
True/False

(10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T **F** $\|\alpha\mathbf{x}\|_2 = \alpha\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

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2. T **F** $(AB)^T = A^T B^T$ for any matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

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3. **T** F Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. If $f(\mathbf{x}) = A\mathbf{x}$, then the codomain of this relation is $\mathbb{R}^m$

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4. T **F** Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^m$. Then $\mathbf{x}^T A = \sum_{i=1}^{n} x_i A(i,:)$.

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5. **T** F If matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times r}$ are equal, then $m = p$ and $n = r$.

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

6. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$. Find the dot product $A(:,3) \cdot A(:,9)$:

A. 1 B. 0 **C.** -2 D. -1 E. 2

7. Define the matrix $B \in \mathbb{R}^{4\times 4}$ by the following product:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Using this definition, we see that b_{42} is given by which of the following:

A. $b_{42} = a_{32}$ B. $b_{42} = a_{42}$ C. $b_{42} = a_{24}$ **D.** $b_{42} = a_{34}$ E. $b_{42} = a_{43}$

8. Let $m, n \in \mathbb{N}$. Suppose $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$ are given. For the solution of the matrix-vector multiplication problem given by $A\mathbf{x} = \mathbf{b}$, which of the following is false:

A. $\mathbf{b}$ is linearly dependent on the columns of A.

B. The vector $\mathbf{b}$ can be written as a linear combination of the columns of A

C. If $f(\mathbf{x}) = A\mathbf{x}$, then $\mathbf{b} \in \text{Rng}(f)$.

D. The columns of A must be linearly independent.

E. $\mathbf{b} = \sum_{j=1}^{n} x_j A(:,j)$ for some $x_1, x_2, ..., x_n \in \mathbb{R}$.

9. Let $E = \{\mathbf{e}_j\}_{j=1}^4$ be the standard basis for $\mathbb{R}^4$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Define the vector $\mathbf{x}$ by taking a linear combination of elements of E below

$$\mathbf{x} = 2\,\mathbf{e}_1 - 5\,\mathbf{e}_2 - 3\,\mathbf{e}_3 + 4\,\mathbf{e}_4.$$

Which of the following gives the value of the dot product $(\mathbf{e}_4 - \mathbf{e}_2) \cdot \mathbf{x}$:

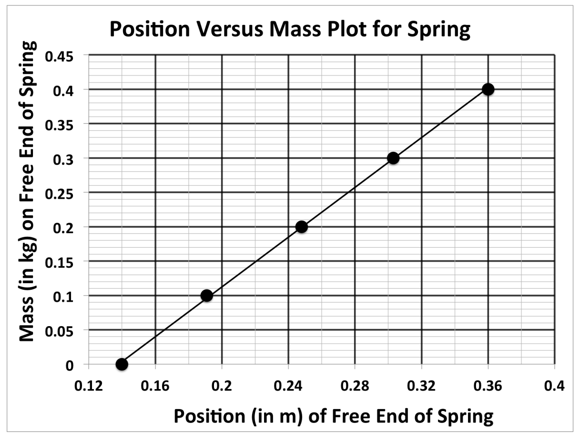
A. 9 B. -9. C. -2 D. 1 E. -1

10. Suppose that $A \in \mathbb{R}^{20\times 5}$ and $\mathbf{x} \in \mathbb{R}^5$. How many total operations on real numbers are necessary to solve the matrix-vector multiplication problem $A\mathbf{x} = \mathbf{b}$? Remember that each multiplication between two real numbers counts as one operation and each addition between two real numbers counts as one operation.

A. 100 **B. 180** C. 90 D. 25 E. 200

11. Hooke's Law is a principle of physics stating that the force needed to extend or compress a spring by some distance is proportional to that distance. Recall from class that we can set up an experiment to verify Hooke's law using a spring, various masses, a scale, and a measuring stick. Below are five collected data points relating to Hooks Law.

Measurement Number	Position x in Meters (m)	Applied mass m in kilograms
1	0.140	0.000
2	0.191	0.100
3	0.248	0.200
4	0.303	0.300
5	0.360	0.400



From this data, we can calculate u_i, the displacement of movable end of spring in measurement i. We can also create a mathematical model in the form

$$y_i = b + k \cdot u_i$$

where y_i is the modeled force associated with displacement u_i. Choose the correct matrix-vector model for generating vector $\mathbf{y} \in \mathbb{R}^5$ given any choice of $b, k \in \mathbb{R}$.

A. $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0.140 \\ 1 & 0.191 \\ 1 & 0.248 \\ 1 & 0.303 \\ 1 & 0.360 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix}$ B. $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0.0 \\ 1 & 0.1 \\ 1 & 0.2 \\ 1 & 0.3 \\ 1 & 0.4 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix}$ **C.** $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0.000 \\ 1 & 0.051 \\ 1 & 0.108 \\ 1 & 0.163 \\ 1 & 0.220 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix}$ D. $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0.140 & 0.0 \\ 0.191 & 0.1 \\ 0.248 & 0.2 \\ 0.303 & 0.3 \\ 0.360 & 0.4 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix}$

12. Which of the following represents the 8-bit binary representation of the number 207?

A. 11110011 B. 11010001 C. 10001011 D. 10111111 **E.** 11001111

13. Let $A \in \mathbb{R}^{12\times 7}$ and $B \in \mathbb{R}^{12\times 6}$. Suppose $C = B^T A$. What are the dimensions of $C(:, 2)$?

A. 7×1 **B.** 6×1 C. 6×7 D. 7×6 E. 1×6

14. Let

$$C([0,1]) = \{f(x)\,|\,\text{function } f:[0,1]\to\mathbb{R} \text{ is a continuous function on interval } [0,1].\ \}$$
$$C^{(1)}([0,1]) = \{f(x)\,|\,\text{function } f:[0,1]\to\mathbb{R} \text{ has a continuous first derivative } f'(x) \text{ on interval } [0,1].\ \}$$

In other words, $C([0,1])$ is the set of continuous functions $f:[0,1]\to\mathbb{R}$ while $C^{(1)}([0,1])$ is the set of functions $f:[0,1]\to\mathbb{R}$ with continuous first derivatives. From Math 1A (Single-Variable, Differential Calculus), we know that if $f(x)$ is differentiable on $[0,1]$, then f is continuous on $[0,1]$. Identify the set theoretic formulation of this theorem:

A. $C([0,1]) \subseteq C^{(1)}([0,1])$
B. $C([0,1]) = C^{(1)}([0,1])$
C. $C([0,1]) \cap C^{(1)}([0,1]) = \emptyset$
D. $C^{(1)}([0,1]) \subseteq C([0,1])$
E. $C([0,1]) \cup C^{(1)}([0,1]) = C^{(1)}([0,1])$

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15. Let A and B be sets. What does it mean if we say that A is a subset of B?

A. Some element x in A is also an element of B.
B. A is an element of B.
C. Every element x in A is contained in some element y of B.
D. Every element x in A is also an element of B.
E. Every element y in B is also an element of A.

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16. Which of the following represents the matrix-matrix product: $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}$:

A. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 1.89 \\ 0 & 3.6 & 3.96 \end{bmatrix}$
B. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 10.89 \\ 0 & 3.6 & 12.96 \end{bmatrix}$
C. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 1.89 \\ 0 & 0 & 3.96 \end{bmatrix}$
D. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0.3 & 10.89 \\ 0 & 0.6 & 12.96 \end{bmatrix}$
E. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0.3 & 1.89 \\ 0 & 0.6 & 3.96 \end{bmatrix}$

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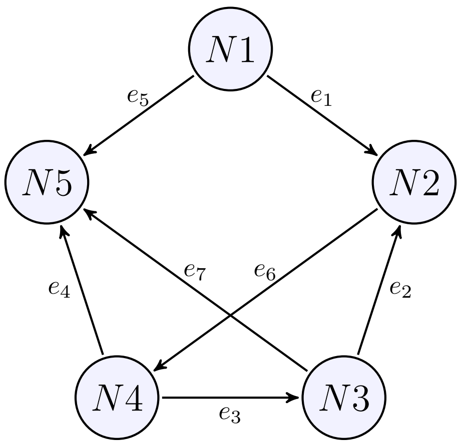
17. Consider the following expression:

$$\begin{bmatrix} 9 & 5 & 3 \\ 8 & 0 & 2 \\ 7 & -6 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -5 & 2 & 0 \end{bmatrix}$$

Using the properties of matrix-matrix multiplication and matrix-matrix addition, which of the following represents the given expression:

A. $\begin{bmatrix} 9 & 5 & 3 \\ 8 & 0 & 2 \\ 7 & -6 & 1 \end{bmatrix}$
B. $\begin{bmatrix} 3 & 5 & 3 \\ -2 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$
C. $\begin{bmatrix} 9 & 5 & -3 \\ 18 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$
D. $\begin{bmatrix} 9 & 5 & -3 \\ -2 & -4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$
E. $\begin{bmatrix} 9 & 5 & -3 \\ -2 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$

Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix. Use your entries for the incidence matrix to identify the correct answer for problems **18 - 19**.



Incidence Matrix

	e_1	e_3	e_3	e_4	e_5	e_6	e_7
N_1							
N_2							
N_3							
N_4							
N_5							

18. Let$A$ represent the 5×7 incidence matrix. Then the entry a_{35} is given by which of the following:

A. $a_{35} = 2$ B. $a_{35} = -1$ **C.** $a_{35} = 0$ D. $a_{35} = 1$ E. $a_{35} = e_7$

19. Let A represent the 5×7 incidence matrix. Then $A(:, 6)$ is given by which of the following:

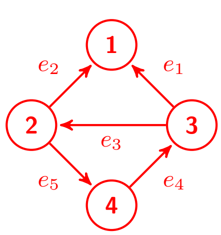
A. $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ C. $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

20. Let the following matrix $A \in \mathbb{R}^{4\times5}$ be the incidence matrix for a directed graph:

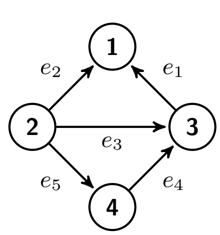
Incidence Matrix

	e_1	e_2	e_3	e_4	e_5
N_1	-1	-1	0	0	0
N_2	0	1	-1	0	1
N_3	1	0	1	-1	0
N_4	0	0	0	1	-1

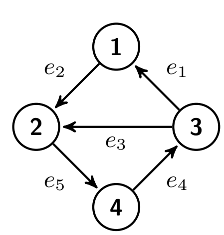
This matrix corresponds to which of the following directed graphs:



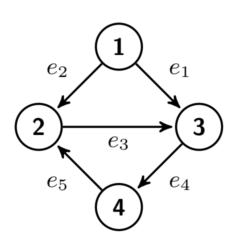
A.



B.



C.



D.

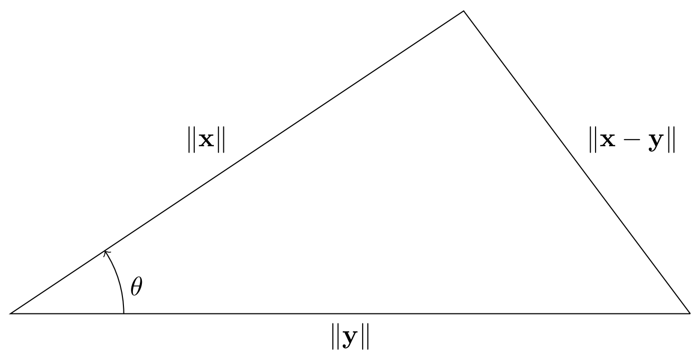
Free Response

21. (10 pts) Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonzero, linearly independent vectors. Prove that

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

Proof. Assume $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other (i.e. assume $\mathbf{x}$ and $\mathbf{y}$ are linearly independent). Suppose we begin with two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider the triangle defined by these vectors. The length of each side of this triangle can be given by the 2−norm of the vectors:



By the Law of Cosines, we know

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\;\|\mathbf{y}\|\;\cos(\theta)$$

Recall, using the algebraic properties of the inner product, we can write

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\
&= \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 - 2\,\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2
\end{aligned}$$

With this we see

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\;\|\mathbf{y}\|\;\cos(\theta) = \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2$$

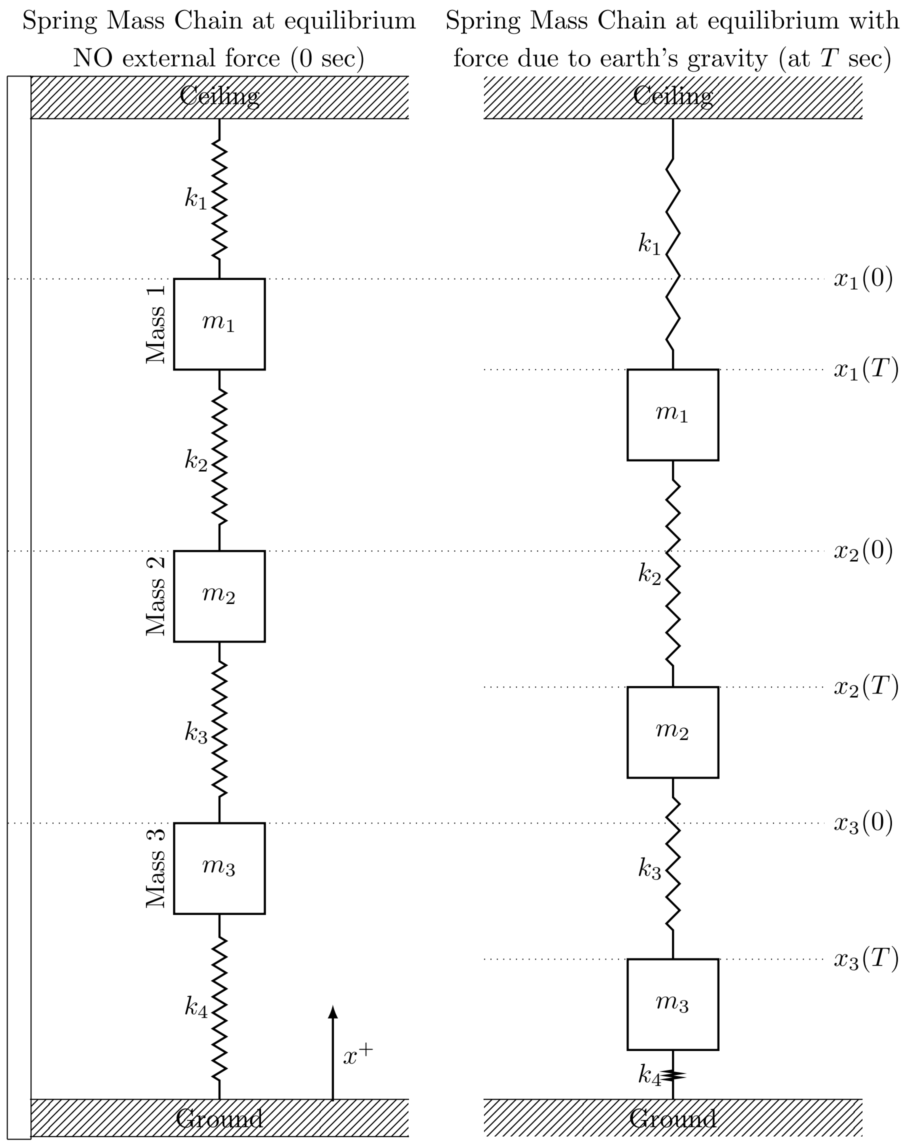
By canceling out the appropriate terms using our knowledge of arithmetic, we see

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\;\|\mathbf{y}\|\;\cos(\theta).$$

Thus we see that the cosine formula for the inner product holds □

Remark: Students who want to earn above a 90% on this problem should also be ready to prove the pythagorean theorem and the law of cosines.

22. (10 pts) Consider the following diagram of the mass-spring chain:



A. Generate vector models (using appropriate matrices and vectors) to define each of the following:

$$\mathbf{u}, \mathbf{e}, \mathbf{F}_s, \mathbf{y},$$

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

Solution: Let's set up our model of the 3-mass, 4-spring chain.

POSITION VECTORS:

Let's define $\mathbf{x}_0$ to be the initial position vector. Also, we will let $\mathbf{x}(t)$ store the positions of each mass at any time t. In this case, we let

$$\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

where $x_i(0)$ represents the position of mass i at time $t = 0$ as shown in the diagram. Further, $x_i(t)$ represents the position of mass i at time $t \in (0, T] \subseteq \mathbb{R}$.

DISPLACEMENT VECTOR:

With this we can set up our displacement vector $\mathbf{u}(t)$. In this case, we have assumed the zero position of our ruler to be on the ground. Moreover, we orient positive position measurements in the upward direction (toward the ceiling). We want $u_i(t)$ to measure the displacement of mass i away from it's initial position. Since $x_i(0) > x_i(t)$ in our diagram, we see that $u_i(t) > 0$ if and only if $x_i(0) - x_i(t) > 0$. Thus, we define our displacement vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \mathbf{x}_0 - \mathbf{x}(t) = \begin{bmatrix} x_1(0) - x_1(t) \\ x_2(0) - x_2(t) \\ x_3(0) - x_3(t) \end{bmatrix}$$

Solution: continued from last page...

Remark (for students who want to earn above a 90%):
In this case, we choose the initial minus final so that positive displacement occurs in the downward position. If we wanted, we could re-orient our model using either of the options:

A. Set $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$ and realize that in this case positive displacement occurs when the masses move upward from their initial position.

B. Force the zero position of our ruler to be on the ceiling of our model. Thus, we measure positive position in the downward position. In this case, we could force $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$ and ensure that positive displacement occurs in the downward position.

In any case, as mathematicians modeling this system we must state our assumptions CLEARLY and make sure to account for our hypothesis correctly throughout our analysis.

ELONGATION VECTOR:

From this vector, we can define the elongation vector

$$\mathbf{e} = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) - u_1(t) \\ u_3(t) - u_2(t) \\ -u_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

where $e_i(t)$ represents the elongation of spring i at time t for $i \in \{1, 2, 3, 4\}$. As discussed in class, we can write $\mathbf{e}$ as a matrix vector product

$$\boxed{\mathbf{e}(t) = A\mathbf{u}(t)} \qquad \text{where } A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{1}$$

FORCE VECTORS FOR SPRINGS:

Now, let's move onto finding the internal forces stored in each spring. To this end, let

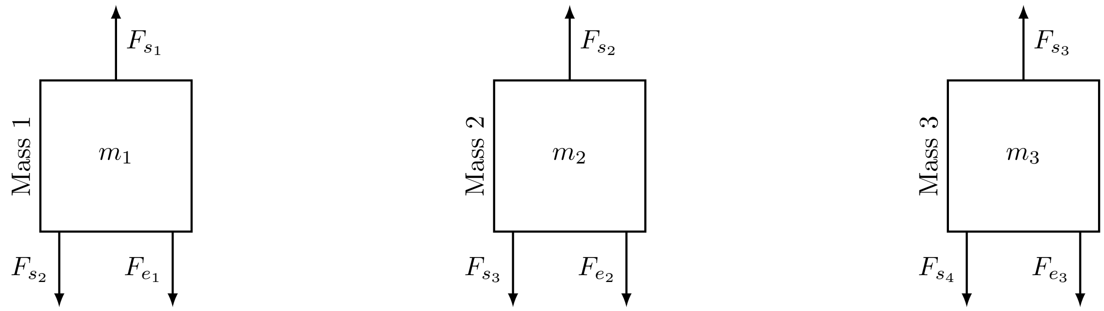
$$\mathbf{F}_s(t) = \begin{bmatrix} F_{s_1}(t) \\ F_{s_2}(t) \\ F_{s_3}(t) \\ F_{s_4}(t) \end{bmatrix} = \begin{bmatrix} k_1\, e_1(t) \\ k_2\, e_2(t) \\ k_3\, e_3(t) \\ k_4\, e_4(t) \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix}$$

Again, we can interpret our vector $\mathbf{F}_s(t)$ using matrix-vector multiplication as

$$\boxed{\mathbf{F}_s(t) = C\mathbf{e}(t)} \qquad \text{where } C = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \tag{2}$$

NET INTERNAL FORCES FOR MASS-SPRING CHAIN:

Let's consider the net internal forces on each mass. To do so, we draw a free-body diagram and focus only on the forces that result from the coupling of the masses and springs.



We now introduce the vector $\mathbf{y}(t)$ to store the net force on each mass. When writing the individual entries of $\mathbf{y}(t)$ we will assume that positive forces result in positive displacements. Since we've oriented positive displacement in the downward direction, we also orient positive force in the downward direction.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} F_{s_2}(t) - F_{s_1}(t) \\ F_{s_3}(t) - F_{s_2}(t) \\ F_{s_4}(t) - F_{s_3}(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} F_{s_1}(t) \\ F_{s_2}(t) \\ F_{s_3}(t) \\ F_{s_4}(t) \end{bmatrix}$$

We transform this into a matrix-vector product

$$\boxed{\mathbf{y}(t) = -A^T \mathbf{F}_s(t)} \tag{3}$$

where A was defined in equation (1) for our model of the elongation vector $\mathbf{e}$. Notice that we've factored out a negative sign in order to make this statement.

Remark (for students who want to earn above a 80%):

- It is important that we have not consider the external force $F_{e_i}(t)$ on mass i when constructing the vector $\mathbf{y}(t)$. We will account for these forces later when considering the vector version of Newton's second law.

- For now, we focus on the internal forces of the system. This enables us to create a description of the mass-spring chain that is independent from the driving forces $F_{e_i}(t)$. As we will see, this is particularly useful when we analyze how the system responds to different external forces based on the internal structure.

B. Using your vector models from above, describe $\mathbf{y}$ as a matrix-vector product with stiffness matrix K and vector $\mathbf{u}$. Demonstrate how to calculate K and explicitly calculate it's value in general.

Solution: In this problem, we will using equations (1), (2), and (3) to create stiffness matrix K. To this end, note

$$\begin{aligned}
\mathbf{y}(t) &= -A^T\mathbf{F}_s(t) && \text{by equation (3)} \\
&= -A^T C\mathbf{e}(t) && \text{by equation (2)} \\
&= -A^T CA\mathbf{u}(t) && \text{by equation (1)} \\
&= -K\mathbf{u}(t)
\end{aligned}$$

If we let $K = A^T CA$, we can then write

$$\boxed{\mathbf{y}(t) = -K\mathbf{u}(t)} \tag{4}$$

We can form our stiffness matrix K explicitly using matrix-matrix multiplication with

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

C. Show how to use Newton's second law leads to an equation of the form

$$K\mathbf{u} = \mathbf{F}_e$$

where $\mathbf{F}_e$ represents the vector of external forces on each mass.

Solution: From Newton's second law, we know that

$$\text{Net Force} = \text{Mass} \times \text{Acceleration}$$

We can apply this law to each mass individually to create a differential equation that describes our system, given by

$$\Sigma\mathbf{F} = \begin{bmatrix} \Sigma F_1 \\ \Sigma F_2 \\ \Sigma F_3 \end{bmatrix} = \begin{bmatrix} m_1\,\ddot{u}_1(t) \\ m_2\,\ddot{u}_2(t) \\ m_3\,\ddot{u}_3(t) \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{bmatrix}$$

where ΣF_i represents the net force on mass i and $\ddot{u}_i(t) = \dfrac{d^2}{dt^2}\Big[u_i(t)\Big]$ for $i \in \{1, 2, 3\}$. We write the matrix-vector multiplication

$$\boxed{\Sigma\mathbf{F} = M\ddot{\mathbf{u}}(t)} \qquad \text{where } M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{5}$$

Further, since all forces are assumed to be positive in the downward direction we see

$$\begin{bmatrix} \Sigma F_1 \\ \Sigma F_2 \\ \Sigma F_3 \end{bmatrix} = \begin{bmatrix} F_{e_1}(t) + F_{s_2}(t) - F_{s_1}(t) \\ F_{e_2}(t) + F_{s_3}(t) - F_{s_2}(t) \\ F_{e_3}(t) + F_{s_4}(t) - F_{s_3}(t) \end{bmatrix} = \begin{bmatrix} F_{e_1}(t) \\ F_{e_2}(t) \\ F_{e_3}(t) \end{bmatrix} + \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

Thus, we can write

$$\boxed{\Sigma \mathbf{F} = \mathbf{F}_e(t) + \mathbf{y}(t)} \tag{6}$$

By combing equations (4), (5), and (6), we see

$$M\ddot{\mathbf{u}}(t) = \mathbf{F}_e(t) + \mathbf{y}(t)$$

$$\implies \qquad M\ddot{\mathbf{u}}(t) = \mathbf{F}_e(t) + -K\mathbf{u}(t)$$

By moving $-K$ onto the other side of the equation, we have

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{F}_e(t) \tag{7}$$

Since we have assume that we study the system at equilibrium for $t = T$, we know $\ddot{\mathbf{u}}(T) = \mathbf{0}$ and we have

$$K\,\mathbf{u}(T) = \mathbf{F}_e(T)$$

Remark (for students who want to earn above a 100%):

- In this derivation, we've used a very general approach to allow $t \in (0, T]$. Only at the very end of our work, did we substitute the value of $t = T$ to represent the case that our masses have settled down to equilibrium. As we will see, this general approach will come in very useful during our discussion of the eigenvalue-eigenvector problem.
- In fact, we have derived a coupled ordinary differential equation in the work above. For those of you that have taken (or will take) Math 2A at Foothill, you may notice that equation (7) is a vector version of the 2nd order differential equation for a harmonic oscillator with no damping and general forcing function.

23. (10 pts) Describe, in detail, each of the following problems. For each problem, your should:

i. Identify the problem statement
ii. Identify the given and unknown quantities (explicitly identify relevant dimensions)
iii. Identify the function description of this problem (explicitly discuss domain, codomain and range)
iv. Describe how each problem is similar to and different from the other two problems in the list below.

A. The Matrix-Vector Multiplication Problem

Solution: The matrix-vector multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, calculate unknown vector $\mathbf{b} \in \mathbb{R}^m$ such that
>
> $$A\mathbf{x} = \mathbf{b}$$

The matrix-vector multiplication is a "forward problem." In particular, let's define the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad\qquad f(\mathbf{x}) = A\mathbf{x} = \sum_{k=1}^{n} x_k \, A(:,k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is $\text{Span}\{A(:,k)\}_{k=1}^{n}$

Matrix-vector multiplication is a forward problem because we start with the function description (defined by matrix A) and we are given one specific input value $\mathbf{x}$ in the domain. From this information, we are asked to find the corresponding output value $\mathbf{b}$ in the range of function $f(\mathbf{x})$. When solving the matrix-vector multiplication problem, we map from the domain forward into the range. Hence, we call this a forward problem.

The matrix-vector multiplication problem is intimately connected with the linear system problem. Matrix-vector multiplication is the forward problem while the linear systems problem represents the backward problem (also known as inverse problem). As discussed below, when solving linear-systems problems, we start with a $\mathbf{b} \in \text{Rng}(f)$ and produce all $\mathbf{x} \in \text{Dom}(f)$ such that $A(\mathbf{x}) = \mathbf{b}$

Remarks (for students who want to earn above a 90%): In addition to the comments above, here are some other remarks about this problem

- For a matrix-vector multiplication problem $\mathbf{b} = A\mathbf{x}$ with $A^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$, solving this problem requires a total of $m \cdot (2n - 1)$ operations between scalars.
- The solution to a matrix vector multiplication is unique. Each output vector $\mathbf{b}$ is given as a linear combination of the columns of A with scalar weights defined by the coefficient entries of $\mathbf{x}$.

B. The Linear-Systems Problem

Solution: The linear-systems multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, find all unknown vectors $\mathbf{x} \in \mathbb{R}^n$ such that
>
> $$A\mathbf{x} = \mathbf{b}$$

Just like the matrix-vector multiplication problem, we can describe the linear-systems using the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad f(\mathbf{x}) = A\mathbf{x} = \sum_{k=1}^{n} x_k\, A(:,k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is $\text{Span}\{A(:,k)\}_{k=1}^n$

The linear-systems problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the range of $f(\mathbf{x})$. From this information, we are asked to find all possible input values $\mathbf{x}$ in the domain of our function such that

$$f(\mathbf{x}) = \mathbf{b}.$$

When solving the linear-systems problem, we begin in the range and work our way backwards to the domain. Hence, we call this a backward problem.

Remarks (for students who want to earn above a 90%): In addition to the comments above, here are some other remarks about this problem

- The solution to a linear-systems problem may not exist. If it does exist, it may not be unique. A great analogy comes from solving backward problems for the nonlinear function $f(x) = x^2$. Let's look at three backward problems:

  A. No solutions: $f(x) = x^2 = -4$

  B. Unique solution: $f(x) = x^2 = 0$

  B. Multiple solutions: $f(x) = x^2 = 4$

Although the theory behind solving linear systems is much different than the theory for solving quadratic equations, analogies about the existence and uniqueness of solutions abound. Linear systems problems may have:

A. No Solution: $\mathbf{b} \notin \text{Rng}(f) = \text{Span}\{A(:,k)\}_{k=1}^n$ (known as least-squares problem)

B. Unique solution: $\mathbf{b} \in \text{Rng}(f) = \text{Span}\{A(:,k)\}_{k=1}^n$ and A has linearly independent columns

C. Non-unique (multiple) solutions: $\mathbf{b} \in \text{Rng}(f)$ and A has linearly dependent columns

C. The Least-Squares Problem

Solution: The least-squares problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, find all unknown vectors $\mathbf{x} \in \mathbb{R}^n$ that minimize the norm of the residual vector $\mathbf{b} - A\mathbf{x}$. In other words, find
>
> $$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{b} - A\mathbf{x}\|_2$$

Just like the matrix-vector multiplication and linear-systems problems, we contextualize the least-squares problem using function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad\qquad f(\mathbf{x}) = A\mathbf{x} = \sum_{k=1}^{n} x_k\, A(:,k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is $\text{Span}\{A(:,k)\}_{k=1}^{n}$

The least-squares problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the CODOMAIN of $f(\mathbf{x})$. From this information, we are asked to find all possible input values $\mathbf{x}$ in the domain that product output value$f(\mathbf{x})$ "as close as possible" the the vector $\mathbf{b}$. When solving the least-squares problem, we begin in the codomain and work our way backwards to the domain. Hence, we call this a backward problem.

Using our definition of the least-squares problem, we can classify all linear-systems problems as least-squares problems. For linear systems problems, since $\mathbf{b}$ begins in the range of $f(\mathbf{x})$, we know that

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{b} - A\mathbf{x}\|_2 = 0$$

for each solution. However, not all least-squares problems are linear systems problems. In the case that we have a $\mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} \notin \text{Rng}(f)$, we know that $f(\mathbf{x}) \neq \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$. This is the most interesting case and is the focus of our discussion of the least-squares problem. Thus, the least-squares is a generalization of the linear systems problem that enables us to create "best-fit" approximations to noisy data.

Challenge Problem

24. (Optional, Extra Credit, Challenge Problem) Prove that the number of linearly independent columns of a general $m \times n$ matrix is equal to the number of linearly independent rows of that matrix.