
True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T **F** Suppose $A \in \mathbb{R}^{4\times 7}$ has three nonpivot columns. Then, we know that there will be exactly three solutions to $A \cdot \mathbf{x} = \mathbf{0}$

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2. **T** F Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{m\times 1}$. Then $\mathbf{x}^T \cdot A = \sum_{i=1}^{m} x_i \, A(i,:)$

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3. T **F** Suppose $A \in \mathbb{R}^{4\times 4}$ with $a_{ii} = 0$ for each $i = 1, 2, 3, 4$. Then $\det(A) = 0$.

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4. **T** F Let $j, n \in \mathbb{N}$ with $1 \leq j \leq n$. If $c \in \mathbb{R}$ is nonzero and $D_j(c) \in \mathbb{R}^{n\times n}$, then

$$\Big(D_j(c)\Big)^{-1} = I_n + \left(\frac{1-c}{c}\right) \mathbf{e}_j \cdot \mathbf{e}_j^T$$

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5. T **F** If $A \in \mathbb{R}^{3\times 3}$ has two pivots, then it is possible to find invertible matrices $E_1, E_2, ..., E_t \in \mathbb{R}^{3\times 3}$ such that

$$E_t \cdot E_{t-1} \cdots E_2 \cdot E_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiple Choice

(50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Let $A^T \in \mathbb{R}^{6\times 5}$, $B \in \mathbb{R}^{5\times 4}$, and $C \in \mathbb{R}^{3\times 6}$. Let the matrix D be formed by the product

$$D^T = B^T \cdot A \cdot C^T$$

What are the dimensions of the matrix $[D(2,:)]^T$?

A. 3×1 B. 4×3 C. 1×3 D. 1×4 **E.** 4×1

7. Let $B \in \mathbb{R}^{3\times 3}$ such that

$$\begin{bmatrix}1&0&0\\0&\frac{3}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}\underbrace{\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}}_{B}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}6&2&-1\\0&1&4\\0&0&-\frac{2}{3}\end{bmatrix}$$

Which of the following gives $\det(B)$:

A. 2 B. -12 C. -8 **D.** 8 E. -2

8. Let $A \in \mathbb{R}^{n\times n}$ with $n > 4$. Suppose that you know A has 1 nonpivot columns. Which of the following must be true:

A. RREF $(A) = I_n$
B. There exists some $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b} \notin$ Span$\{A(:,1), A(:,2), ..., A(:,n)\}$
C. Span$\{A(:,1), A(:,2), ..., A(:,n)\} = \mathbb{R}^n$
D. $a_{ii} = 0$ for at least one index i for $1 \le i \le n$
E. $\det(A) \neq 0$

For the next two questions, consider the following general linear-systems problems:

$$\underbrace{\begin{bmatrix} -2 & 1 & 4 & 2 & 14 & 2 & 12 \\ 2 & -1 & 2 & 4 & -2 & 4 & 6 \\ 2 & -1 & -2 & 0 & -10 & 1 & -7 \\ 2 & -1 & 4 & 6 & 2 & 8 & 10 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 10 \\ -4 \\ -10 \\ -6 \end{bmatrix}}_{\mathbf{b}}$$

9. Which of the following vectors is a solution to $A \cdot \mathbf{x} = \mathbf{0}$?

A. $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ E. None of these

10. Vector $\mathbf{b} \in \mathbb{R}^4$ is NOT in the span of which of the following sets?

A. $\{A(:,1),\ A(:,3),\ A(:,6)\}$ B. $\{A(:,2),\ A(:,3),\ A(:,6)\}$ C. $\{A(:,2),\ A(:,4),\ A(:,7)\}$

D. $\{A(:,1),\ A(:,3),\ A(:,5)\}$ E. $\{A(:,4),\ A(:,5),\ A(:,6)\}$

11. Suppose we want to solve the linear systems problem

$$\underbrace{\begin{bmatrix} 2 & -4 & 0 & 2 \\ 4 & -4 & -2 & 5 \\ -6 & 4 & 5 & -6 \\ 2 & 0 & -4 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 6 \\ 8 \\ -5 \\ -4 \end{bmatrix}}_{\mathbf{b}}$$

If $\mathbf{x} \in \mathbb{R}^4$ is the solution to this problem, then find the value of the dot product:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A. -3 B. 1 **C.** -1 D. 3 E. -5

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12. Define four vectors in $\mathbb{R}^4$ as

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 10 \\ 15 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 25 \\ 100 \\ 225 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} 5 \\ -45 \\ -45 \\ 5 \end{bmatrix}$$

We can confirm that $\mathbf{a}_4 = 5 \cdot \mathbf{a}_1 - 15 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$. Choose the vector $\mathbf{x} \in \mathbb{R}^4$ such that

$$\left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array}\right] \cdot \mathbf{x} = \mathbf{0}$$

A. $\begin{bmatrix} 45 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ B. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} -5 \\ 15 \\ -1 \\ -1 \end{bmatrix}$ **D.** $\begin{bmatrix} 5 \\ -15 \\ 1 \\ -1 \end{bmatrix}$ E. The product will never be zero

13. Consider the following nonsingular linear-systems problem

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

Let $A \in \mathbb{R}^{3\times 3}$ be the coefficient matrix in this problem and $\mathbf{b} \in \mathbb{R}^3$ be the vector on the right-hand side. Find the matrices $L, U \in \mathbb{R}^{3\times 3}$, where the L and U factors of A, respectively, from the the LU factorization of A. Now, solve the two linear system problems

$$L \cdot \mathbf{y} = \mathbf{b}, \qquad\qquad U \cdot \mathbf{x} = \mathbf{y}$$

Find the value of $\mathbf{y} \cdot \mathbf{x}$:

A. 9 B. -14 C. 4 D. -39 E. -159

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14. Let matrix $B \in \mathbb{R}^{4\times 4}$ be given as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

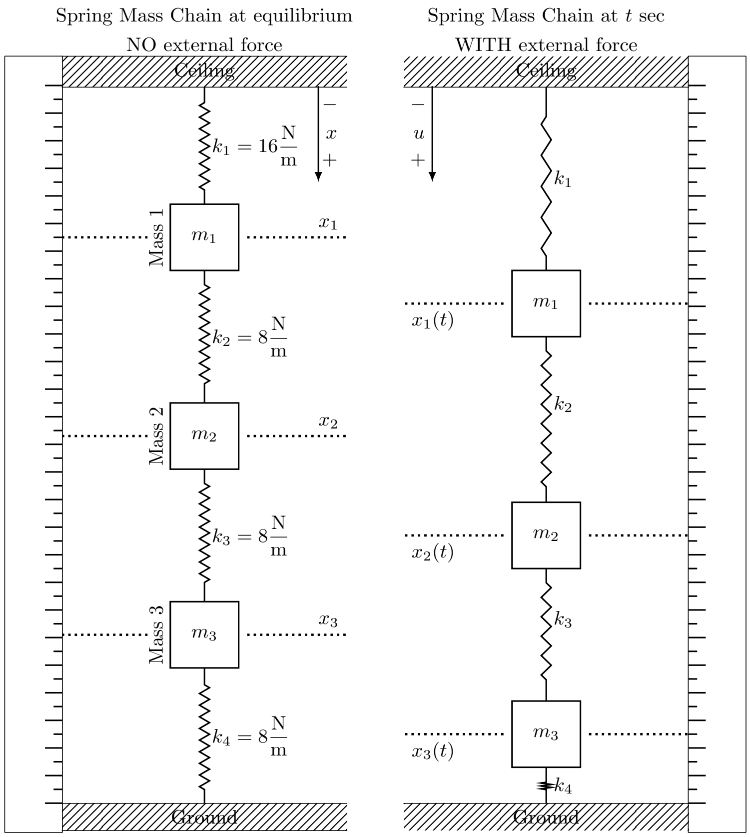
In symbols, we can write

$$B = D_2\left(\frac{1}{2}\right) \cdot A \cdot S_{12}(3)$$

Using this definition, we see that a_{22} is equal to which of the following:

A. $\frac{3}{2}\, b_{21} + \frac{1}{2}\, b_{22}$ **B.** $-6\, b_{21} + 2\, b_{22}$ C. $6\, b_{21} + 2 b_{22}$ D. $2\, b_{22}$ E. $\frac{1}{2}\, b_{22}$

For Problems 15, consider the following model for a 3-mass, 4-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly.



15. Recall that the initial position vector $\mathbf{x}_0$ and the mass vector $\mathbf{m}$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and the mass measurements, measured in kg, respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.7 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.080 \\ 0.040 \\ 0.080 \end{bmatrix}$$

Which of the following gives the vector $\mathbf{x}(T) = \begin{bmatrix} x_1(T) & x_2(T) & x_3(T) \end{bmatrix}^T$ as measured in meters, used to store the positions of each mass at equilibrium when when $t = T$? If necessary, please round your answers to the nearest 3 places after the decimal.

A. $\begin{bmatrix} 0.230 \\ 0.388 \\ 0.595 \end{bmatrix}$ B. $\begin{bmatrix} 0.245 \\ 0.310 \\ 0.327 \end{bmatrix}$ **C.** $\begin{bmatrix} 0.370 \\ 0.612 \\ 0.805 \end{bmatrix}$ D. $\begin{bmatrix} 0.545 \\ 0.810 \\ 1.027 \end{bmatrix}$ E. $\begin{bmatrix} 0.070 \\ 0.112 \\ 0.105 \end{bmatrix}$

Free Response

16. (10 pts) Consider the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A. Use a sequence of two matrix multiplications to transform A into upper-triangular U. Specifically identify the 3×3 matrices E_1 and E_2.

Solution: The first thing we will do is to choose pivot 1 in entry $(1,1)$ and zero out all strictly lower-triangular entries in column 1. To do so, we will multiply matrix A on the left by a shear matrix:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \cdot \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{L_1 \cdot A}$$

Now that we have canceled out all nonpivot elements in column 1, we move onto column 2. In this case, we choose the nonzero coefficient in entry $(2,2)$ to be pivot 2 and annihilate all nonpivot entries below this pivot element:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}}_{L_2} \cdot \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{L_1 \cdot A} = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U}$$

The factor U above is the upper-triangular matrix we are looking for.

B. Find the LU factorization of the matrix A from above.

Solution: To form the corresponding lower-triangular matrix L we recall

$$L_2 \cdot L_1 \cdot A = U \qquad \Longrightarrow \qquad A = \left(L_1^{-1} \cdot L_2^{-1}\right) \cdot U$$

Moreover, since each L_i has an easily calculated inverse, we write

$$L = L_1^{-1} \cdot L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$$

Then, we can write the LU factorization of A as

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U}$$

17. (10 pts) Let $\ell_{32}, \ell_{42} \in \mathbb{R}$ and define the matrix $L_2 \in \mathbb{R}^{4\times 4}$ be given by

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} = I_4 + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T, \qquad \text{where } \boldsymbol{\tau}_2 = \begin{bmatrix} 0 \\ 0 \\ \ell_{32} \\ \ell_{42} \end{bmatrix}.$$

Prove $L_2^{-1} = \left(I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right)$.

Solution: Methods 1: Given a matrix $A \in \mathbb{R}^{n\times n}$, in order to prove matrix $C \in \mathbb{R}^{n\times n}$ is the inverse of A, we need to show that $A \cdot C = I_n$. To this end, let $C = I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T$ and consider the product:

$$\begin{aligned}
L_2 \cdot C &= \left(I_4 + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \cdot \left(I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \\
&= I_4 \cdot \left(I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T \cdot \left(I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \\
&= I_4 \cdot I_4 - I_4 \cdot \left(\boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) + \left(\boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \cdot I_4 - \left(\boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \cdot \left(\boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T\right) \\
&= I_4 \cdot I_4 - \left(I_4 \cdot \boldsymbol{\tau}_2\right) \cdot \mathbf{e}_2^T + \boldsymbol{\tau}_2 \cdot \left(\mathbf{e}_2^T \cdot I_4\right) - \boldsymbol{\tau}_2 \cdot \left(\mathbf{e}_2^T \cdot \boldsymbol{\tau}_2\right) \cdot \mathbf{e}_2^T \\
&= I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T - \boldsymbol{\tau}_2 \cdot \left(\mathbf{e}_2^T \cdot \boldsymbol{\tau}_2\right) \cdot \mathbf{e}_2^T \\
&= I_4 - \boldsymbol{\tau}_2 \cdot \left(\mathbf{e}_2^T \cdot \boldsymbol{\tau}_2\right) \cdot \mathbf{e}_2^T \\
&= I_4 - \boldsymbol{\tau}_2 \cdot 0 \cdot \mathbf{e}_2^T \\
&= I_4
\end{aligned}$$

The last equality results from adding the matrices $-\boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T + \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T = 0 \in \mathbb{R}^{n\times n}$. We also executed the product

$$\mathbf{e}_2^T \cdot \boldsymbol{\tau}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \ell_{32} \\ \ell_{42} \end{bmatrix} = 0$$

This verifies that $C = L_2^{-1}$ as was claimed.

Solution: Methods 2: Given a matrix $A \in \mathbb{R}^{n\times n}$, in order to prove matrix $C \in \mathbb{R}^{n\times n}$ is the inverse of A, we need to show that $A \cdot C = I_n$. To this end, let $C = I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T$. We can write the entry-by-entry definition of this matrix as

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\ell_{32} & 1 & 0 \\ 0 & -\ell_{42} & 0 & 1 \end{bmatrix}$$

Then, consider the matrix product:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\ell_{32} & 1 & 0 \\ 0 & -\ell_{42} & 0 & 1 \end{bmatrix} = L_1 \cdot C$$

In this case, let's use the column-partition version of matrix-matrix multiplication to find each column of B. To this end consider, calculate $\text{Column}_1(B) = L_1 \cdot \text{Column}_1\Big(C\Big)$ as

$$B(:,1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ \ell_{32} \\ \ell_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, the second column of B is $\text{Column}_2(B) = L_1 \cdot \text{Column}_2\Big(C\Big)$ given by

$$B(:,2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -\ell_{32} \\ -\ell_{42} \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ \ell_{32} \\ \ell_{42} \end{bmatrix} - \ell_{32} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \ell_{42} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The third column $\text{Column}_3(B) = L_1 \cdot \text{Column}_3\Big(C\Big)$ of our product is

$$B(:,3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ \ell_{32} \\ \ell_{42} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, the last column $\text{Column}_4(B) = L_1 \cdot \text{Column}_4\Big(C\Big)$ is a linear combination of the columns of A with scalars defined by the entries in the fourth column of C. This column vector is calculated as follows

$$B(:,4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell_{32} & 1 & 0 \\ 0 & \ell_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ \ell_{32} \\ \ell_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Matrix B results from combining each of these outputs and is given by

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This confirms that $L_2^{-1} = I_4 - \boldsymbol{\tau}_2 \cdot \mathbf{e}_2^T$.

18. (10 pts) Let $B = A \cdot X$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}.$$

A. Use the row-partition version of matrix-matrix multiplication to find $B(3,:)$. Show your steps.

Solution: Let's use the row-partition version of matrix-matrix multiplication to find the third row of B. To this end consider, calculate $\text{Row}_3(B) = \text{Row}_3\Big(A\Big) \cdot X$ as

$$B(3,:) = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} 2 & 0 \end{bmatrix} + -1 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & -2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & -3 \end{bmatrix}}$$

B. Use the column-partition version of matrix-matrix multiplication to find $B(:,2)$. Show your steps.

Solution: Now, let's use the column-partition version of matrix-matrix multiplication to the second column of B. To this end consider, calculate $\text{Column}_2(B) = A \cdot \text{Column}_2\Big(X\Big)$ as

$$B(:,2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

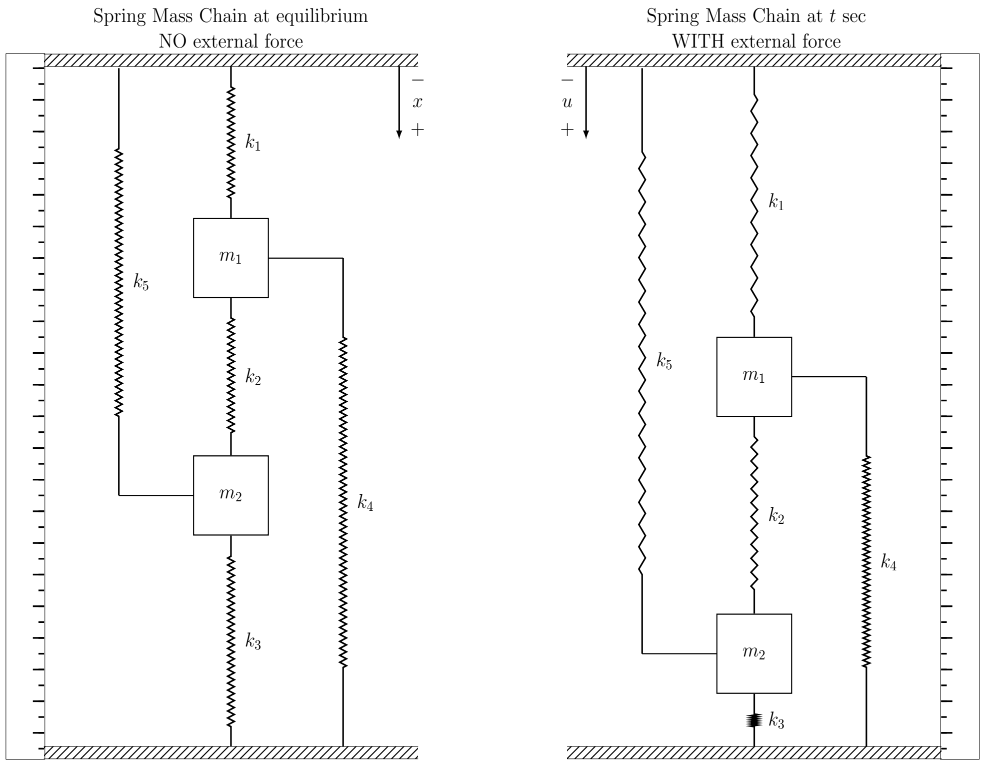
$$= 0 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + -2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}}.$$

C. Use the entry-by-entry version of matrix-matrix multiplication to find b_{42}. Show your steps.

Solution: Now, let's use the entry-by-entry definition of matrix-matrix multiplication to find b_{42}. To this end consider, calculate $b_{42} = \text{Enty}_{42}(B) = \text{Row}_4\Big(A\Big) \cdot \text{Column}_2\Big(X\Big)$ as

$$b_{42} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \boxed{2}$$

19. (10 pts) Consider the following spring-mass system:



A. Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t, and the displacement vector, respectively (as discussed in class and in our lesson notes).

Solution: Let's set up our model of the 2-mass, 5-spring chain.

POSITION VECTORS:

Let's define $\mathbf{x}_0$ to be the equilibrium position vector. Also, we will let $\mathbf{x}(t)$ store the positions of each mass at any time t. In this case, we let

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where x_i represents the position of mass i at equilibrium when masses are at rest with no external force applied, as shown in the diagram. Further, $x_i(t)$ represents the position of mass i at time $t \in [t_0, T) \subseteq \mathbb{R}$.

DISPLACEMENT VECTOR:

With this we can set up our displacement vector $\mathbf{u}(t)$. In this case, we have assumed the zero position of our ruler to be on the ceiling at the top of our apparatus. Moreover, we orient positive position measurements in the downward direction (toward the ground below the masses). We want $u_i(t)$ to measure the displacement of mass i away from it's initial position. Since $x_i > x_i(t)$ in our diagram, we see that $u_i(t) > 0$ if and only if $x_i(t) - x_i > 0$. Thus, we define our displacement vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_1 \\ x_2(t) - x_2 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}(t) - \mathbf{x}_0$$

ORIENTATION OF RULER:

Remark (for students who want to earn above a 90%):

A. In this case, we choose to orient our ruler so that the zero position was on the top of the apparatus and positive position measurements are in the downward direction. This guarantees that positive displacement measurements are oriented in the downward position calculated as the position of mass i at time t minus the equilibrium position.This follows the convention in physics to calculate displacement as "final" position minus "initial position." In other words, $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}_0$.

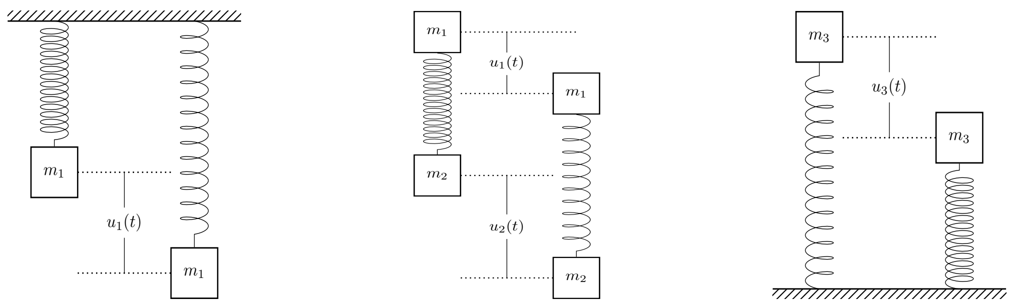
B. In this derivation, we assumed that the position of mass i at time t was described by a continuous function $x_i(t)$. As we will see, this very general modeling paradigm gives rise to an eigenvalue problem.

B. Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below.

Solution: The elongation vector $\mathbf{e}(t)$ is a 5×1 vector. The ith entry $e_i(t)$ of this elongation vector represents the "elongation" of spring i at time t. To measure elongation of the ith spring, we subtract the length of spring i when the system is at equilibrium from the length of this spring again at time t. To find the elongation of each spring, consider the following diagrams:



Positive $e_i(t)$ values occur when the length of this spring at time t is larger than the length of this spring at equilibrium.

ELONGATION VECTOR:

Using these diagrams, we see that our desired elongation vector is given by

$$\mathbf{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) - u_1(t) \\ -u_2(t) \\ -u_1(t) \\ u_2(t) \end{bmatrix} = u_1(t) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + u_2(t) \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We can write this linear combination as a matrix-vector product as following

$$\mathbf{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where $\mathbf{u}$ is the 3×1 displacement vector from part (A) above. In this case, the matrix $A \in \mathbb{R}^{4 \times 3}$ is given by

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, we write $\mathbf{e}(t)$ as a matrix vector product

$$\boxed{\mathbf{e}(t) = A \cdot \mathbf{u}(t)}. \qquad (1)$$

C. Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

Solution: Recall that Hooke's law states that the change in internal force stored inside spring i is directly proportional to the elongation of the spring. In other words, for a spring with spring constant k_i, Hooke's law states that

$$f_{s_i}(t) = k_i \cdot e_i(t)$$

The vector $\mathbf{f}_s(t)$ can stores the internal forces in each of the four springs in our system due to the elongations discussed in part (B) above.

FORCE VECTORS FOR SPRINGS:

Now, let's move onto finding the internal forces stored in each spring. To this end, let

$$\mathbf{f}_s(t) = \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \end{bmatrix} = \begin{bmatrix} k_1\, e_1(t) \\ k_2\, e_2(t) \\ k_3\, e_3(t) \\ k_4\, e_4(t) \\ k_5\, e_5(t) \end{bmatrix} = e_1(t) \cdot \begin{bmatrix} k_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_2(t) \cdot \begin{bmatrix} 0 \\ k_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_3(t) \cdot \begin{bmatrix} 0 \\ 0 \\ k_3 \\ 0 \\ 0 \end{bmatrix} + e_4(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_4 \\ 0 \end{bmatrix} + e_5(t) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_5 \end{bmatrix}$$

The force vector $\mathbf{f}_s(t)$ as the matrix-vector product

$$\mathbf{f}_s(t) = \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix}$$

where $\mathbf{e}$ is our elongation vector from above. The diagonal matrix $C \in \mathbb{R}^{4\times 4}$ is defined as

$$C = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{bmatrix}$$

We write

$$\boxed{\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)} \tag{2}$$

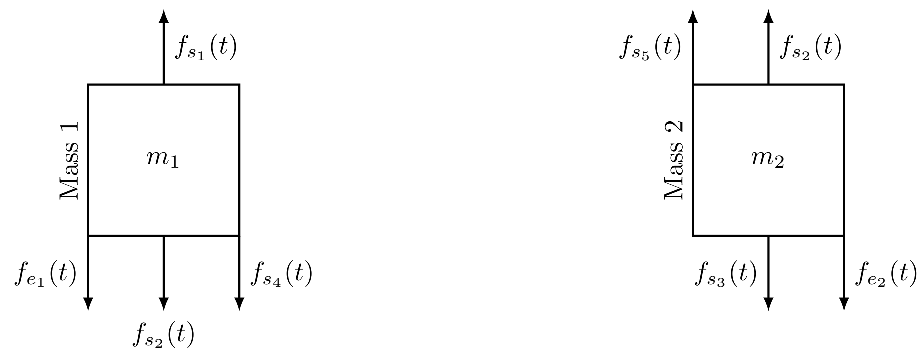
D. Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K. You should also find the entry-by-entry definition of K.

Solution:

Let's consider the net internal forces on each mass. To do so, we draw a free-body diagram for each mass in our system, as seen below.



When analyzing the net internal forces, we work to calculated the net force acting on each mass when considering ONLY the springs within the system. In other words, if the coupled chain of the masses and springs is our system, we want only to focus on the net forces within this system (and ignore external forces that may act on each mass for now).

NET INTERNAL FORCES FOR MASS-SPRING CHAIN:

We now introduce the vector $\mathbf{y}(t)$ to store the net force on each mass. Each entry $y_i(t)$ represents the difference between $f_s i(t)$ and $f_{s_{i+1}}(t)$. When writing the individual entries of $\mathbf{y}(t)$ we will assume that positive net forces result in positive displacements. Since we've oriented positive displacement in the downward direction, we also orient positive force in the downward direction.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} f_{s_2}(t) + f_{s_4}(t) - f_{s_1}(t) \\ f_{s_3}(t) - f_{s_2}(t) - f_{s_5}(t) \end{bmatrix} = -\begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \end{bmatrix}$$

We transform this into a matrix-vector product

$$\boxed{\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)} \tag{3}$$

where A was defined in equation (1) for our model of the elongation vector $\mathbf{e}(t)$. Notice that we've factored out a negative sign in order to make this statement.

Solution: In this problem, we will using equations (1), (2), and (3) to create stiffness matrix K. To this end, note

$$\begin{aligned}
\mathbf{y}(t) &= -A^T \cdot \mathbf{f}_s(t) && \text{by equation (3)} \\
&= -A^T \cdot C \cdot \mathbf{e}(t) && \text{by equation (2)} \\
&= -A^T \cdot C \cdot A \cdot \mathbf{u}(t) && \text{by equation (1)} \\
&= -K \cdot \mathbf{u}(t)
\end{aligned}$$

If we let $K = A^T \cdot C \cdot A$, we can then write

$$\boxed{\mathbf{y}(t) = -K \cdot \mathbf{u}(t)} \tag{4}$$

We can form our stiffness matrix K explicitly using matrix-matrix multiplication with

$$\begin{aligned}
K &= \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & 0 & 0 & k_5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 \\ -k_2 & k_2 + k_3 + k_5 \end{bmatrix}
\end{aligned}$$

This is a tridiagonal, symmetric matrix.

E. Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

Solution: From Newton's second law, we know that

$$\text{Net Force acting on an object} = \text{ Mass of object} \times \text{ Acceleration of object}$$

In order to state this law for each mass in our system, let

$$\ddot{u}_i(t) = \frac{d^2}{dt^2}\Big[u_i(t)\Big]$$

be the acceleration of mass i at time t for $i \in \{1, 2, 3\}$. Then, for mass i, the coefficient

$$m_i \cdot \ddot{u}_i(t)$$

measures the mass multiplied by the acceleration. Organizing all three of these values into a column vector yields

$$\begin{bmatrix} m_1\, \ddot{u}_1(t) \\ m_2\, \ddot{u}_2(t) \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{bmatrix}$$

By defining the 3×3 mass matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

these mass times acceleration calculations can be written as a matrix-vector multiplication

$$M \cdot \ddot{\mathbf{u}}(t) \tag{5}$$

The next step is to calculate the net force on mass i. To do so, let's look back at our free-body diagram from part D above. The net force on mass i is given by

$$f_{s_{i+1}}(t) - f_{s_i}(t) + f_{e_i}(t) = y_i(t) + f_{e_i}(t)$$

We can organize all three net force calculations as a column vector

$$\begin{bmatrix} f_{s_2}(t) - f_{s_1}(t) \\ f_{s_3}(t) - f_{s_2}(t) \end{bmatrix} + \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \end{bmatrix}$$

Thus, the net force calculations for each mass in this system is given by

$$\mathbf{y}(t) + \mathbf{f}_e(t) \tag{6}$$

where vector $\mathbf{f}_e(t)$ gives the ne Newton's second law indicates that the mass times acceleration vector from equation (5) is equal to the net force vector from equation (6)

$$M \cdot \ddot{\mathbf{u}}(t) = \mathbf{y}(t) + \mathbf{f}_e(t)$$

Solution: Now, use the stiffness matrix K from equation (4) to represent the net internal force vector $\mathbf{y}(t)$, yielding

$$M \cdot \ddot{\mathbf{u}}(t) = -K \cdot \mathbf{u}(t) + \mathbf{f}_e(t)$$

By moving $-K$ onto the other side of the equation, we have

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}_e(t) \tag{7}$$

Since we have assume that we study the system at equilibrium for $t = T$, we know $\ddot{\mathbf{u}}(T) = \mathbf{0}$ and we have

$$K\,\mathbf{u}(T) = \mathbf{f}_e(T)$$

Remark (for students who want to earn above a 100%):

- In this derivation, we've used a very general approach to allow $t \in [t_0, T)$. Only at the very end of our work, did we substitute the value of $t = T$ to represent the case that our masses have settled down to equilibrium. As we will see, this general approach will come in very useful during our discussion of the eigenvalue-eigenvector problem.
- In fact, we have derived a coupled ordinary differential equation in the work above. For those of you that have taken (or will take) Math 2A at Foothill, you may notice that equation (7) is a vector version of the 2nd order differential equation for a harmonic oscillator with no damping and general forcing function.

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem) Let $m, n \in \mathbb{N}$. Suppose $A \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{m\times m}$ where the diagonal elements of C are positive $c_{ii} > 0$ for all $i \in \{1, 2, ..., m\}$. Let $K = A^T \cdot C \cdot A$. Prove that

$$K \cdot \mathbf{x} = \mathbf{0} \text{ if and only if } A \cdot \mathbf{x} = \mathbf{0}.$$

Solution: To prove this statement, we need to show:

i. If $A \cdot \mathbf{x} = \mathbf{0}$, then $K \cdot \mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$

ii. If $K \cdot \mathbf{x} = \mathbf{0}$, then $A \cdot \mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$

Case i: $A \cdot \mathbf{x} = \mathbf{0} \Longrightarrow K \cdot \mathbf{x} = \mathbf{0}$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ such that $A \cdot \mathbf{x} = \mathbf{0}$. Then, by definition of matrix K we know

$$\begin{aligned}
K \cdot \mathbf{x} &= \Big(A^T \cdot C \cdot A\Big) \cdot \mathbf{x} && \text{by definition of } K \\
&= A^T \cdot C \cdot \Big(A \cdot \mathbf{x}\Big) && \text{by associativity of matrix multiplication} \\
&= A^T \cdot C \cdot \mathbf{0} && \text{since } A \cdot \mathbf{x} = 0 \\
&= \mathbf{0}
\end{aligned}$$

This is what we wanted to show for this direction of the proof. $\square$

Case ii: $K \cdot \mathbf{x} = \mathbf{0} \Longrightarrow A \cdot \mathbf{x} = \mathbf{0}$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ such that $K \cdot \mathbf{x} = \mathbf{0}$. Then

$$\begin{aligned}
0 &= \mathbf{x}^T \cdot K \cdot \mathbf{x} \\
&= \mathbf{x}^T \cdot \Big(A^T \cdot C \cdot A\Big) \cdot \mathbf{x} && \text{by definition of } K \\
&= \left(\mathbf{x}^T \cdot A^T\right) \cdot C \cdot (A \cdot \mathbf{x}) && \text{by associativity of matrix multiplication} \\
&= \mathbf{y}^T \cdot C \cdot \mathbf{y} && \text{setting } \mathbf{y} = A \cdot \mathbf{x}
\end{aligned}$$

We know that the diagonal elements of C are positive. Thus, when we consider

$$\mathbf{y}^T \cdot C \cdot \mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{mm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= c_{11} \cdot y_1^2 + c_{22} \cdot y_2^2 + \cdots + c_{mm} \cdot y_m^2$$

Since $0 = \mathbf{y}^T \cdot C \cdot \mathbf{y} = \sum_{i=1}^{m} c_{ii} \cdot y_i^2$, we know that $y_i^2 = 0$ for all $i \in \{1, 2, ..., m\}$. In other words, we must have

$$\mathbf{y} = A \cdot \mathbf{x} = \mathbf{0}$$

This is exactly what we wanted to show. □

Since we have proved both both directions of our biconditional statement, we conclude that

$$A \cdot \mathbf{x} = \mathbf{0} \Longleftrightarrow K \cdot \mathbf{x} = \mathbf{0}.$$