. $\begin{bmatrix} 300 & -200 & 0 \\ -200 & 200 & -200 \\ 0 & -200 & 300 \end{bmatrix}$ C. $\begin{bmatrix} 500 & -200 & -200 \\ -200 & 400 & -200 \\ -200 & -200 & 500 \end{bmatrix}$

D. $\begin{bmatrix} -500 & 200 & 0 \\ 200 & -400 & 200 \\ 0 & 200 & -500 \end{bmatrix}$ E. $\begin{bmatrix} 300 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$

17. Suppose that you are given the displacement vector when $t = T$ at equilibrium to find

$$\mathbf{u} = \begin{bmatrix} u_1(T) \\ u_2(T) \\ u_3(T) \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1.96 \\ 0.98 \end{bmatrix}$$

measured in meters. Then, which of the following gives the mass vector $\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T$ as measured in kg? Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy Hooke's law exactly.

A. $\begin{bmatrix} 10 \\ 40 \\ 10 \end{bmatrix}$ B. $\begin{bmatrix} 9.8 \\ 39.2 \\ 9.8 \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ D. $\begin{bmatrix} 0.1 \\ 0.4 \\ 0.1 \end{bmatrix}$ E. $\begin{bmatrix} 29.4 \\ 39.2 \\ 29.4 \end{bmatrix}$

18. Define matrix A by

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 & 6 \\ 1 & -2 & 3 & 2 & 0 \\ -4 & 1 & 0 & 5 & 7 \\ 6 & -2 & 8 & 0 & -1 \\ -7 & -2 & -1 & 3 & 1 \end{bmatrix}$$

For which of the following matrices E below will the matrix product

$$EA = C$$

not have a zero in the first column?

A. $S_{21}(-0.5)$ B. $S_{31}(2)$ C. $S_{41}(-3)$ D. $S_{51}(3.5)$ E. $S_{41}(3)$

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19. Which of the following sets of vectors is linearly dependent?

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

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20. Recall that we used a spring in class modeled by the equation $f(e) = ke + b$ where $k = 17.57$ N/m and $b = 0.064$N. Which of the following gives an ideal version of vector $\mathbf{e}$ (where entries are measured in m) if we hang masses encoded in the mass vector

$$\mathbf{m} = \begin{bmatrix} 0.00 \\ 0.10 \\ 0.20 \\ 0.30 \\ 0.40 \end{bmatrix}$$

In this case, assume elongation measurements are given in meters (m) and are rounded to 4 digits to the right of the decimal place. Each entry of $\mathbf{m}$ is measured in units of kilograms (kg). Remember the unit equation $1 \text{ N} = 1\dfrac{\text{kg} \cdot \text{m}}{\text{s}^2}$. Also, sssume the acceleration due to gravity is 9.8 m/s^2.

A. $\begin{bmatrix} 0.064 \\ 1.821 \\ 3.578 \\ 5.335 \\ 7.092 \end{bmatrix}$ B. $\begin{bmatrix} -0.0036 \\ 0.0521 \\ 0.1079 \\ 0.1637 \\ 0.2195 \end{bmatrix}$ C. $\begin{bmatrix} -0.0036 \\ 0.0020 \\ 0.0077 \\ 0.0134 \\ 0.0191 \end{bmatrix}$ D. $\begin{bmatrix} 0.00 \\ 0.98 \\ 1.96 \\ 2.94 \\ 3.92 \end{bmatrix}$ E. $\begin{bmatrix} 0.064 \\ 17.2826 \\ 34.5012 \\ 51.7198 \\ 68.9384 \end{bmatrix}$

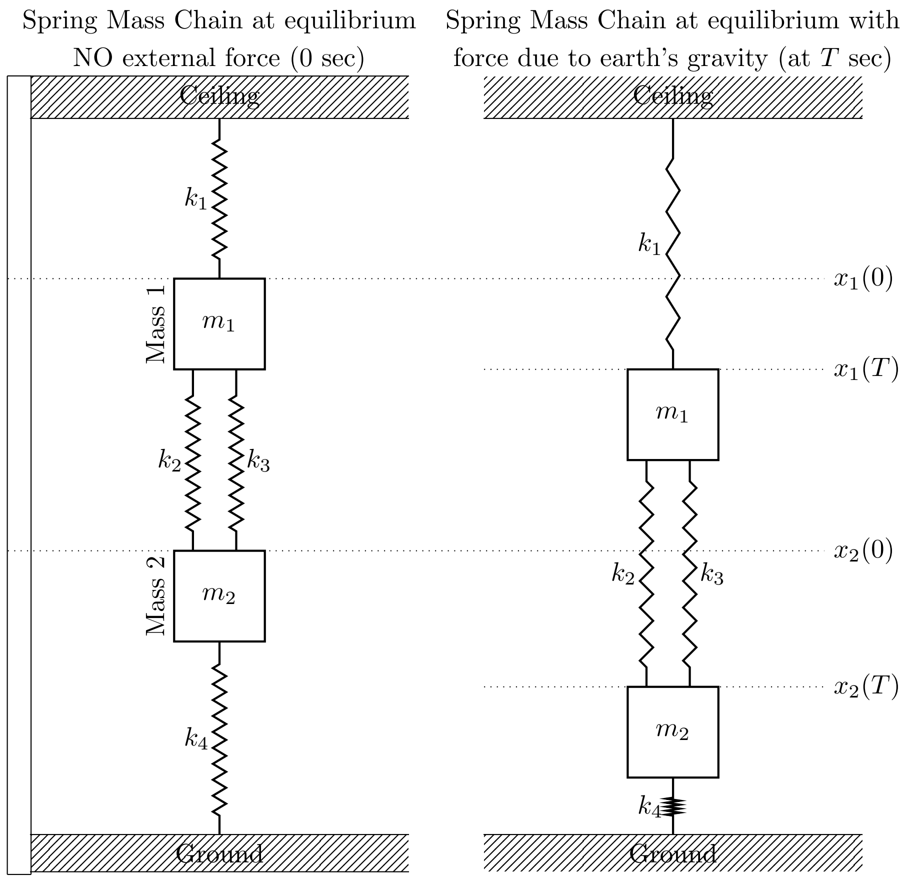
Free Response

$\boxed{10}$ 21. Let $A \in \mathbb{R}^{m \times n}$. Suppose that $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$. The superposition principle of matrix-vector multiplication is given by

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2$$

Prove this theorem.

10 22. Consider the mass-spring system below:



A. Generate vector models (using appropriate matrices and vectors) to define each of the following:

$$\mathbf{u}, \mathbf{e}, \mathbf{F}_s, \mathbf{y},$$

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

B. Using your vector models from above, describe $\mathbf{y}$ as a matrix-vector product with stiffness matrix K and vector $\mathbf{u}$. Demonstrate how to calculate K and explicitly calculate it's value in general.

C. Show how to use Newton's second law leads to an equation of the form

$$K\mathbf{u} = \mathbf{F}_e$$

where $\mathbf{F}_e$ represents the vector of external forces on each mass.

23. Let's consider the space of 4×4 matrices. Let $S_{ik}(c) \in \mathbb{R}^{4\times 4}$ be a shear matrix, as defined in class. Then, find vector $\boldsymbol{\tau} \in \mathbb{R}^4$ such that

$$S_{41}(-4) \cdot S_{31}(2) \cdot S_{21}(-5) = I_4 - \boldsymbol{\tau}\, \mathbf{e}_1^T$$

Show your work. Explain how you found vector $\boldsymbol{\tau}$ and demonstrate your mastery of the matrix-matrix multiplication and outer product operations.

Challenge Problem

24. (Optional, Extra Credit, Challenge Problem) Let $\mathbf{x} \in \mathbb{R}^n$ be a column vector. Recall that we defined the 2-norm of $\mathbf{x}$ to be

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

We can create a different norm, called the $\infty-$norm (read "infinity norm"), using the following definition

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

which takes the maximum value of the absolute values of all entries of $\mathbf{x}$. Using these definitions, prove

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{n} \cdot \|\mathbf{x}\|_\infty$$

Use for Scratch Work