1. (8 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then use the algebraic properties of the inner product and 2-norm to prove

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 = 2\left(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\right)$$

Draw a diagram associated with this problem and interpret this result geometrically.

Solution: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider:

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x}+\mathbf{y}) + \mathbf{y} \cdot (\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|_2^2 + 2\ \mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|_2^2\end{aligned}$$

Using the same reasoning, we can find

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|_2^2 &= (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x}-\mathbf{y}) - \mathbf{y} \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|_2^2 - 2\ \mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|_2^2\end{aligned}$$

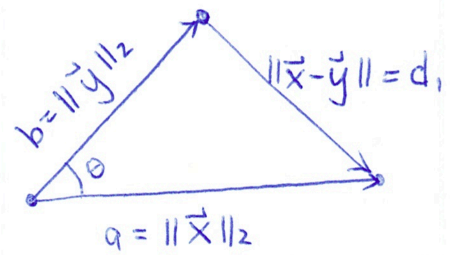
Then, combining these two results, we see

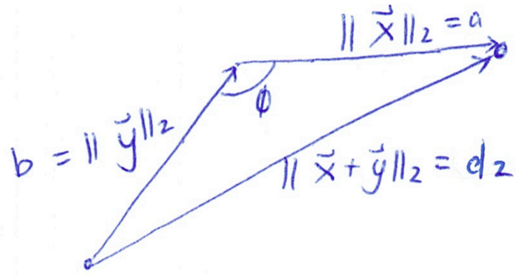
$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 &= \|\mathbf{x}\|^2 + 2\ \mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 - 2\ \mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|_2^2 \\ &= 2\|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2^2 \\ &= 2\Big(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\Big)\end{aligned}$$

This is exactly what we wanted to show.

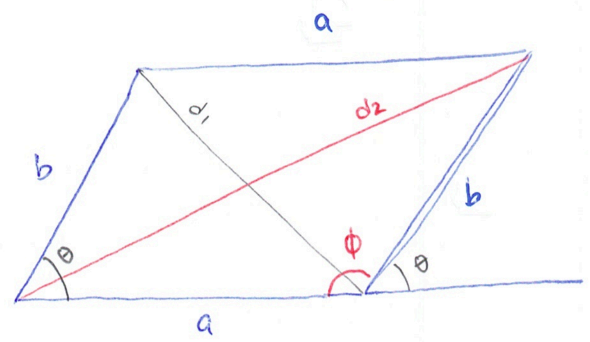
Geometric Interpretation : Suppose $\vec{x} \notin \text{Span}\{\vec{y}\}$.

Let's consider the two triangles





Now, lets combine these together on same diagram



Notice that $\theta + \phi = 180 \implies \phi = 180 - \theta$

By the law of cosines, note that

- $d_1^2 = a^2 + b^2 - 2ab\cos(\theta)$
- $d_2^2 = a^2 + b^2 - 2ab\cos(\theta)$

In this problem, we showed that

$$d_1^2 + d_2^2 = \|\vec{x} - \vec{y}\|_2^2 + \|\vec{x} + \vec{y}\|_2^2 = 2\left(\|\vec{x}\|_2^2 + \|\vec{y}\|_2^2\right) = 2\left(a^2 + b^2\right)$$

This is what we expect via law of cosines since

$$d_1^2 + d_2^2 = a^2 + b^2 - 2ab\cos(\theta) + a^2 + b^2 - 2ab\cos(\phi)$$

$$= 2(a^2 + b^2) - 2ab\left(\cos(\theta) + \cos(\phi)\right)$$

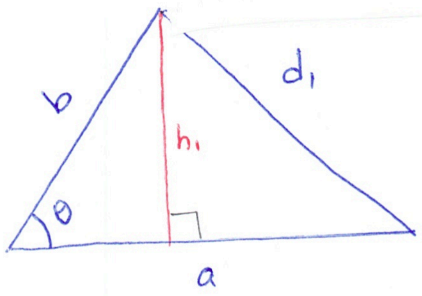
$$= 2(a^2 + b^2) - 2ab\left(\cos(\theta) + \cos(180 - \theta)\right)$$

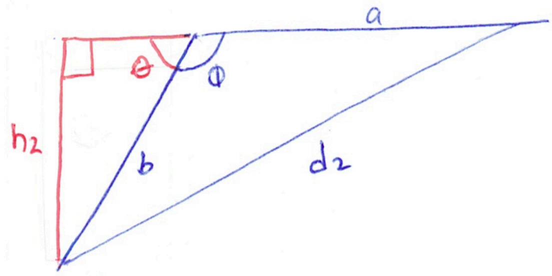
$$= 2(a^2 + b^2) - 2ab\left(\underbrace{\cos(\theta) - \cos(\theta)}_{\text{equals } 0}\right)$$

$$= 2(a^2 + b^2) + 0$$

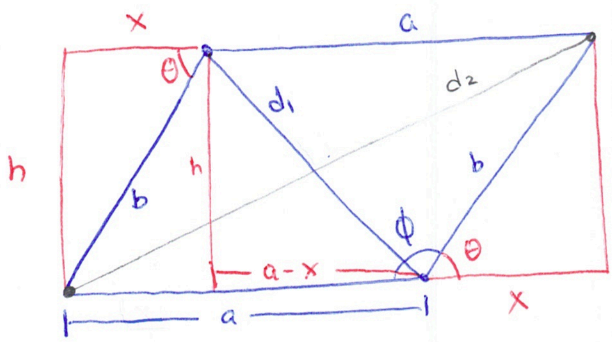
$$= 2(a^2 + b^2)$$

Notice, in the diagrams below we have $h_1 = h_2$

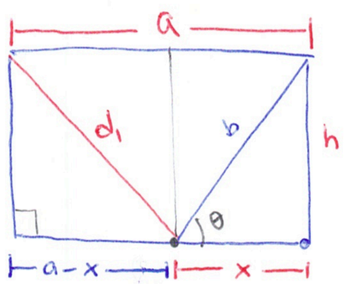




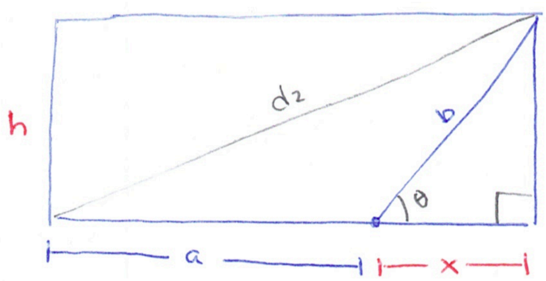
We can combine diagrams



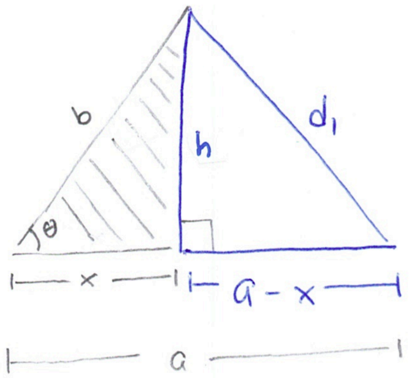
Here we have two rectangles:

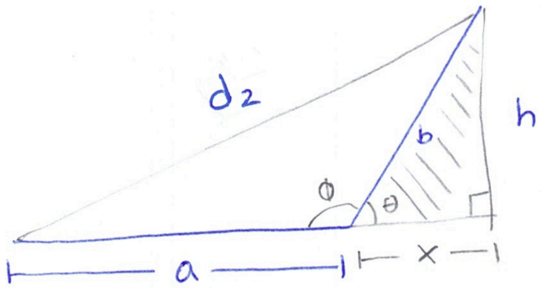


small rectangle



large rectangle





$$d_1^2 = h^2 + (a-x)^2$$

$$d_2^2 = h^2 + (a+x)^2$$

$$\Rightarrow \quad d_1^2 + d_2^2 = a^2 - 2ax + x^2 + h^2 + a^2 + 2ax + x^2 + h^2$$

$$= 2a^2 + 2(x^2 + h^2)$$

$$= 2a^2 + 2b^2$$

$\Rightarrow$ the sum of the squares of the lengths of a parallelogram equals 2 times the sum of the squares of the lengths of the sides of that parallelogram.

2. (8 points) Let $n \in \mathbb{N}$ with $i, k \in [n]$ and $i \neq k$. Use the definition of the transposition matrix P_{ik} to find the output of the product $\mathbf{e}_k^T \cdot P_{ik} \cdot \mathbf{e}_i$. Show your work.

Solution: Let's consider a specific case to build intuition. To this end, assume $n = 6$ with $i = 2$ and $k = 5$. Then, we consider the product

$$\mathbf{e}_5^T \cdot P_{25} \cdot \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's read left-to-right and focus on the first product:

$$\mathbf{e}_5^T \cdot P_{25} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, we take linear combinations of the rows of P_{25} using scalar coefficients from the entries of $\mathbf{e}_5^T$. Based on the structure of the vector $\mathbf{e}_5^T$, we see that we pick off the fifth row of P_{25} in this operation. Now when we multiply on the right by $\mathbf{e}_2$, we see

$$\mathbf{e}_5^T \cdot P_{25} \cdot \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$$

Thus, we see that

$$\mathbf{e}_5^T \cdot P_{25} \cdot \mathbf{e}_2 = 1.$$

We could have equivalently began with the product $P_{25} \cdot \mathbf{e}_2$ via the column partition version of matrix-vector multiplication and the proceeded to multiply the output on the left by $\mathbf{e}_5^T$. The result is the same either way.

3. (8 points) Here Consider the list set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}$$

A. (2 points) Show that these vectors are linear dependent by demonstrating that you can write one of these vectors as a linear combination of the other three.

Solution: We see that we can write $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ as follows:

$$\mathbf{a}_4 = -2 \cdot \mathbf{a}_1 + -2 \cdot \mathbf{a}_2 + -\frac{1}{2} \cdot \mathbf{a}_3$$

Notice, that with the same technique we discussed in lecture, we can find a set of nonzero scalars that combine to produce the zero vector with

$$\mathbf{0} = 2 \cdot \mathbf{a}_1 + 2 \cdot \mathbf{a}_2 + \frac{1}{2} \cdot \mathbf{a}_3 + \mathbf{a}_4$$

If we set $\mathbf{x}^T = \begin{bmatrix} 2 & 2 & \frac{1}{2} & 1 \end{bmatrix}$ and we define the column-partition of a matrix $A \in \mathbb{R}^{6\times 4}$ with $A(:,k) = \mathbf{a}_k$ for $k \in [4]$, then we see that

$$A \cdot \mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ \frac{1}{2} \\ 1 \end{bmatrix} = \mathbf{0} \in \mathbb{R}^6$$

B. (2 points) For $k \in [4]$, define $A \in \mathbb{R}^{6\times 4}$ with $A(:,k) = \mathbf{a}_k$. Let $B = A^T$. Find a nonzero vector $\mathbf{y} \in \mathbb{R}^4$ such that

$$\mathbf{y}^T \cdot B = \mathbf{0}$$

Solution: This problem is asking us to re-interpret our observation from part A by setting $\mathbf{y} = \mathbf{x}$. Specifically, we see that for $i \in [4]$, we have $B(i,:) = [A(:,i)]^T$. Thus, the nonzero linear combination that sends the column vectors of A to the zero vector in $\mathbb{R}^6$ must be the same as the nonzero linear combination of the rows of B that get sent to the zero vector in $\mathbb{R}^{1\times 6}$. In other words, let's set

$$\mathbf{y}^T = \begin{bmatrix} 2 & 2 & \frac{1}{2} & 1 \end{bmatrix}$$

Then, when we consider

$$\mathbf{y}^T \cdot B = \begin{bmatrix} 2 & 2 & \frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 4 & 0 \\ 2 & -2 & 2 & 2 & -2 & 4 \end{bmatrix}$$

Using the row-partition version of (row) vector-matrix multiplication, we can take linear combinations of the rows of B using scalar coefficients from the vector $\mathbf{y}^T$, yielding

$$\begin{aligned}\mathbf{y}^T \cdot B = &+ 2 \cdot \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \\ &+ \frac{1}{2} \cdot \begin{bmatrix} 0 & 0 & -4 & 0 & 4 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 & -2 & 2 & 2 & -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

C. (2 points) Is the vector $\mathbf{y} \in \mathbb{R}^4$ that you found in part B above unique? Explain your reasoning.

Solution: No, the scalars we found above are not unique. We can take any scalar multiple of the vector $\mathbf{y}$ and the result will still produce a linear combination equal to zero:

$$(\alpha \mathbf{y})^T \cdot B = \alpha \cdot \mathbf{y}^T \cdot B = \alpha \cdot \mathbf{0} = \mathbf{0} \in \mathbb{R}^{1\times 6}$$

Thus, we see there are an infinite number of nonzero vectors that send the rows of B to zero via linear combination.

D. (2 points) Find a vector in $\mathbb{R}^4$ that is not in $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. Justify your answer.

Solution: Let A be the matrix defined above in Part B of this problem. Recall the definition of the span of a set of vectors is given by

$$\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \left\{\mathbf{b} \,:\, \mathbf{b} = A \cdot \mathbf{x} \text{ for any } \mathbf{x} \in \mathbb{R}^4\right\}$$

In other words, if we define the matrix-vector multiplication function $f(\mathbf{x}) = A \cdot \mathbf{x}$, then $\operatorname{Rng}(f) = \operatorname{Span}\{\mathbf{a}_k\}_{k=1}^4 \subseteq \mathbb{R}^6$. In this case, we want to find an element $\mathbf{z} \in \mathbb{R}^6$ such that for all $\mathbf{x} \in \mathbb{R}^4$ we have $\mathbf{z} \neq A \cdot \mathbf{x}$. With this in mind, let $\mathbf{x}$ be any vector in $\mathbb{R}^4$ and consider

$$\mathbf{z} = A \cdot \mathbf{x} = x_1 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 2 \\ 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_4 \\ x_2 - 2x_4 \\ -4x_3 + 2x_4 \\ -x_2 + 2x_4 \\ 4x_3 - 2x_4 \\ -2x_1 + 4x_4 \end{bmatrix}$$

With this, we notice the following patterns:

$$\operatorname{Entry}_6(\mathbf{z}) = 2 \cdot \operatorname{Entry}_1(\mathbf{z}), \qquad \operatorname{Entry}_4(\mathbf{z}) = - \cdot \operatorname{Entry}_2(\mathbf{z}), \qquad \operatorname{Entry}_5(\mathbf{z}) = - \cdot \operatorname{Entry}_3(\mathbf{z}).$$

To construct a vector $\mathbf{z}$ not in the span, we simply need to violate these patters. One way to do this is to set

$$\mathbf{z}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the first and sixth entry of this vector violate the structure of vectors in the span, we see that this vector cannot be in said span. This is exactly what we were asked to show.

4. (8 points) Suppose we let $n = 5$. Recall that if $i, j, k \in [n]$ with $i \neq k$, we defined the elementary matrices $S_{ik}(c), D_j(c)$, and P_{ik} for a general scalar coefficient $c \in \mathbb{R}$. Using these definitions, find the matrix sum given by

$$D_3(4) - D_2(-2) + S_{52}(3) - S_{25}(3) + P_{24} + P_{15}$$

Solution: Consider the difference of dilation matrices:

$$D_3(4) - D_3(4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We also know that the desired difference of shear matrices is given by:

$$S_{52}(3) - S_{25}(3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Finally, we have the sum of transposition matrices can be written as

$$P_{24} + P_{15} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

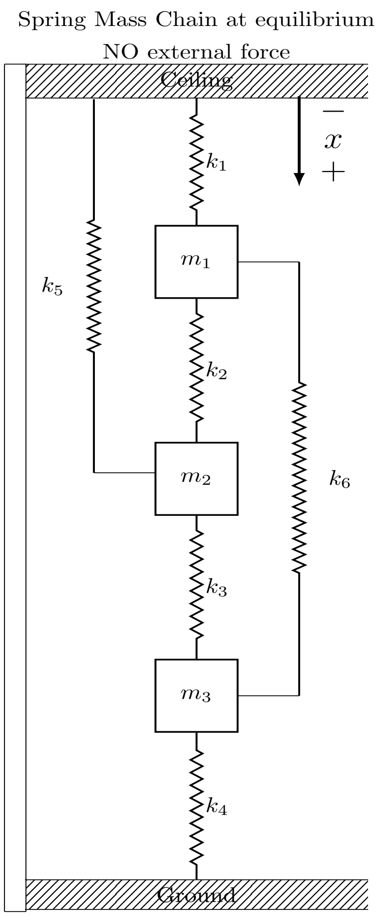
Then, our desired sum is given as

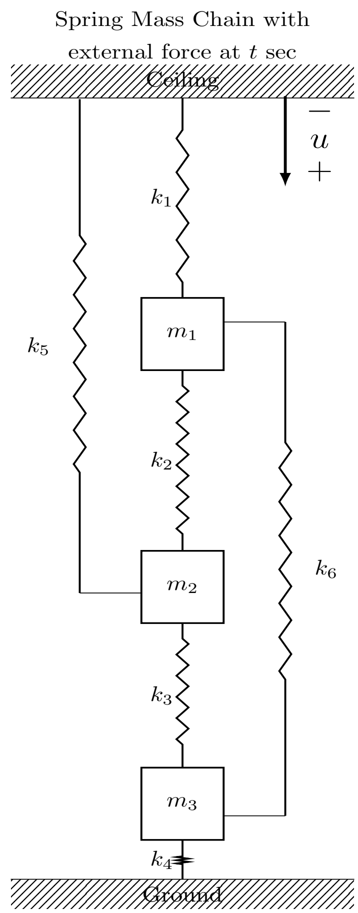
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which can be written succinctly as the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 & 3 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

5. (10 points) For the problem below, consider the following model for a 3-mass, 6-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly. Finally, assume that the massed move only in one axis and that the masses do not rotate in this system.





A. (2 points) Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t, and the displacement vector, respectively (as discussed in class and in our lesson notes).

Solution: Let's set up our model of the 2-mass, 5-spring chain.

POSITION VECTORS:

Let's define $\mathbf{x}_0 \in \mathbb{R}^3$ store the equilibrium positions of the center of each mass at initial time $t = t_0 = 0$ sec. In other words, for $i \in [3]$ we suppose x_i stores the measured position of the center of mass when the system is resting at equilibrium with no external force, zero acceleration, zero velocity, and zero net internal force on each mass. Since there are three masses, we track all three equilibrium positions and set

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

On the other hand, suppose we apply an external force to our system at some time and let the masses move dynamically. For $i \in [3]$, let's use the real-valued function $x_i(t)$ to store the measured position of the center of mass i along the metric ruler at time $t \in [t_0, T) \subseteq \mathbb{R}$. We can then store the position of the center of all three masses as a vector valued function $\mathbf{x}(t)$ where $\mathbf{x} : [t_0, T) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ with

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

DISPLACEMENT VECTOR:

With this we can set up our displacement vector $\mathbf{u}(t)$. In this case, we have assumed the zero position of our ruler to be on the ceiling at the top of our apparatus. Moreover, we orient positive position measurements in the downward direction (toward the ground below the masses). We want $u_i(t)$ to measure the displacement of mass i away from it's initial position. Since $x_i > x_i(t)$ in our diagram, we see that $u_i(t) > 0$ if and only if $x_i(t) - x_i > 0$. Thus, we define our displacement vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_1 \\ x_2(t) - x_2 \\ x_3(t) - x_3 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}(t) - \mathbf{x}_0$$

Solution: Remark (for students who want to earn top marks): **ORIENTATION OF RULER:**

A. In this case, we choose to orient our ruler so that the zero position was on the top of the apparatus and positive position measurements are in the downward direction. This guarantees that positive displacement measurements are oriented in the downward position calculated as the position of mass i at time t minus the equilibrium position.This follows the convention in physics to calculate displacement as "final" position minus "initial position." In other words, $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$.

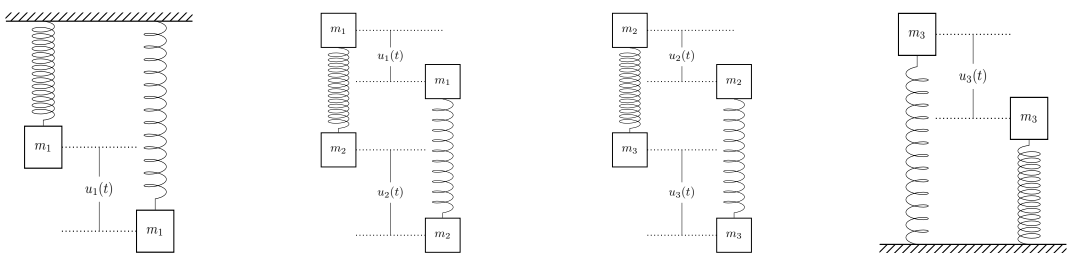
B. In this derivation, we assumed that the position of mass i at time t was described by a continuous function $x_i(t)$. As we will see, this very general modeling paradigm gives rise to an eigenvalue problem.

B. (2 points) Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

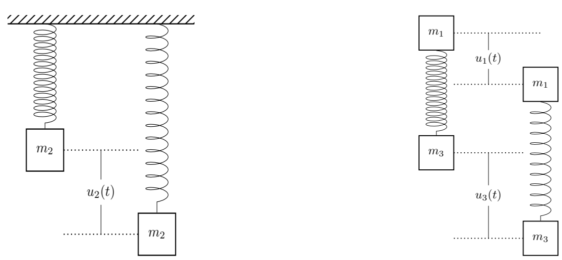
$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below. As a hint, remember there should be one entry of $\mathbf{e}(t)$ for each spring in the system.

Solution: Since the elongation vector $\mathbf{e}(t)$ has one entry for each spring and there are 6 springs in this system, we know that $\mathbf{e}(t)$ must output a 6×1 vector. The ith entry $e_i(t)$ of this elongation vector represents the "elongation" of spring i at time t. To measure elongation of the ith spring, we subtract the length of spring i when the system is at equilibrium from the length of this spring again at time t. To find the elongation of each spring, consider the following diagrams:



The four diagrams above follow the same pattern as discussed in lecture for our 2−mass, 3−spring system. However, in this system we have two additional springs labeled k_5 and k_6 which have a different pattern. To analyze these spring elongations, let's draw diagrams that simplify the problem and focus on the changes for these springs in particular:



Solution: Positive $e_i(t)$ values occur when the length of this spring at time t is larger than the length of this spring at equilibrium.

ELONGATION VECTOR:

Using these diagrams, we see that our desired elongation vector is given by

$$\mathbf{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \\ e_6(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) - u_1(t) \\ u_3(t) - u_2(t) \\ -u_3(t) \\ u_2(t) \\ u_3(t) - u_1(t) \end{bmatrix} = u_1(t) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + u_2(t) \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u_3(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We can write this linear combination as a matrix-vector product as following

$$\mathbf{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \\ e_6(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

where $\mathbf{u}$ is the 3×1 displacement vector from part (A) above. In this case, the matrix $A \in \mathbb{R}^{4\times 3}$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Thus, we write $\mathbf{e}(t)$ as a matrix vector product

$$\boxed{\mathbf{e}(t) = A \cdot \mathbf{u}(t)}. \tag{1}$$

C. (2 points) Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

Solution: Recall that Hooke's law states that the change in internal force stored inside spring i is directly proportional to the elongation of the spring. In other words, for a spring with spring constant k_i, Hooke's law states that

$$f_{s_i}(t) = k_i \cdot e_i(t)$$

The vector $\mathbf{f}_s(t)$ can stores the internal forces in each of the four springs in our system due to the elongations discussed in part (B) above.

FORCE VECTORS FOR SPRINGS:

Now, let's move onto finding the internal forces stored in each spring. To this end, let

$$\begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \\ f_{s_6}(t) \end{bmatrix} = e_1(t) \cdot \begin{bmatrix} k_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_2(t) \cdot \begin{bmatrix} 0 \\ k_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_3(t) \cdot \begin{bmatrix} 0 \\ 0 \\ k_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_4(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_4 \\ 0 \\ 0 \end{bmatrix} + e_5(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_5 \\ 0 \end{bmatrix} + e_6(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k_6 \end{bmatrix}$$

The force vector $\mathbf{f}_s(t)$ as the matrix-vector product

$$\mathbf{f}_s(t) = \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \\ f_{s_6}(t) \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \\ e_6(t) \end{bmatrix}$$

where $\mathbf{e}$ is our elongation vector from above. The diagonal matrix $C \in \mathbb{R}^{4\times 4}$ is defined as

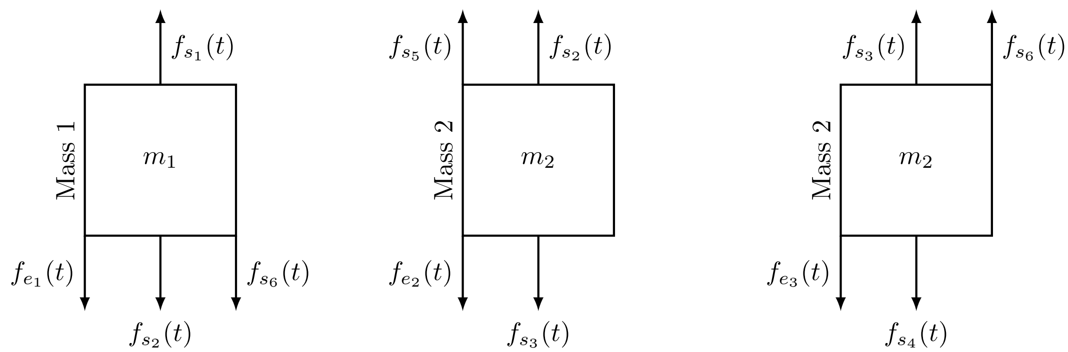
$$C = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}$$

We write

$$\boxed{\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)} \tag{2}$$

D. (2 points) Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector
$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$
of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K.

Solution:
Let's consider the net internal forces on each mass. To do so, we draw a free-body diagram for each mass in our system, as seen below.



When analyzing the net internal forces, we work to calculated the net force acting on each mass when considering ONLY the springs within the system. In other words, if the coupled chain of the masses and springs is our system, we want only to focus on the net forces within this system (and ignore external forces that may act on each mass for now).

NET INTERNAL FORCES FOR MASS-SPRING CHAIN:

We now introduce the vector $\mathbf{y}(t)$ to store the net force on each mass. Each entry $y_i(t)$ represents the difference between $f_s i(t)$ and $f_{s_{i+1}}(t)$. When writing the individual entries of $\mathbf{y}(t)$ we will assume that positive net forces result in positive displacements. Since we've oriented positive displacement in the downward direction, we also orient positive force in the downward direction.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} f_{s_2}(t) + f_{s_6}(t) - f_{s_1}(t) \\ f_{s_3}(t) - f_{s_2}(t) - f_{s_5}(t) \\ f_{s_4}(t) - f_{s_3}(t) - f_{s_6}(t) \end{bmatrix} = -\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \\ f_{s_6}(t) \end{bmatrix}$$

We transform this into a matrix-vector product

$$\boxed{\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)} \tag{3}$$

where A was defined in equation (1) for our model of the elongation vector $\mathbf{e}(t)$. Notice that we've factored out a negative sign in order to make this statement.

Solution: In this problem, we will using equations (1), (2), and (3) to create stiffness matrix K. To this end, note

$$\begin{aligned}
\mathbf{y}(t) &= -A^T \cdot \mathbf{f}_s(t) && \text{by equation (3)} \\
&= -A^T \cdot C \cdot \mathbf{e}(t) && \text{by equation (2)} \\
&= -A^T \cdot C \cdot A \cdot \mathbf{u}(t) && \text{by equation (1)} \\
&= -K \cdot \mathbf{u}(t)
\end{aligned}$$

If we let $K = A^T \cdot C \cdot A$, we can then write

$$\boxed{\mathbf{y}(t) = -K \cdot \mathbf{u}(t)} \tag{4}$$

We can form our stiffness matrix K explicitly using matrix-matrix multiplication with

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 + k_6 & -k_2 & -k_6 \\ -k_2 & k_2 + k_3 + k_5 & -k_3 \\ -k_6 & -k_3 & k_3 + k_4 + k_6 \end{bmatrix}$$

This is a tridiagonal, symmetric matrix.

E. (2 points) Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

Solution: From Newton's second law, we know that

$$\text{Net Force acting on an object} = \text{ Mass of object} \times \text{ Acceleration of object}$$

In order to state this law for each mass in our system, let

$$\ddot{u}_i(t) = \frac{d^2}{dt^2}\Big[u_i(t)\Big]$$

be the acceleration of mass i at time t for $i \in \{1, 2, 3\}$. Then, for mass i, the coefficient

$$m_i \cdot \ddot{u}_i(t)$$

measures the mass multiplied by the acceleration. Organizing all three of these values into a column vector yields

$$\begin{bmatrix} m_1\,\ddot{u}_1(t) \\ m_2\,\ddot{u}_2(t) \\ m_3\,\ddot{u}_3(t) \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{bmatrix}$$

By defining the 3×3 mass matrix

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

these mass times acceleration calculations can be written as a matrix-vector multiplication

$$M \cdot \ddot{\mathbf{u}}(t) \tag{5}$$

The next step is to calculate the net force on mass i. To do so, let's look back at our free-body diagram from part D above. We can calculate the net forces on all masses simultaneously using vector notation. To do so, we isolate internal net forces from external forces to find that We can organize all three net force calculations as a column vector

$$\begin{bmatrix} f_{s_2}(t) + f_{s_6}(t) - f_{s_1}(t) \\ f_{s_3}(t) - f_{s_2}(t) - f_{s_5}(t) \\ f_{s_4}(t) - f_{s_3}(t) - f_{s_6}(t) \end{bmatrix} + \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \\ f_{e_3}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \\ f_{e_3}(t) \end{bmatrix}$$

Thus, the net force calculations for each mass in this system is given by

$$\mathbf{y}(t) + \mathbf{f}_e(t) \tag{6}$$

where vector $\mathbf{f}_e(t)$ gives the external forces on each mass in our system. Newton's second law indicates that the mass times acceleration vector from equation (5) is equal to the net force vector from equation (6)

$$M \cdot \ddot{\mathbf{u}}(t) = \mathbf{y}(t) + \mathbf{f}_e(t)$$

Solution: Now, use the stiffness matrix K from equation (4) to represent the net internal force vector $\mathbf{y}(t)$, yielding

$$M \cdot \ddot{\mathbf{u}}(t) = -K \cdot \mathbf{u}(t) + \mathbf{f}_e(t)$$

By moving $-K$ onto the other side of the equation, we have

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}_e(t) \tag{7}$$

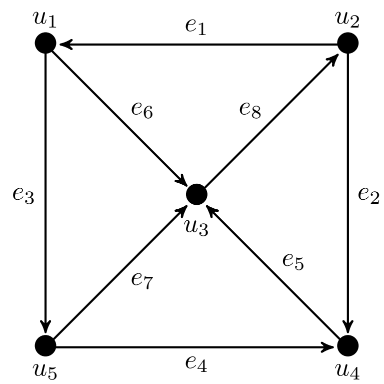
Since we have assume that we study the system at equilibrium for $t = T$, we know $\ddot{\mathbf{u}}(T) = \mathbf{0}$ and we have

$$K\,\mathbf{u}(T) = \mathbf{f}_e(T)$$

Remark (for students who want to earn top marks):

- In this derivation, we've used a very general approach to allow $t \in [t_0, T)$. Only at the very end of our work, did we substitute the value of $t = T$ to represent the case that our masses have settled down to equilibrium. As we will see, this general approach will come in very useful during our discussion of the eigenvalue-eigenvector problem.
- In fact, we have derived a coupled ordinary differential equation in the work above. For those of you that have taken (or will take) Math 2A at Foothill, you may notice that equation (7) is a vector version of the 2nd order differential equation for a harmonic oscillator with no damping and general forcing function.

6. (6 points) Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix table. Let $A \in \mathbb{R}^{8\times 5}$ be the incidence matrix corresponding to this directed graph.



Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_5
e_1	-1	1	0	0	0
e_2	0	1	0	-1	0
e_3	1	0	0	0	-1
e_4	0	0	0	-1	1
e_5	0	0	-1	1	0
e_6	1	0	-1	0	0
e_7	0	0	-1	0	1
e_8	0	-1	1	0	0

Find a nonzero vector $\mathbf{x} \in \mathbb{R}^5$ such that

$$A \cdot \mathbf{x} = \mathbf{0}$$

Then determine whether the columns of A are linearly independent vectors. Explain your reasoning.

Solution: From our table above, we notice that the entry-by-entry definition of our incidence matrix $A \in= \mathbb{R}^{8\times 5}$ corresponding to the given directed graph is

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

This matrix has very special structure that arises from our definition. In particular, in each row of A we notice that there are exactly two nonzero entries whose values are 1 and -1. Thus, if we were to sum all coefficients in each row, we would produce the zero scalar. However, when summing all values in each row is equivalent to multiplying the matrix A on the right by the vector $\mathbf{1} \in \mathbb{R}^5$ that has five entries, each of which is equal to one. In other words, we see

$$A \cdot \mathbf{1} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is exactly what we wanted to produce. Moreover, we know that the columns of A are linearly dependent since we can find a nonzero linear combination of these columns that sends the columns to zero.

Challenge Problem

7. (Optional, Extra Credit, Challenge Problem) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be a column vectors. Recall that we defined the 2-norm of $\mathbf{x}$ to be

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

This is one example of a much larger class of vector norms, known as $p-$norms. To create a $p-$norm, we choose a real number $p \geq 1$ and set

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Using this definition, suppose we have two real numbers $p, q \in \mathbb{R}$ such that $p > 1$, $q > 1$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, show that

$$|\mathbf{x}^T \cdot \mathbf{y}| \leq \|\mathbf{x}\|_p \cdot \|\mathbf{x}\|_q$$

