Name : ______________________ **Class Number**: ______

Exam 2, Version 3A

Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones or put your cell phones into airplane mode during this exam. Please place your cell phones inside your bag. No cell phones will be allowed on your desk.
- Close your bag and put it under your seat.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 7 sheets of paper (14 pages front and back) including this cover page.
- There are a total of 20 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 10 Multiple Choice Questions (50 points)
  - 4 Free-Response Questions (40 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger than 8.5 inches by 11 inches This note card must be handwritten. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T F Any system of n linear equations in n variables has at most n solutions.

2. T F Let $m, n \in \mathbb{N}$. Suppose that $A \in \mathbb{R}^{m\times n}$. Then, $\mathbf{0} \in \mathbb{R}^m$ is always in the span of the columns of A.

3. T F If a square matrix $A \in \mathbb{R}^{n\times n}$ has a zero on its main diagonal, then it is singular.

4. T F Let $n \in \mathbb{N}$ and $A, B \in \mathbb{R}^{n\times n}$. If $A \cdot B = 0 \in \mathbb{R}^{n\times n}$, then either $A = 0$ or $B = 0$.

5. T F Let $A \in \mathbb{R}^{5\times 5}$ such that $A = A^T$. Then

$$A \cdot P_{24} = P_{24} \cdot A$$

for permutation matrix $P_{24} \in \mathbb{R}^{5\times 5}$.

Multiple Choice (50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

6. We can use matrix-matrix multiplication to verify the following equation:

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&2&1\end{bmatrix}}_{L_3}\cdot\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&-1&0&1\end{bmatrix}}_{L_2}\cdot\underbrace{\begin{bmatrix}1&0&0&0\\-2&1&0&0\\3&0&1&0\\-1&0&0&1\end{bmatrix}}_{L_1}\cdot\underbrace{\begin{bmatrix}2&-4&0&2\\4&-4&-2&5\\-6&4&5&-6\\2&0&-4&1\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}2&-4&0&2\\0&4&-2&1\\0&0&1&2\\0&0&0&2\end{bmatrix}}_{U}$$

Choose the unit lower triangular factor $L \in \mathbb{R}^{4\times 4}$ from the LU factorization of A:

A. $\begin{bmatrix}1&0&0&0\\-2&1&0&0\\3&2&1&0\\-1&-1&2&1\end{bmatrix}$ B. $\begin{bmatrix}1&0&0&0\\2&1&0&0\\-7&-2&1&0\\17&5&-2&1\end{bmatrix}$ C. $\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&2&1&0\\-1&3&2&1\end{bmatrix}$ D. $\begin{bmatrix}1&0&0&0\\2&1&0&0\\-3&-2&1&0\\1&1&-2&1\end{bmatrix}$

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7. Suppose we know that

$$\underbrace{\begin{bmatrix}1&2&-1&0\\-3&-5&6&1\\-1&2&8&-2\\2&4&-2&-1\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0&0\\-3&1&0&0\\-1&4&1&0\\2&0&0&1\end{bmatrix}}_{L}\cdot\underbrace{\begin{bmatrix}1&2&-1&0\\0&1&3&1\\0&0&-5&-6\\0&0&0&-1\end{bmatrix}}_{U}$$

Suppose we want to solve the linear-systems problem

$$\underbrace{\begin{bmatrix}1&2&-1&0\\-3&-5&6&1\\-1&2&8&-2\\2&4&-2&-1\end{bmatrix}}_{A}\cdot\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{\mathbf{x}}=\underbrace{\begin{bmatrix}2\\-7\\-13\\2\end{bmatrix}}_{\mathbf{b}}$$

Let $A \in \mathbb{R}^{4\times 4}$ be the coefficient matrix in this problem and $\mathbf{b} \in \mathbb{R}^4$ be the vector on the right-hand side. Using the matrices $L, U \in \mathbb{R}^{4\times 4}$, where the L and U factors of A, respectively, from the the LU factorization of A, solve the two linear system problems

$$L \cdot \mathbf{y} = \mathbf{b}, \qquad\qquad U \cdot \mathbf{x} = \mathbf{y}$$

Find the value of $\mathbf{x} \cdot \mathbf{y}$:

A. 5 B. 13 C. 4 D. -9 E. None of these

8. Which one of the following statements about elementary matrices in $\mathbb{R}^{4\times 4}$ is true?

A. $\left[S_{31}(2)\right]^{-1} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\left[P_{13}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

C. $\left[S_{24}(4)\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\left[D_4\left(\frac{1}{5}\right)\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

E. $\left[S_{34}(1/4)\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}$

9. Define the matrix $B \in \mathbb{R}^{4\times 4}$ by the following product:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Using this definition, which of the following gives a_{13}?

A. b_{14} B. b_{34} C. b_{43} D. b_{13} E. b_{41}

10. Let $A \in \mathbb{R}^{n\times n}$. Suppose you multiply A on the left by a sequence of elementary matrices, with

$$E_t \cdots E_2 \cdot E_1 \cdot A = U$$

where $U \in \mathbb{R}^{n\times n}$ is upper-triangular matrix U and $u_{nn} = 0$. Which of the following must be true:

A. A^{-1} exists

B. There exists some $\mathbf{b} \in \mathbb{R}^n$ that cannot be written as a linear combination of the columns of A.

C. $\text{Span}\{A(:,1), A(:,2), ..., A(:,n)\} = \mathbb{R}^n$

D. There exists a nonsingular matrix $E \in \mathbb{R}^{n\times n}$ such that $E \cdot A = I_n$

E. U has n pivot elements

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11. Let $A \in \mathbb{R}^{4\times 4}$ be defined by the following column partition:

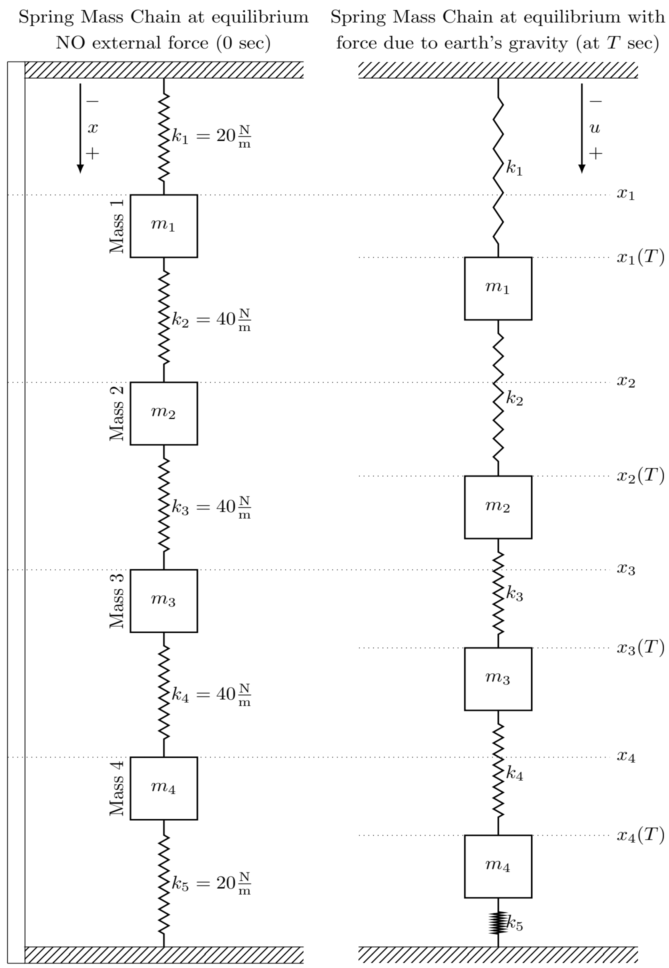
$$A(:,1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad A(:,2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad A(:,3) = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \qquad A(:,4) = \begin{bmatrix} 11 \\ -1 \\ 7 \\ 9 \end{bmatrix}$$

We can confirm that $A(:,4) = 4 \cdot A(:,1) - A(:,2) + 2 \cdot A(:,3)$. Choose a nonzero vector $\mathbf{x} \in \mathbb{R}^4$ such that

$$A \cdot \mathbf{x} = \mathbf{0}$$

A. $\begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ B. $\begin{bmatrix} -4 \\ 1 \\ -2 \\ -1 \end{bmatrix}$ C. $\begin{bmatrix} 4 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ D. $\begin{bmatrix} 8 \\ -2 \\ 4 \\ -2 \end{bmatrix}$ E. Only $\mathbf{x} = \mathbf{0}$ solves this problem

For the following two problems, consider the following model for a 4-mass, 5-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly.



12. Recall our model for the mass-spring chain above is given by $M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}_e(t)$. Also, remember that the initial position vector $\mathbf{x}_0$ and the final position vector $\mathbf{x}(T)$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and when $t = T$ respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.150 \\ 0.350 \\ 0.600 \\ 0.800 \end{bmatrix} \qquad \mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \\ x_4(T) \end{bmatrix} = \begin{bmatrix} 0.353 \\ 0.630 \\ 0.859 \\ 0.989 \end{bmatrix}$$

where each entry is given in meters. Which of the following gives the mass vector $\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix}^T$. as measured in kg, used to produce this position data? If necessary, please round your answers to the nearest 3 places after the decimal.

A. $\begin{bmatrix} 0.98 \\ 3.92 \\ 1.96 \\ 0.98 \end{bmatrix}$ B. $\begin{bmatrix} 0.203 \\ 0.280 \\ 0.259 \\ 0.189 \end{bmatrix}$ C. $\begin{bmatrix} 0.100 \\ 0.971 \\ 1.786 \\ 2.029 \end{bmatrix}$ D. $\begin{bmatrix} 0.100 \\ 0.400 \\ 0.200 \\ 0.100 \end{bmatrix}$ E. $\begin{bmatrix} -1.729 \\ 0.200 \\ 0.957 \\ 0.057 \end{bmatrix}$

13. Recall that the initial position vector $\mathbf{x}_0$ and the mass vector $\mathbf{m}$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and the mass measurements, measured in kg, respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.5 \\ 0.7 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.100 \\ 0.200 \\ 0.200 \\ 0.100 \end{bmatrix}$$

measured in kg. Under the same assumptions in the problem above, which of the following gives the position vector $\mathbf{x}(T)$ as measured in meters, at $t = T$ when the system is at equilibrium under the force of gravity on earth. If necessary, please round your answers to the nearest 3 places after the decimal.

A. $\begin{bmatrix} 0.147 \\ 0.196 \\ 0.196 \\ 0.147 \end{bmatrix}$ B. $\begin{bmatrix} 0.015 \\ 0.020 \\ 0.020 \\ 0.015 \end{bmatrix}$ C. $\begin{bmatrix} 0.247 \\ 0.496 \\ 0.696 \\ 0.847 \end{bmatrix}$ D. $\begin{bmatrix} 0.115 \\ 0.320 \\ 0.520 \\ 0.715 \end{bmatrix}$ E. None of these

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14. Let $A \in \mathbb{R}^{5\times 4}$, $B^T \in \mathbb{R}^{6\times 5}$, and $C \in \mathbb{R}^{3\times 6}$. Let the matrix D be formed by the product

$$D^T = A^T \cdot B \cdot C^T$$

What are the dimensions of the matrix $[D(:,2)]^T$?

A. 3×1 B. 4×1 C. 1×3 D. 1×4 E. Product D is undefined

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15. Let A be a given $\mathbb{R}^{3\times 3}$ matrix such that the following holds

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} = \begin{bmatrix} 5 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 4/5 \end{bmatrix}$$

Which of the following cannot be true:

A. A has linearly independent columns
B. A is nonsingular
C. Any $\mathbf{b} \in \mathbb{R}^3$ will be linearly dependent on the columns of A
D. $\|A\mathbf{x} - \mathbf{b}\|_2 = 0$ for all $\mathbf{x} \in \mathbb{R}^3$
E. There is a nonzero $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{0}$

Free Response

16. Let $L, M \in \mathbb{R}^{4\times 4}$ be lower-triangular matrices. Prove that the product

$$A = L \cdot M$$

is also a lower-triangular matrix.

17. Find the LU Factorization of the matrix $A \in \mathbb{R}^{3\times 3}$. Show your work and explain your steps.

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

18. Let $\mathbf{u} \in \mathbb{R}^n$ be a vector such that $\mathbf{u}^T\mathbf{u} = 1$. The $n \times n$ matrix

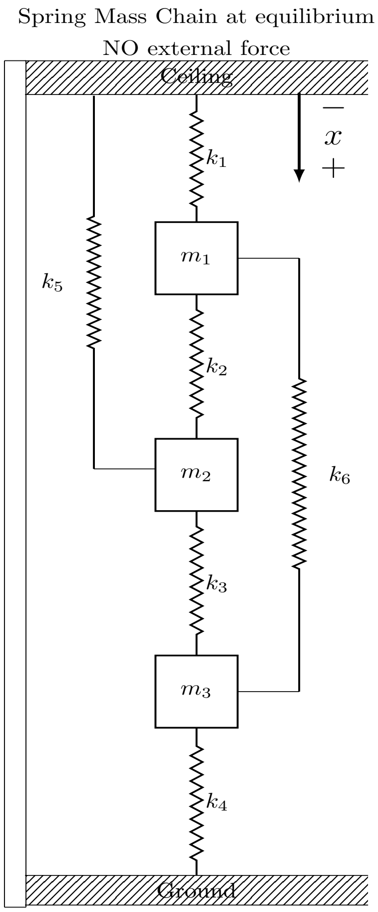
$$H = I_n - 2\mathbf{u}\mathbf{u}^T.$$

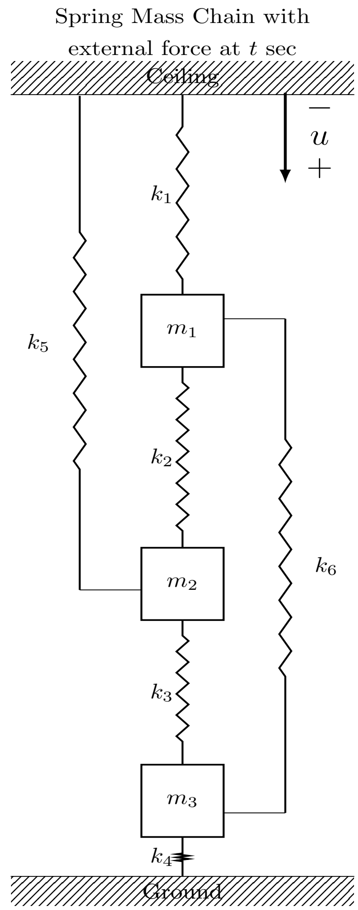
is called a **Householder matrix**.

A. Show that $H = H^T$ (in other words, show that H is symmetric).

B. Show that $H^{-1} = H$.

19. For the problem below, consider the following model for a 3-mass, 6-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly.





A. Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t, and the displacement vector, respectively (as discussed in class and in our lesson notes).

B. Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below.

C. Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

D. Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K. You should also find the entry-by-entry definition of K.

E. Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem)

    Let $n \in \mathbb{N}$ and $K \in \mathbb{R}^{n\times n}$ with $K^T = K$. Suppose that $\mathbf{x}^T \cdot K \cdot \mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$. Define the quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ using the matrix equation

    $$q(\mathbf{x}) = \mathbf{x}^T \cdot K \cdot \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c$$

    for constant $c \in \mathbb{R}$ and constant vector $\mathbf{f} \in \mathbb{R}^n$.

    A. Show that K is invertible.
    B. Show that $q(\mathbf{x})$ has a unique minimizer $\mathbf{x}^\star \in \mathbb{R}^n$ such that $q(\mathbf{x}^\star) < q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.