# Sample Exam 1, Version 1A

Math 2B: Linear Algebra

#### What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

#### How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 5 sheets of paper (10 pages front and back) including this cover page.
- There are a total of 25 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 15 Multiple Choice Questions (60 points)
  - 3 Free-Response Questions (30 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

#### What can I use on this exam?

- You may use one note card that is no larger than 8.5 inches by 11 inches. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

#### How will I be graded on the Free-Response Questions?

- Read the directions carefully. Show all your work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

| 1. T F $\ \alpha \mathbf{x}\ _2 = \alpha \ \mathbf{x}\ _2$ for a | all $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ . |
|--|---|
|--|---|

2. T F  $(AB)^T = A^T B^T$  for any matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$ .

3. T F Let  $A \in \mathbb{R}^{m \times n}$  and  $\mathbf{x} \in \mathbb{R}^n$ . If  $f(\mathbf{x}) = A\mathbf{x}$ , then the codomain of this relation is  $\mathbb{R}^m$ 

4. T F Let  $A \in \mathbb{R}^{m \times n}$  and  $\mathbf{x} \in \mathbb{R}^m$ . Then  $\mathbf{x}^T A = \sum_{i=1}^n x_i A(i, :)$ .

5. T F If matrices  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{p \times r}$  are equal, then m = p and n = r.

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

7. Define the matrix  $B \in \mathbb{R}^{4 \times 4}$  by the following product:

| $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ |   | [0 | 0 | 0 | 1 | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | [0 | 0 | 1 | 0] |
|----------|----------|----------|----------|---|----|---|---|---|----------|----------|----------|----------|----|---|---|----|
| $b_{21}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | _ | 1  | 0 | 0 | 0 | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | 0  | 0 | 0 | 1  |
| $b_{31}$ | $b_{32}$ | $b_{33}$ | $b_{34}$ | _ | 0  | 1 | 0 | 0 | $a_{31}$ | $a_{32}$ | $a_{33}$ | $a_{34}$ | 1  | 0 | 0 | 0  |
| $b_{41}$ | $b_{42}$ | $b_{43}$ | $b_{44}$ |   | 0  | 0 | 1 | 0 | $a_{41}$ | $a_{42}$ | $a_{43}$ | $a_{44}$ | 0  | 1 | 0 | 0  |

Using this definition, we see that  $b_{42}$  is given by which of the following:

A. 
$$b_{42} = a_{32}$$
 B.  $b_{42} = a_{42}$  C.  $b_{42} = a_{24}$  D.  $b_{42} = a_{34}$  E.  $b_{42} = a_{43}$ 

- 8. Let  $m, n \in \mathbb{N}$ . Suppose  $A \in \mathbb{R}^{m \times n}$  and  $\mathbf{x} \in \mathbb{R}^n$  are given. For the solution of the matrix-vector multiplication problem given by  $A\mathbf{x} = \mathbf{b}$ , which of the following is false:
  - A. **b** is linearly dependent on the columns of A.
  - B. The vector  ${\bf b}$  can be written as a linear combination of the columns of A
  - C. If  $f(\mathbf{x}) = A\mathbf{x}$ , then  $\mathbf{b} \in \operatorname{Rng}(f)$ .
  - D. The columns of A must be linearly independent.
  - E.  $\mathbf{b} = \sum_{j=1}^{n} x_j A(:, j)$  for some  $x_1, x_2, ..., x_n \in \mathbb{R}$ .
- 9. Let  $E = {\mathbf{e}_j}_{j=1}^4$  be the standard basis for  $\mathbb{R}^4$ :

$$\mathbf{e}_1 = \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix}, \qquad \mathbf{e}_4 = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}.$$

Define the vector  $\mathbf{x}$  by taking a linear combination of elements of E below

$$\mathbf{x} = 2\,\mathbf{e}_1 - 5\,\mathbf{e}_2 - 3\,\mathbf{e}_3 + 4\,\mathbf{e}_4.$$

Which of the following gives the value of the dot product  $(\mathbf{e}_4 - \mathbf{e}_2) \cdot \mathbf{x}$ :

A. 9 B. -9. C. -2 D. 1 E. -1

10. Suppose that  $A \in \mathbb{R}^{20 \times 5}$  and  $\mathbf{x} \in \mathbb{R}^5$ . How many total operations on real numbers are necessary to solve the matrix-vector multiplication problem  $A\mathbf{x} = \mathbf{b}$ ? Remember that each multiplication between two real numbers counts as one operation and each addition between two real numbers counts as one operation.

11. Hooke's Law is a principle of physics stating that the force needed to extend or compress a spring by some distance is proportional to that distance. Recall from class that we can set up an experiment to verify Hooke's law using a spring, various masses, a scale, and a measuring stick. Below are five collected data points relating to Hooks Law.



From this data, we can calculate  $u_i$ , the displacement of movable end of spring in measurement i. We can also create a mathematical model in the form

#### $y_i = b + k \cdot u_i$

where  $y_i$  is the modeled force associated with displacement  $u_i$ . Choose the correct matrix-vector model for generating vector  $\mathbf{y} \in \mathbb{R}^5$  given any choice of  $b, k \in \mathbb{R}$ .

| A. | $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ | = | 1<br>1<br>1<br>1 | $\begin{array}{c} 0.140 \\ 0.191 \\ 0.248 \\ 0.303 \\ 0.360 \end{array}$ | $\begin{bmatrix} b\\k \end{bmatrix}$ | В. | $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} =$ | $\begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix}$ | $\begin{array}{c} 0.0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{array}$ | $\begin{bmatrix} b\\k \end{bmatrix}$ | C. | $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ | = | $\begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix}$ | $\begin{array}{c} 0.000\\ 0.051\\ 0.108\\ 0.163\\ 0.220 \end{array}$ | $\begin{bmatrix} b \\ k \end{bmatrix}$ | D. | $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ | = | $\begin{bmatrix} 0.140 \\ 0.191 \\ 0.248 \\ 0.303 \\ 0.360 \end{bmatrix}$ | $\begin{bmatrix} 0.0 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$ | $\begin{bmatrix} b\\k \end{bmatrix}$ |
|----|---|---|------------------|--|--------------------------------------|----|---|--|--|--------------------------------------|----|---|---|--|--|--|----|---|---|---|---|--------------------------------------|
|----|---|---|------------------|--|--------------------------------------|----|---|--|--|--------------------------------------|----|---|---|--|--|--|----|---|---|---|---|--------------------------------------|

12. Which of the following represents the 8-bit binary representation of the number 207?

| A. 11110011 D. 11010001 C. 10001011 D. 10111111 E. 11001111 | A. 11110011 | B. 11010001 | C. 10001011 | D. 10111111 | E. 11001111 |
|---|-------------|-------------|-------------|-------------|-------------|
|---|-------------|-------------|-------------|-------------|-------------|

13. Let  $A \in \mathbb{R}^{12 \times 7}$  and  $B \in \mathbb{R}^{12 \times 6}$ . Suppose  $C = B^T A$ . What are the dimensions of C(:,2)?

A.  $7 \times 1$  B.  $6 \times 1$  C.  $6 \times 7$  D.  $7 \times 6$  E.  $1 \times 6$ 

14. Let

 $C([0,1]) = \{f(x) \mid \text{function } f:[0,1] \to \mathbb{R} \text{ is a continuous function on interval } [0,1]. \}$  $C^{(1)}([0,1]) = \{f(x) \mid \text{function } f:[0,1] \to \mathbb{R} \text{ has a continuous first derivative } f'(x) \text{ on interval } [0,1]. \}$ 

In other words, C([0,1]) is the set of continuous functions  $f:[0,1] \to \mathbb{R}$  while  $C^{(1)}([0,1])$  is the set of functions  $f:[0,1] \to \mathbb{R}$  with continuous first derivatives. From Math 1A (Single-Variable, Differential Calculus), we know that if f(x) is differentiable on [0,1], then f is continuous on [0,1]. Identify the set theoretic formulation of this theorem:

A. 
$$C([0,1]) \subseteq C^{(1)}([0,1])$$
 B.  $C([0,1]) = C^{(1)}([0,1])$  C.  $C([0,1]) \cap C^{(1)}([0,1]) = \emptyset$   
D.  $C^{(1)}([0,1]) \subseteq C([0,1])$  E.  $C([0,1]) \cup C^{(1)}([0,1]) = C^{(1)}([0,1])$ 

15. Let A and B be sets. What does it mean if we say that A is a subset of B?

- A. Some element x in A is also an element of B.
- B. A is an element of B.
- C. Every element x in A is contained in some element y of B.
- D. Every element x in A is also an element of B.
- E. Every element y in B is also an element of A.



17. Consider the following expression:

$$\begin{bmatrix} 9 & 5 & 3 \\ 8 & 0 & 2 \\ 7 & -6 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} -5 & 2 & 0 \end{bmatrix}$$

Using the properties of matrix-matrix multiplication and matrix-matrix addition, which of the following represents the given expression:

A.
$$\begin{bmatrix} 9 & 5 & 3 \\ 8 & 0 & 2 \\ 7 & -6 & 1 \end{bmatrix}$$
B. $\begin{bmatrix} 3 & 5 & 3 \\ -2 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 9 & 5 & -3 \\ 18 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 9 & 5 & -3 \\ -2 & -4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 9 & 5 & -3 \\ -2 & 4 & 2 \\ 7 & -6 & 1 \end{bmatrix}$ 

Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix. Use your entries for the incidence matrix to identify the correct answer for problems 18 - 19.



18. Let A represent the  $5 \times 7$  incidence matrix. Then the entry  $a_{35}$  is given by which of the following:

| A. $a_{35} = 2$ | B. $a_{35} = -1$ | C. $a_{35} = 0$ | D. $a_{35} = 1$ | E. $a_{35} = e_7$ |
|-----------------|------------------|-----------------|-----------------|-------------------|
|-----------------|------------------|-----------------|-----------------|-------------------|

19. Let A represent the  $5 \times 7$  incidence matrix. Then A(:, 6) is given by which of the following:

| A. $\begin{bmatrix} 0\\ -1\\ 0\\ 1\\ 0 \end{bmatrix}$ | B. $\begin{bmatrix} 1\\0\\0\\0\\-1 \end{bmatrix}$ | C. $\begin{bmatrix} 0\\0\\1\\0\\-1 \end{bmatrix}$ | D. $\begin{bmatrix} -1\\0\\0\\0\\1 \end{bmatrix}$ | E. $\begin{bmatrix} 0\\1\\0\\-1\\0\end{bmatrix}$ |
|---|---|---|---|--|
|---|---|---|---|--|

20. Let the following matrix  $A \in \mathbb{R}^{4 \times 5}$  be the incidence matrix for a directed graph:

| Incidence Matrix |       |       |       |       |       |  |  |  |  |  |
|------------------|-------|-------|-------|-------|-------|--|--|--|--|--|
|                  | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |  |  |  |  |  |
| $N_1$            | -1    | -1    | 0     | 0     | 0     |  |  |  |  |  |
| $N_2$            | 0     | 1     | -1    | 0     | 1     |  |  |  |  |  |
| $N_3$            | 1     | 0     | 1     | -1    | 0     |  |  |  |  |  |
| $N_4$            | 0     | 0     | 0     | 1     | -1    |  |  |  |  |  |

This matrix corresponds to which of the following directed graphs:



## Free Response

21. (10 pts) Suppose that  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  are nonzero, linearly independent vectors. Prove that

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

where  $\theta$  is the angle between **x** and **y**.

*Proof.* Assume  $\mathbf{x}$  and  $\mathbf{y}$  are not scalar multiples of each other (i.e. assume  $\mathbf{x}$  and  $\mathbf{y}$  are linearly independent). Suppose we begin with two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Consider the triangle defined by these vectors. The length of each side of this triangle can be given by the 2-norm of the vectors:



By the Law of Cosines, we know

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

Recall, using the algebraic properties of the inner product, we can write

$$\|\mathbf{x} - \mathbf{y}\|^{2} = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$$
$$= \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{y})$$
$$= \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y}$$
$$= \|\mathbf{x}\|^{2} - 2 \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^{2}$$

With this we see

$$\|\mathbf{x}\|^{2} + \|\mathbf{y}\|^{2} - 2\|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta) = \|\mathbf{x}\|^{2} - 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^{2}$$

By canceling out the appropriate terms using our knowledge of arithmetic, we see

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta).$$

Thus we see that the cosine formula for the inner product holds

Remark: Students who want to earn above a 90% on this problem should also be ready to prove the pythagorean theorem and the law of cosines.



### 22. (10 pts) Consider the following diagram of the mass-spring chain:

A. Generate vector models (using appropriate matrices and vectors) to define each of the following:

 $\mathbf{u}, \mathbf{e}, \mathbf{F}_s, \mathbf{y},$ 

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

B. Using your vector models from above, describe  $\mathbf{y}$  as a matrix-vector product with stiffness matrix K and vector  $\mathbf{u}$ . Demonstrate how to calculate K and explicitly calculate it's value in general.

C. Show how to use Newton's second law leads to an equation of the form

 $K\mathbf{u}=\mathbf{F}_e$ 

where  $\mathbf{F}_{e}$  represents the vector of external forces on each mass.

23. (10 pts) Describe, in detail, each of the following problems. For each problem, your should:

- i. Identify the problem statement
- ii. Identify the given and unknown quantities (explicitly identify relevant dimensions)
- iii. Identify the function description of this problem (explicitly discuss domain, codomain and range)
- iv. Describe how each problem is similar to and different from the other two problems in the list below.
- A. The Matrix-Vector Multiplication Problem

B. The Linear-Systems Problem

C. The Least-Squares Problem

# Challenge Problem

24. (Optional, Extra Credit, Challenge Problem) Prove that the number of linearly independent columns of a general  $m \times n$  matrix is equal to the number of linearly independent rows of that matrix.

Use for Scratch Work