Name : ____________________ **Class Number**: ______

Exam 2, Version 2A
Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note sheet), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones or put your cell phones into airplane mode during this exam. Please place your cell phones inside your bag. No cell phones will be allowed on your desk.
- Close your bag and put it under your seat.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 7 sheets of paper (14 pages front and back) including this cover page.
- There are a total of 20 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 10 Multiple Choice Questions (50 points)
  - 4 Free-Response Questions (40 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note sheet that is no larger than 8.5 inches by 11 inches This note sheet must be handwritten. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTESHEET WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T F Suppose $A \in \mathbb{R}^{4\times 7}$ has three nonpivot columns. Then, we know that there will be exactly three solutions to $A \cdot \mathbf{x} = \mathbf{0}$

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2. T F Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{m\times 1}$. Then $\mathbf{x}^T \cdot A = \sum_{i=1}^{m} x_i\, A(i,:)$

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3. T F Suppose $A \in \mathbb{R}^{4\times 4}$ with $a_{ii} = 0$ for each $i = 1, 2, 3, 4$. Then $\det(A) = 0$.

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4. T F Let $j, n \in \mathbb{N}$ with $1 \leq j \leq n$. If $c \in \mathbb{R}$ is nonzero and $D_j(c) \in \mathbb{R}^{n\times n}$, then

$$\Big(D_j(c)\Big)^{-1} = I_n + \left(\frac{1-c}{c}\right) \mathbf{e}_j \cdot \mathbf{e}_j^T$$

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5. T F If $A \in \mathbb{R}^{3\times 3}$ has two pivots, then it is possible to find invertible matrices $E_1, E_2, ..., E_t \in \mathbb{R}^{3\times 3}$ such that

$$E_t \cdot E_{t-1} \cdots E_2 \cdot E_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiple Choice

(50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Let $A^T \in \mathbb{R}^{6\times 5}$, $B \in \mathbb{R}^{5\times 4}$, and $C \in \mathbb{R}^{3\times 6}$. Let the matrix D be formed by the product

$$D^T = B^T \cdot A \cdot C^T$$

What are the dimensions of the matrix $[D(2,:)]^T$?

A. 3×1 B. 4×3 C. 1×3 D. 1×4 E. 4×1

7. Let $B \in \mathbb{R}^{3\times 3}$ such that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

Which of the following gives $\det(B)$:

A. 2 B. -12 C. -8 D. 8 E. -2

8. Let $A \in \mathbb{R}^{n\times n}$ with $n > 4$. Suppose that you know A has 1 nonpivot columns. Which of the following must be true:

A. $\text{RREF}(A) = I_n$
B. There exists some $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b} \notin \text{Span}\{A(:,1), A(:,2), ..., A(:,n)\}$
C. $\text{Span}\{A(:,1), A(:,2), ..., A(:,n)\} = \mathbb{R}^n$
D. $a_{ii} = 0$ for at least one index i for $1 \le i \le n$
E. $\det(A) \neq 0$

For the next two questions, consider the following general linear-systems problems:

$$\underbrace{\begin{bmatrix} -2 & 1 & 4 & 2 & 14 & 2 & 12 \\ 2 & -1 & 2 & 4 & -2 & 4 & 6 \\ 2 & -1 & -2 & 0 & -10 & 1 & -7 \\ 2 & -1 & 4 & 6 & 2 & 8 & 10 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 10 \\ -4 \\ -10 \\ -6 \end{bmatrix}}_{\mathbf{b}}$$

9. Which of the following vectors is a solution to $A \cdot \mathbf{x} = \mathbf{0}$?

A. $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ E. None of these

10. Vector $\mathbf{b} \in \mathbb{R}^4$ is NOT in the span of which of the following sets?

A. $\{A(:,1),\ A(:,3),\ A(:,6)\}$ B. $\{A(:,2),\ A(:,3),\ A(:,6)\}$ C. $\{A(:,2),\ A(:,4),\ A(:,7)\}$

D. $\{A(:,1),\ A(:,3),\ A(:,5)\}$ E. $\{A(:,4),\ A(:,5),\ A(:,6)\}$

11. Suppose we want to solve the linear systems problem

$$\underbrace{\begin{bmatrix} 2 & -4 & 0 & 2 \\ 4 & -4 & -2 & 5 \\ -6 & 4 & 5 & -6 \\ 2 & 0 & -4 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 6 \\ 8 \\ -5 \\ -4 \end{bmatrix}}_{\mathbf{b}}$$

If $\mathbf{x} \in \mathbb{R}^4$ is the solution to this problem, then find the value of the dot product:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A. -3 B. 1 C. -1 D. 3 E. -5

12. Define four vectors in $\mathbb{R}^4$ as

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 10 \\ 15 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 25 \\ 100 \\ 225 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} 5 \\ -45 \\ -45 \\ 5 \end{bmatrix}$$

We can confirm that $\mathbf{a}_4 = 5 \cdot \mathbf{a}_1 - 15 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$. Choose the vector $\mathbf{x} \in \mathbb{R}^4$ such that

$$\left[\; \mathbf{a}_1 \; \middle| \; \mathbf{a}_2 \; \middle| \; \mathbf{a}_3 \; \middle| \; \mathbf{a}_4 \; \right] \cdot \mathbf{x} = \mathbf{0}$$

A. $\begin{bmatrix} 45 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ B. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} -5 \\ 15 \\ -1 \\ -1 \end{bmatrix}$ D. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ -1 \end{bmatrix}$ E. The product will never be zero

13. Consider the following nonsingular linear-systems problem

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

Let $A \in \mathbb{R}^{3\times 3}$ be the coefficient matrix in this problem and $\mathbf{b} \in \mathbb{R}^3$ be the vector on the right-hand side. Find the matrices $L, U \in \mathbb{R}^{3\times 3}$, where the L and U factors of A, respectively, from the the LU factorization of A. Now, solve the two linear system problems

$$L \cdot \mathbf{y} = \mathbf{b}, \qquad\qquad U \cdot \mathbf{x} = \mathbf{y}$$

Find the value of $\mathbf{y} \cdot \mathbf{x}$:

A. 9 B. -14 C. 4 D. -39 E. -159

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14. Let matrix $B \in \mathbb{R}^{4\times 4}$ be given as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

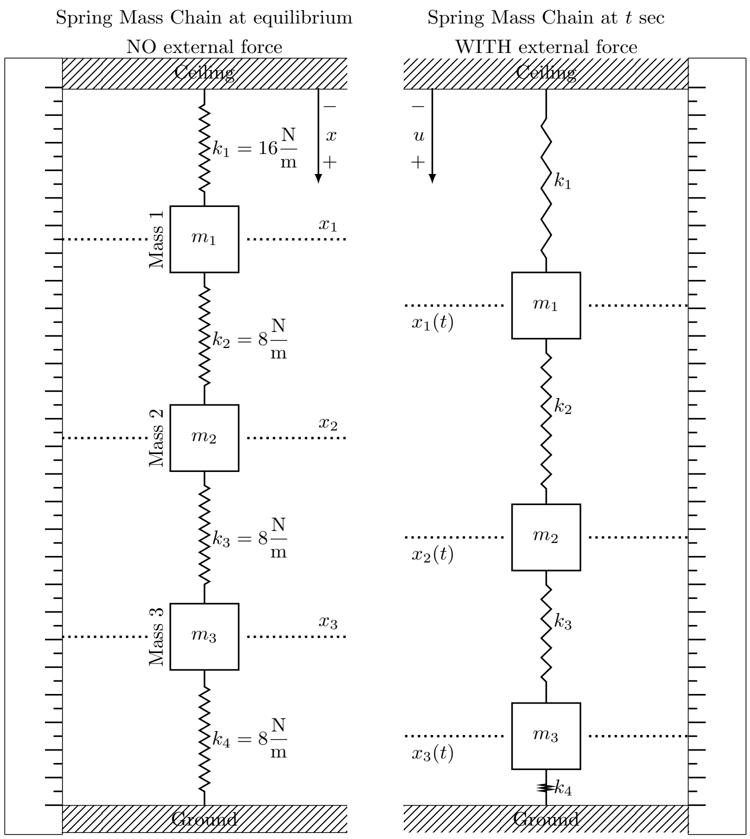
In symbols, we can write

$$B = D_2\left(\frac{1}{2}\right) \cdot A \cdot S_{12}(3)$$

Using this definition, we see that a_{22} is equal to which of the following:

A. $\frac{3}{2}\, b_{21} + \frac{1}{2}\, b_{22}$ B. $-6\, b_{21} + 2\, b_{22}$ C. $6\, b_{21} + 2b_{22}$ D. $2\, b_{22}$ E. $\frac{1}{2}\, b_{22}$

For Problems 15, consider the following model for a 3-mass, 4-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly.



15. Recall that the initial position vector $\mathbf{x}_0$ and the mass vector $\mathbf{m}$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and the mass measurements, measured in kg, respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.7 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.080 \\ 0.040 \\ 0.080 \end{bmatrix}$$

Which of the following gives the vector $\mathbf{x}(T) = \begin{bmatrix} x_1(T) & x_2(T) & x_3(T) \end{bmatrix}^T$ as measured in meters, used to store the positions of each mass at equilibrium when when $t = T$? If necessary, please round your answers to the nearest 3 places after the decimal.

A. $\begin{bmatrix} 0.230 \\ 0.388 \\ 0.595 \end{bmatrix}$ B. $\begin{bmatrix} 0.245 \\ 0.310 \\ 0.327 \end{bmatrix}$ C. $\begin{bmatrix} 0.370 \\ 0.612 \\ 0.805 \end{bmatrix}$ D. $\begin{bmatrix} 0.545 \\ 0.810 \\ 1.027 \end{bmatrix}$ E. $\begin{bmatrix} 0.070 \\ 0.112 \\ 0.105 \end{bmatrix}$

Free Response

$\boxed{10}$ 16. Consider the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A. Use a sequence of two matrix multiplications to transform A into upper-triangular U. Specifically identify the 3×3 matrices E_1 and E_2.

B. Find the LU factorization of the matrix A from above.

$\boxed{10}$ 17. Let $\ell_{32}, \ell_{42} \in \mathbb{R}$ and define the matrix $L_2 \in \mathbb{R}^{4\times 4}$ be given by

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} = I_4 + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T, \qquad \text{where } \boldsymbol{\tau}_2 = \begin{bmatrix} 0 \\ 0 \\ \ell_{32} \\ \ell_{42} \end{bmatrix}.$$

Prove $L_2^{-1} = \left(I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right)$.

$\boxed{10}$ 18. Let $B = A \cdot X$ where

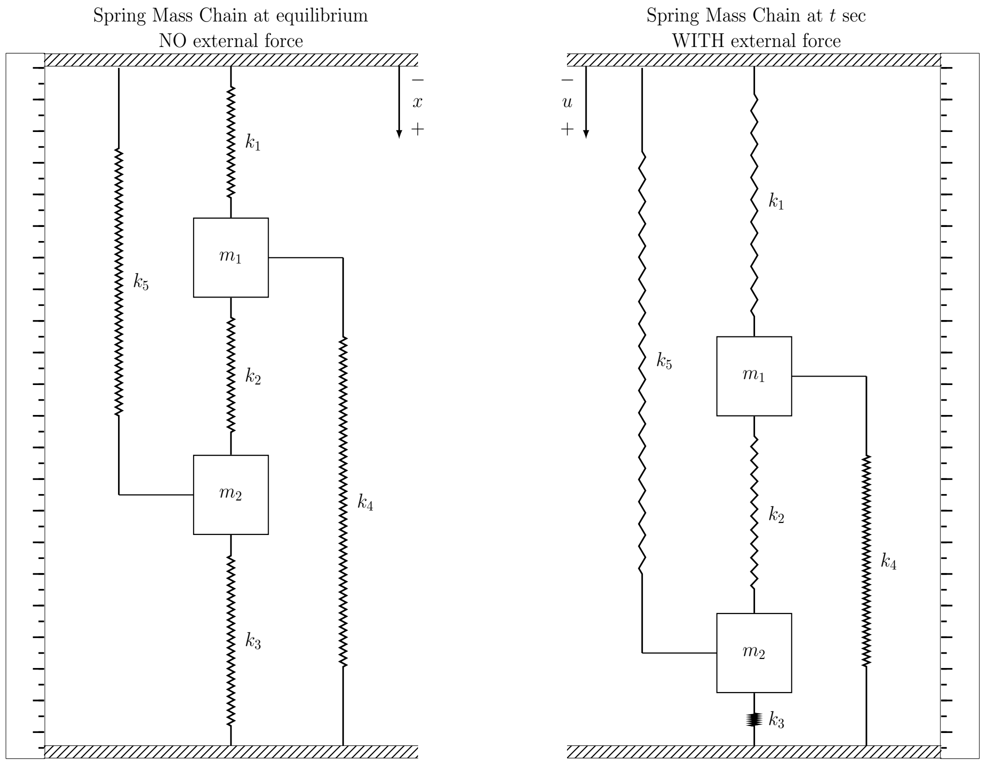
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}.$$

A. Use the row-partition version of matrix-matrix multiplication to find $B(3,:)$. Show your steps.

B. Use the column-partition version of matrix-matrix multiplication to find $B(:,2)$. Show your steps.

C. Use the entry-by-entry version of matrix-matrix multiplication to find b_{42}. Show your steps.

$\boxed{10}$ 19. Consider the following spring-mass system:



A. Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t, and the displacement vector, respectively (as discussed in class and in our lesson notes).

B. Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below.

C. Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

D. Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K. You should also find the entry-by-entry definition of K.

E. Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem) Let $m, n \in \mathbb{N}$. Suppose $A \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{m \times m}$ where the diagonal elements of C are positive $c_{ii} > 0$ for all $i \in \{1, 2, ..., m\}$. Let $K = A^T \cdot C \cdot A$. Prove that

$$K \cdot \mathbf{x} = \mathbf{0} \text{ if and only if } A \cdot \mathbf{x} = \mathbf{0}.$$