
True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T **F** Let $A \in \mathbb{R}^{9\times 6}$ and $X \in \mathbb{R}^{6\times 7}$. Set

$$B = A \cdot X.$$

Then $B(:,3) = A(3,:) \cdot X$

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2. **T** F For matrices in $\mathbb{R}^{4\times 4}$, $D_3(6) - D_3(5) = \mathbf{e}_3 \cdot \mathbf{e}_3^T$

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3. T **F** Any square matrix with nonzero diagonal entries must be invertible.

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4. T **F** Suppose we are given

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

Then, there exists an $\mathbf{x} \in \mathbb{R}^2$ such that $\|A \cdot \mathbf{x} - \mathbf{b}\|_2 = 0$

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5. **T** F Let $n \in \mathbb{N}$. Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in V$. Suppose

$$W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}.$$

Then $\dim(W) \le n$.

Multiple Choice

(60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

For problems 6 and 7 below, consider the wireframe model for a begin polygon V defined by vertex matrix and edge table below.

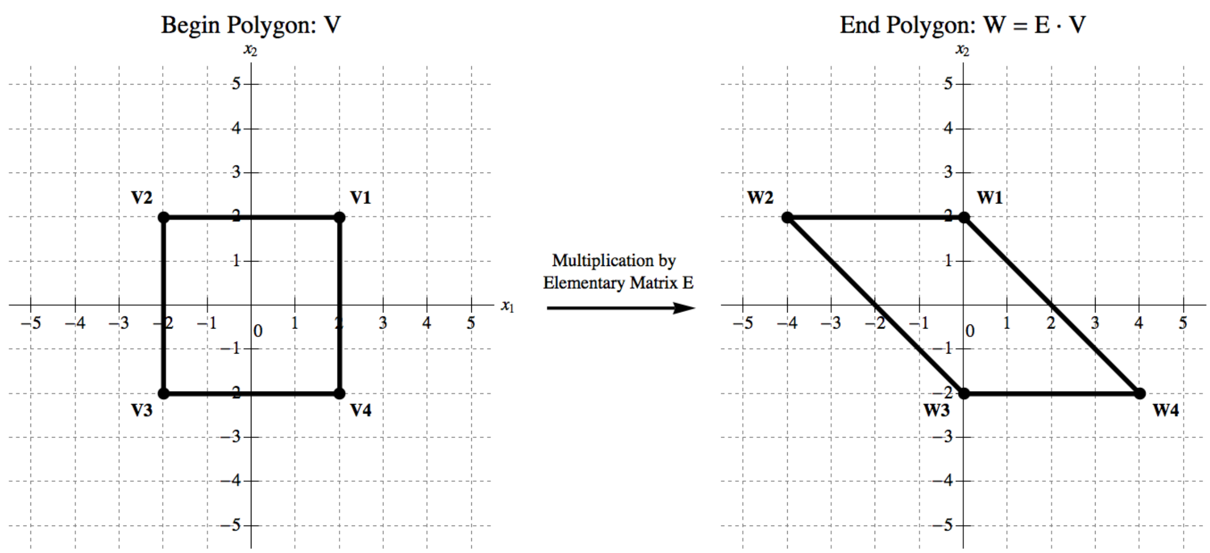
$$V = \begin{bmatrix} 2 & -2 & -2 & 2 \\ 2 & 2 & -2 & -2 \end{bmatrix}$$

Edge #	Start Vertex	End Vertex
1	1	2
2	2	3
3	3	4
4	4	1

Let W be a wireframe model for an end polygon given by

$$W = \begin{bmatrix} 0 & -4 & 0 & 4 \\ 2 & 2 & -2 & -2 \end{bmatrix}$$

Assume W formed by multiplying V by some matrix $E \in \mathbb{R}^{2\times 2}$ with $W = E \cdot V$. Also, assume that the edge tables of V and W are identical. Under these assumptions, the wireframe model for both V and W are given below.



6. Choose the matrix E used to produce W in this situation:

A. $S_{21}(-2)$ B. $S_{12}(-2)$ C. $S_{21}(-1)$ D. $S_{12}(1)$ **E.** $S_{12}(-1)$

7. Find the length of edge 4 from the wireframe model for the end polygon $W = E \cdot V$ in the problem above.

A. 2 B. $\sqrt{20}$ C. 4 **D.** $4\sqrt{2}$ E. 0

8. Let matrix $B \in \mathbb{R}^{4\times 4}$ be given as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

In symbols, we can write

$$B = S_{41}(-2) \cdot A \cdot D_4(3)$$

Using this definition, we see that b_{44} is equal to which of the following:

A. $-6a_{44}$ B. $-2a_{14} + 3a_{44}$ **C.** $-6a_{14} + 3a_{44}$ D. $3a_{14} - 6a_{44}$ E. $3a_{14} - 2a_{44}$

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9. Let $n \in \mathbb{N}$ with $n \geq 3$. Suppose that we define the matrix

$$B = I_n + c_1\, \mathbf{e}_2\, \mathbf{e}_1^T - c_2\, \mathbf{e}_3\, \mathbf{e}_1^T$$

where $\mathbf{e}_k = I_n(:, k)$. Which of the following is equivalent to B^{-1}?

A. $S_{21}(c_1) \cdot S_{31}(-c_2)$ B. $S_{31}(c_2) - S_{21}(c_1)$ **C.** $S_{31}(c_2) \cdot S_{21}(-c_1)$

D. $S_{21}\left(\frac{1}{c_1}\right) \cdot S_{31}\left(\frac{-1}{c_2}\right)$ E. $S_{12}(c_1) \cdot S_{13}(-c_2)$

10. Consider the following matrix equation

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-5&1\end{bmatrix}}_{L_3}\cdot\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-3&1&0\\0&3&0&1\end{bmatrix}}_{L_2}\cdot\underbrace{\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-3&0&1&0\\-1&0&0&1\end{bmatrix}}_{L_1}\cdot\underbrace{\begin{bmatrix}1&1&2&2\\2&1&3&1\\3&0&2&2\\1&4&0&1\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&1&2&2\\0&-1&-1&-3\\0&0&-1&5\\0&0&0&-35\end{bmatrix}}_{U}$$

Find the lower-triangular matrix $L \in \mathbb{R}^{4\times 4}$ from the LU factorization of the matrix A:

A. $\begin{bmatrix}1&0&0&0\\2&1&0&0\\3&3&1&0\\1&-3&5&1\end{bmatrix}$

B. $\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-3&-3&1&0\\-1&3&-5&1\end{bmatrix}$

C. $\begin{bmatrix}1&0&0&0\\-2&1&0&0\\3&-3&1&0\\-22&18&-5&1\end{bmatrix}$

D. $\begin{bmatrix}1&0&0&0\\2&1&0&0\\9&3&1&0\\40&12&5&1\end{bmatrix}$

E. $\begin{bmatrix}1&0&0&0\\1&-1&0&0\\2&-1&-1&0\\2&-3&5&-35\end{bmatrix}$

11. Suppose that $A \in \mathbb{R}^{3\times 3}$ with inverse given by

$$A^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\-2&1&0\\1&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\cdot\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}0.5&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Then, find $\det(A)$:

A. $\frac{1}{6}$

B. $-\frac{1}{6}$

C. -6

D. 6

E. $\frac{2}{3}$

12. Let $A \in \mathbb{R}^{n\times n}$ be given. Suppose that you know $\dim(\mathrm{Nul}(A)) \neq 0$. Which of the following must be true:

   A. $\det(A) \neq 0$

   B. There exists some $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b} \notin \mathrm{Col}(A)$

   C. $\dim(\mathrm{Col}(A)) > 0$

   D. $\mathrm{Col}(A) = \mathbb{R}^n$

   E. $a_{ii} = 0$ for at least one index i for $1 \leq i \leq n$

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13. Consider the 3×3 matrices given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

As discussed in class, we can multiply the matrix A by a sequence of three elementary matrices E_1, E_2, E_3 to produce the upper-triangular matrix $U \in \mathbb{R}^{3\times 3}$ with

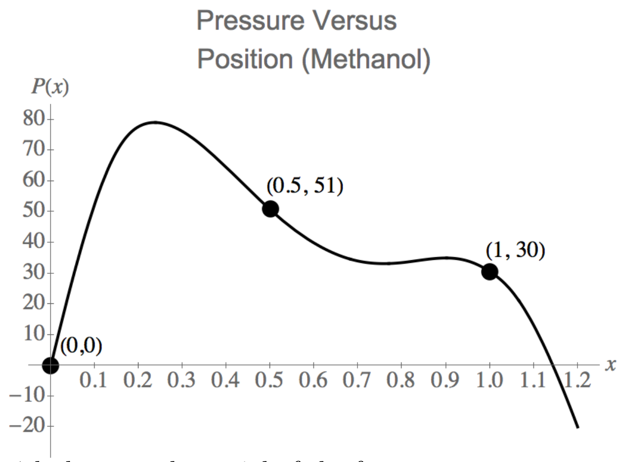
$$E_3 \cdot E_2 \cdot E_1 \cdot A = U.$$

Which of the following matrices is NOT one of the elementary matrices E_i we used to accomplish this transformation?

A. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.5 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

For Problems 14 - 15, we consider the following data set: A combustion-driven potato cannon is a small-scale projectile launcher used for physics demonstrations. In such a device, we can burn methanol to produce high pressures in the combustion chamber that forces a projectile out of the barrel of the cannon. Below is a partial data set that describes the pressure at different distances from the end of the combustion chamber that results from burning methanol.

Position x in meters	Pressure $P(x)$ in kPA
0.0	0
0.5	51
1.0	30



Suppose we choose to fit this data using a 4th degree polynomial of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

We can use this assumption to generate the linear-systems problem

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 51 \\ 30 \end{bmatrix}}_{\mathbf{b}}$$

14. Which of the following does not give a basis for Col(A)? Choose all that apply.

A. $\{A(:,1), A(:,2), A(:,3)\}$

B. $\{A(:,1), A(:,4), A(:,5)\}$

C. $\{A(:,1), A(:,2), A(:,4)\}$

D. $\{A(:,1), A(:,2), A(:,5)\}$

E. $\{A(:,2), A(:,3), A(:,4)\}$

15. Let $c_1, c_2, c_3 \in \mathbb{R}$. Which of the following is not a solution to the linear-systems problem given above?

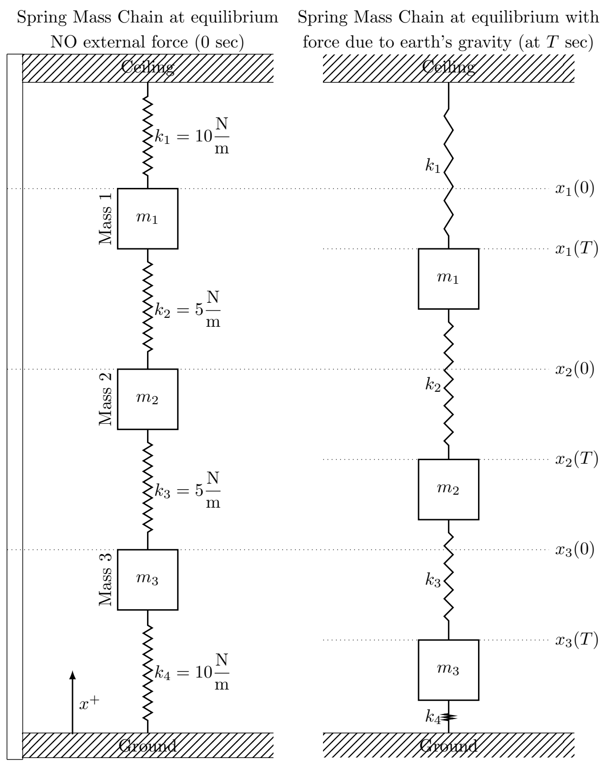
A. $\begin{bmatrix} 0 \\ 174 \\ -144 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0.00 \\ -0.75 \\ 1.75 \\ 0.00 \\ -1.00 \end{bmatrix}$

B. $\begin{bmatrix} 0 \\ 174 \\ -144 \\ 0 \\ 0 \end{bmatrix} + c_1 \cdot \begin{bmatrix} 0.0 \\ -0.5 \\ 1.5 \\ -1.0 \\ 0.0 \end{bmatrix}$

C. $c_1 \cdot \begin{bmatrix} 0.0 \\ -0.5 \\ 1.5 \\ -1.0 \\ 0.0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0.00 \\ -0.75 \\ 1.75 \\ 0.00 \\ -1.00 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 174 \\ -144 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ -2 \\ 6 \\ -4 \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 3 \\ -7 \\ 0 \\ 4 \end{bmatrix}$

For Problems 16, consider the following model for a 3-mass, 4-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly.



16. Recall that the initial position vector $\mathbf{x}_0$ and the mass vector $\mathbf{m}$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and the mass measurements, measured in kg, respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.75 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.100 \\ 0.200 \\ 0.100 \end{bmatrix}$$

Which of the following gives the vector $\mathbf{x}(T) = \begin{bmatrix} x_1(T) & x_2(T) & x_3(T) \end{bmatrix}^T$ as measured in meters, used to store the positions of each mass at equilibrium when when $t = T$? If necessary, please round your answers to the nearest 3 places after the decimal.

A. $\begin{bmatrix} 0.196 \\ 0.392 \\ 0.196 \end{bmatrix}$ **B.** $\begin{bmatrix} 0.446 \\ 0.892 \\ 0.946 \end{bmatrix}$ C. $\begin{bmatrix} 0.020 \\ 0.400 \\ 0.200 \end{bmatrix}$ D. $\begin{bmatrix} 0.270 \\ 0.540 \\ 0.770 \end{bmatrix}$ E. $\begin{bmatrix} 0.054 \\ 0.108 \\ 0.554 \end{bmatrix}$

17. Let $B = A^{-1}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

Then, which of the following gives $\Big(B(1,:)\Big)^T$?

A. A^{-1} does not exist   B. $\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$ C. $\begin{bmatrix} 3 & 4 & -5 \end{bmatrix}$ **D.** $\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$ E. $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

18. Consider the 3×3 matrix A from the problem above. Suppose we use this matrix in the following linear-systems problem

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

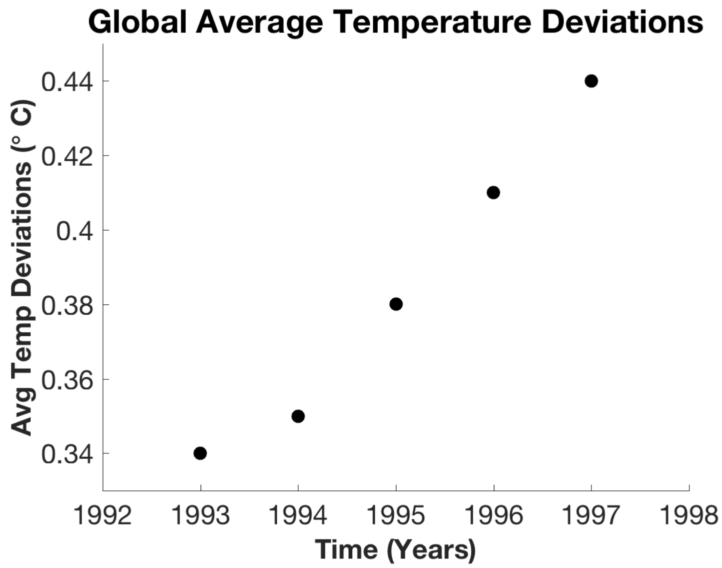
If $\mathbf{x}$ is the solution to this linear-system problem, then which of the following gives the value of

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}?$$

A. -3 **B.** -2 C. -1 D. 0 E. 2

For Problems 19 - 20, we consider the following data set: As part of an effort to understand the effects of human activity on earth's climate, scientists study the changes in the average temperature around the globe. Below, we see a partial data set that presents global mean temperature deviations during the 1990's. The larger the deviations, the more likely it is that the climate is changing over time.

Year	Global Average Temperature Deviations (in °C)
1993	0.3400
1994	0.3500
1995	0.3800
1996	0.4100
1997	0.4400



We can model this partial data set using a quadratic function

$$D(t) = a_0 + a_1 t + a_2 t^2$$

where $D(t)$ represents the global average temperature deviations t years after 1993.

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19. Choose the correct model for the residual vector $\mathbf{r} = A \cdot \mathbf{x} - \mathbf{b}$ associated with the least-squares problem.

A. $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} 0.34 \\ 0.35 \\ 0.38 \\ 0.41 \\ 0.44 \end{bmatrix}}_{\mathbf{b}}$

B. $\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} 0.34 \\ 0.35 \\ 0.38 \\ 0.41 \\ 0.44 \end{bmatrix}}_{\mathbf{b}}$

C. $\underbrace{\begin{bmatrix} 1 & 1993 \\ 1 & 1994 \\ 1 & 1995 \\ 1 & 1996 \\ 1 & 1997 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} 0.34 \\ 0.35 \\ 0.38 \\ 0.41 \\ 0.44 \end{bmatrix}}_{\mathbf{b}}$

D. $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} 0.34 \\ 0.35 \\ 0.38 \\ 0.41 \\ 0.44 \end{bmatrix}}_{\mathbf{b}}$

E. $\underbrace{\begin{bmatrix} 1 & 1993 & 1993^2 \\ 1 & 1994 & 1994^2 \\ 1 & 1995 & 1995^2 \\ 1 & 1996 & 1996^2 \\ 1 & 1997 & 1997^2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} 0.34 \\ 0.35 \\ 0.38 \\ 0.41 \\ 0.44 \end{bmatrix}}_{\mathbf{b}}$

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20. Solve the least-square problem associated with this temperature data. Make a prediction for the global average temperature deviation in the year 2000 (where your prediction is in °C rounded to the nearest 2 decimal places):

A. 0.54 B. 0.53 **C. 0.58** D. 0.51 E. 0.64

Free Response

$\boxed{10}$ 21. Suppose $m, n, p \in \mathbb{N}$. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. Define the product

$$C = A \cdot B.$$

Prove that calculating $C \in \mathbb{R}^{m\times p}$ via matrix-matrix multiplication by rows is equivalent to finding the product $C \in \mathbb{R}^{m\times p}$ using matrix-matrix multiplication by columns.

Solution: Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. In this proof, we will calculate matrix

$$C = A \cdot B$$

using matrix-matrix multiplication by rows. On the other hand, we calculate

$$D = A \cdot B$$

using matrix-matrix multiplication by columns. We will show that these two methods are identical by demonstrating that $c_{ik} = d_{ik}$ for all possible values of i, k. To begin, assume $i, k \in \mathbb{N}$ with $1 \leq i \leq m$ and $1 \leq k \leq p$.

By Rows: Consider

$$\begin{aligned} C(i,:) = A(i,:) \cdot B = \sum_{j=1}^{n} a_{ij} B(j,:) \\ &= a_{i1} \begin{bmatrix} b_{11} & \cdots & b_{1k} & \cdots & b_{1p} \end{bmatrix} \\ &+ a_{i2} \begin{bmatrix} b_{21} & \cdots & b_{2k} & \cdots & b_{2p} \end{bmatrix} \\ &\qquad\qquad\vdots \\ &+ a_{in} \begin{bmatrix} b_{n1} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix} \end{aligned}$$

Using this expansion, we conclude that

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}.$$

Solution: By Columns: Consider

$$\begin{aligned} D(:,k) = A \cdot B(:,k) &= \sum_{j=1}^{n} b_{jk} A(:,j) \\ &= b_{1k} \begin{bmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2k} \begin{bmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{nk} \begin{bmatrix} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix} \end{aligned}$$

Using this expansion, we conclude that

$$d_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}.$$

We have just proved that $c_{ik} = d_{ik}$ for all possible values of i, k. Thus, $C = D$ and we conclude that both forms of matrix-matrix multiplication are identical.

22. (10 pts) Let $A \in \mathbb{R}^{m\times n}$. Prove that $\mathrm{Nul}\left(A^T\right)$ is a subspace of $\mathbb{R}^m$.

Recall $\mathrm{Nul}(A^T) = \{ \vec{x} \in \mathbb{R}^m : A^T \cdot \vec{x} = \vec{0} \}$

We want to prove $\mathrm{Nul}(A^T) \subseteq \mathbb{R}^m$ is a subspace.

To do so, we need to show

i. $\vec{0} \in \mathrm{Nul}(A^T)$:

Consider for $\vec{0} \in \mathbb{R}^m$, we have

$$A^T \cdot \vec{0} = \begin{bmatrix} A(1,:)^T & \cdots & A(m,:)^T \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \sum_{k=1}^{m} 0 \cdot [A(k,:)]^T$$

$$= \vec{0} \in \mathbb{R}^n \checkmark$$

Thus $\vec{0} \in \mathrm{Nul}(A^T)$.

ii. $\mathrm{Nul}(A^T)$ is closed under addition:

Suppose $\vec{x}_1, \vec{x}_2 \in \mathrm{Nul}(A^T)$. Consider

$$A^T \cdot (\vec{x}_1 + \vec{x}_2) = A^T \cdot \vec{x}_1 + A^T \vec{x}_2$$

$$= \vec{0} + \vec{0}$$

$$= \vec{0} \in \mathbb{R}^n \checkmark$$

iii. $\mathrm{Nul}(A^T)$ is closed under scalar-vector mult

Suppose $\alpha \in \mathbb{R}$ and $\vec{x}_1 \in \mathrm{Nul}(A^T)$. Consider

$$A^T \cdot (\alpha \cdot \vec{x}_1) = \alpha \cdot A^T \cdot \vec{x}_1$$

$$= \alpha \cdot \vec{0}$$

$$= \vec{0} \in \mathbb{R}^n \checkmark$$

Thus $\mathrm{Nul}(A^T) \subseteq \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$.

23. (10 pts) Let $A \in \mathbb{R}^{m\times n}$. Suppose $U = \text{RREF}(A)$. Prove that $\text{Nul}(A) = \text{Nul}(U)$. To do so, verify each of the following

i. $\text{Nul}(A) \subseteq \text{Nul}(U)$
ii. $\text{Nul}(U) \subseteq \text{Nul}(A)$

Part i. Let $\vec{x} \in \text{Nul}(A)$ and suppose $U = \text{RREF}(A)$.

We know there exists invertible matrices $E_1, \cdots, E_t \in \mathbb{R}^{m\times m}$

such that $E_t \cdots E_1 \cdot A = U$

$$\Rightarrow \quad E \cdot A = U \quad \text{for } E = E_t \cdots E_1 \text{ where } E \text{ invertible}$$

$$\Rightarrow \quad U\vec{x} = (E\cdot A)\cdot \vec{x}$$

$$= E\cdot(A\cdot\vec{x})$$

$$= E\cdot\vec{0}$$

$$= \vec{0}$$

$$\Rightarrow \quad \cancel{\vec{x} \in \text{Nul}(A}\quad \vec{x} \in \text{Nul}(U)$$

$$\Rightarrow \quad \text{Nul}(A) \subseteq \text{Nul}(U) \checkmark$$

↗

Part ii) Let $\vec{x} \in Nul(U)$ and $U = RREF$

Consider $U\vec{x} = \vec{0}$

$$\Rightarrow (E \cdot A)\vec{x} = \vec{0}$$

$$\Rightarrow E \cdot (A \cdot \vec{x}) = \vec{0}$$

$$\Rightarrow A \cdot \vec{x} = \vec{0} \quad \text{since } Nul(E) = \{\vec{0}\}$$

$$\Rightarrow \vec{x} \in Nul(A)$$

$$\Rightarrow Nul(U) \subseteq Nul(A)$$

With i) and ii) combined, we have

$$Nul(A) = Nul(U) \quad \square$$

Challenge Problem

24. (Optional, Extra Credit, Challenge Problem) Suppose that square matrix $A \in \mathbb{R}^{n\times n}$ is a strictly upper-triangular matrix. In other words, suppose that $a_{ik} = 0$ for all $i \geq k$ where $1 \leq i, k \leq n$. Then prove that $A^n = 0$.

Suppose $A \in \mathbb{R}^{n\times n}$ is strictly upper-triangular with

$a_{ik} = 0$ for all $i \geq k$.

We want to show $A^n = 0$. To do so, we notice

$A^n = 0 \iff A^n \cdot \vec{x} = \vec{0} \in \mathbb{R}^n$ for all $\vec{x} \in \mathbb{R}^n$

To this end, suppose $\vec{x} \in \mathbb{R}^n$.

Consider $A^n \cdot \vec{x} = A^{n-1}(A \cdot \vec{x})$

Setting $\vec{y}_1 = A \cdot \vec{x} \Rightarrow \text{row}_n(\vec{y}_1) = \text{row}_n(A \cdot \vec{x})$

$$= A(n,:) \cdot \vec{x}$$

$$= \begin{bmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \checkmark$$

Then assume $A^{k-1} \vec{x} = \vec{y}_{k-1}$ with $\text{row}_{n-k+1}(\vec{y}_{k-1}) = 0$

Consider $\vec{y}_k = A^k \vec{x} = A \cdot (A^{k-1} \vec{x})$

$$= A \cdot \vec{y}_{k-1}$$

Then $\text{row}_{n-k}(\vec{y}_k) = \text{row}_{n-k}(A) \cdot \vec{y}_{k-1}$

$$= \begin{bmatrix} a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= 0$$

$$= \sum_{i=1}^{n} a_{n-k,i} \, y_i$$

$$= \sum_{i=1}^{n-k} a_{n-k,i} \, y_i + \sum_{i=n-k+1}^{n} a_{n-k,i} \, y_i$$

$$= 0$$