Name : ____________ © Jeffrey A. Anderson **Class Number**: ______

Math 2B: Applied Linear Algebra
Exam 2, Version 4B

How long is this exam?

- This exam is scheduled for a 135 minute period.
- Make sure you have 5 sheets of paper (10 pages front and back) including this cover page.
- There are a total of 8 separate questions on this exam including:
  - 6 Free-response questions (50 points)
  - 1 Optional, extra credit challenge problem (5 points)

How will your written work be graded on these questions?

- Your work should show evidence of original thought and deep understanding. Work that too closely resembles the ideas presented in Jeff's lesson notes will likely NOT earn top scores. Work that does not demonstrate individualized, nuanced understanding will likely NOT earn top scores.
- Read the directions carefully. Your work will be graded based on what you are being asked to do.
- In order to earn a top score, please show all your work. In most cases, a correct answer with no supporting work will NOT earn top scores. What you write down and how you write it are the most important means of getting a good score on this exam.
- Neatness and organization are IMPORTANT! Do your best to make your work easy to read.
- You will be graded on accurate use of the notation we studied in this class.

What can you use on this exam?

- You may use no more than six note sheets (double-sided) or twelve note sheets (single-sided).
- Each note sheet is to be no larger then 11-inches by 8.5-inches (standard U.S. letter-sized paper).
- You must be the author of your own notes. and your note sheets must be handwritten (in YOUR OWN handwriting).
- PLEASE SUBMIT ALL OF YOUR NOTE SHEETS WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

What other rules govern your participation during this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note sheets), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

1. (8 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then use the algebraic properties of the inner product and 2-norm to prove

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 = 2\left(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\right)$$

Draw a diagram associated with this problem and interpret this result geometrically.

2. (8 points) Let $n \in \mathbb{N}$ with $i, k \in [n]$ and $i \neq k$. Use the definition of the transposition matrix P_{ik} to find the output of the product $\mathbf{e}_k^T \cdot P_{ik} \cdot \mathbf{e}_i$. Show your work.

3. (8 points) Here Consider the list set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}$$

A. (2 points) Show that these vectors are linear dependent by demonstrating that you can write one of these vectors as a linear combination of the other three.

B. (2 points) For $k \in [4]$, define $A \in \mathbb{R}^{6\times 4}$ with $A(:,k) = \mathbf{a}_k$. Let $B = A^T$. Find a nonzero vector $\mathbf{y} \in \mathbb{R}^4$ such that

$$\mathbf{y}^T \cdot B = \mathbf{0}$$

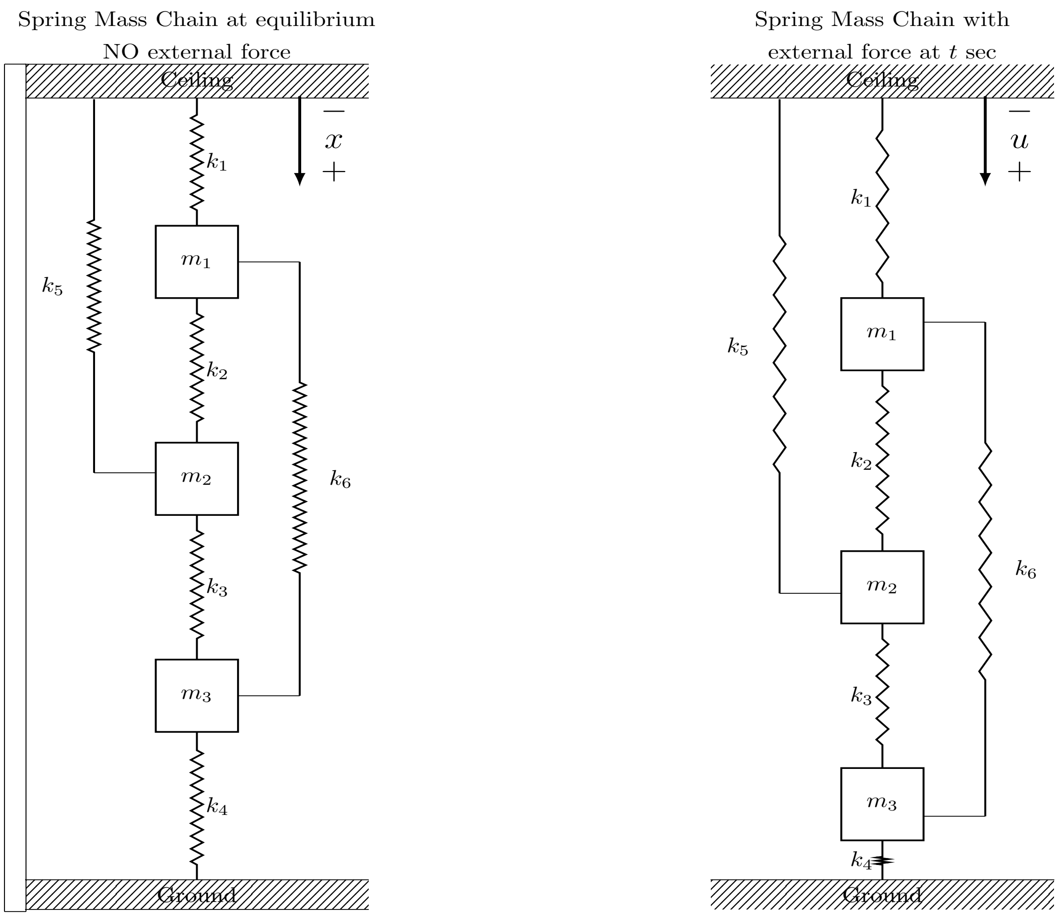
C. (2 points) Is the vector $\mathbf{y} \in \mathbb{R}^4$ that you found in part B above unique? Explain your reasoning.

D. (2 points) Find a vector in $\mathbb{R}^4$ that is not in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. Justify your answer.

4. (8 points) Suppose we let $n = 5$. Recall that if $i, j, k \in [n]$ with $i \neq k$, we defined the elementary matrices $S_{ik}(c), D_j(c)$, and P_{ik} for a general scalar coefficient $c \in \mathbb{R}$. Using these definitions, find the matrix sum given by

$$D_3(4) - D_2(-2) + S_{52}(3) - S_{25}(3) + P_{24} + P_{15}$$

5. (10 points) For the problem below, consider the following model for a 3-mass, 6-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly. Finally, assume that the massed move only in one axis and that the masses do not rotate in this system.



A. (2 points) Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t, and the displacement vector, respectively (as discussed in class and in our lesson notes).

B. (2 points) Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below. As a hint, remember there should be one entry of $\mathbf{e}(t)$ for each spring in the system.

C. (2 points) Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

D. (2 points) Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$

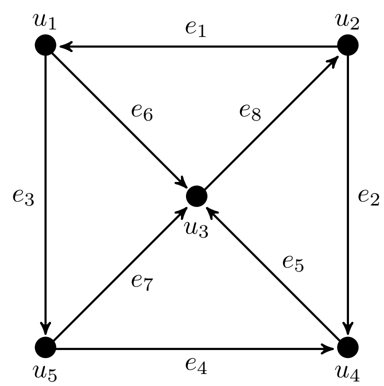
of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K.

E. (2 points) Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

6. (6 points) Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix table. Let $A \in \mathbb{R}^{8\times 5}$ be the incidence matrix corresponding to this directed graph.



Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_5
e_1					
e_2					
e_3					
e_4					
e_5					
e_6					
e_7					
e_8					

Find a nonzero vector $\mathbf{x} \in \mathbb{R}^5$ such that

$$A \cdot \mathbf{x} = \mathbf{0}$$

Then determine whether the columns of A are linearly independent vectors. Explain your reasoning.

Challenge Problem

7. (Optional, Extra Credit, Challenge Problem) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be a column vectors. Recall that we defined the 2-norm of $\mathbf{x}$ to be

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

This is one example of a much larger class of vector norms, known as $p-$norms. To create a $p-$norm, we choose a real number $p \geq 1$ and set

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Using this definition, suppose we have two real numbers $p, q \in \mathbb{R}$ such that $p > 1$, $q > 1$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, show that

$$|\mathbf{x}^T \cdot \mathbf{y}| \leq \|\mathbf{x}\|_p \cdot \|\mathbf{x}\|_q$$