Name : _____ © Jeffrey A. Anderson **Class Number**: _____

Exam 1, Version 3A
Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones or put your cell phones into airplane mode during this exam. Please place your cell phones inside your bag. No cell phones will be allowed on your desk.
- Close your bag and put it under your seat.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 7 sheets of paper (14 pages front and back) including this cover page.
- There are a total of 20 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 10 Multiple Choice Questions (50 points)
  - 4 Free-Response Questions (40 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger than 8.5 inches by 11 inches This note card must be handwritten. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T F Let $m, n \in \mathbb{N}$ and suppose that matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$ are given. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be defined as the matrix-vector multiplication function given by $f(\mathbf{x})$. Then $\mathbf{b} \in \text{Rng}(f)$ if and only if there is some $\mathbf{x}^\star \in \mathbb{R}^n$ such that $\|A \cdot \mathbf{x}^\star - \mathbf{b}\|_2 = 0$.

2. T F The vector sum $\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ equals the zero vector.

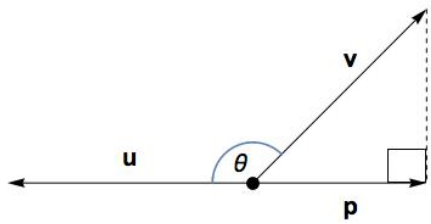
3. T F Let $n \in \mathbb{N}$ and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Suppose that $\{\mathbf{x}, \mathbf{y}\}$ is a linearly independent set of vectors and suppose $\mathbf{z} \in \text{Span}\{\mathbf{x}, \mathbf{y}\}$. Then,the set of vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.

4. T F If $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^n$ is nonzero, then $\| -1 \cdot \mathbf{x}\|_2 + \|1 \cdot \mathbf{x}\|_2 = \| -1 \cdot \mathbf{x} + 1 \cdot \mathbf{x}\|_2 = 0$

5. T F The domain space of the function $f(x) = \dfrac{1}{x}$ is $\mathbb{R}$.

Multiple Choice (50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Let $n \in \mathbb{N}$ and suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Suppose $\mathbf{p} \in \mathbb{R}^n$ is the vector shown in the diagram below:



Then, find $\|\mathbf{p}\|_2$:

A. $-\cos(\theta)$ B. $\cos(\theta)$ C. $\|\mathbf{v}+\mathbf{u}\|_2$ D. $\mathbf{u}\cdot\mathbf{v}$ E. $\|\mathbf{v}-\mathbf{u}\|_2$

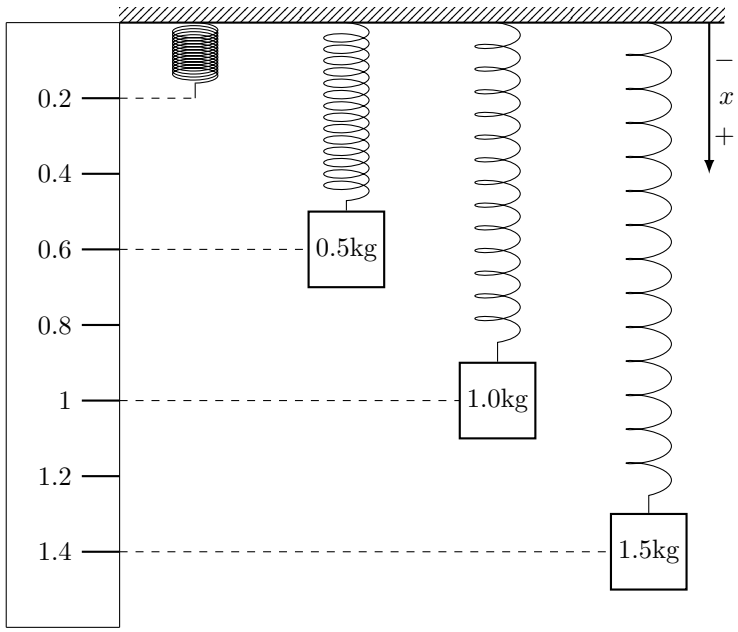
7. Consider the 8−bit binary numbers given by

$$b_1 = 10011001, \qquad b_2 = 00101010$$

Assume that these binary numbers encode integers. Suppose d_i is the decimal conversion of b_i for $i = 1, 2$. In other words, suppose d_i is the base 10 representation of the binary number b_i. Then, find the binary representation of the following expression $d_1 + d_2 + 60$:

A. 11000011 B. 00111100 C. 11111111 D. 11100001 E. 100101001

8. Consider the ideal depiction of a Hooke's law experiment depicted below. Suppose we hang three masses on the same ideal extension spring and record the position data for that spring using a metric ruler so that all positions are measured in meters. Assume also that the acceleration due to earth's gravity is $g = 9.8N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfy Hooke's law exactly.



Suppose that we represent the known spring constant as k N/m. Find a valid vector model that describes Hooke's law for this situation.

A. $\begin{bmatrix} 0.0 \\ 0.4 \\ 0.8 \\ 1.2 \end{bmatrix} = k \cdot \begin{bmatrix} 0.0 \\ 4.9 \\ 9.8 \\ 14.7 \end{bmatrix}$

B. $\begin{bmatrix} 0.00 \\ 0.50 \\ 1.00 \\ 1.50 \end{bmatrix} = k \cdot \begin{bmatrix} 0.2 \\ 0.6 \\ 1.0 \\ 1.4 \end{bmatrix}$

C. $\begin{bmatrix} 0.00 \\ 0.50 \\ 1.00 \\ 1.50 \end{bmatrix} = k \cdot \begin{bmatrix} 0.0 \\ 0.4 \\ 0.8 \\ 1.2 \end{bmatrix}$

D. $\begin{bmatrix} 0.0 \\ 4.9 \\ 9.8 \\ 14.7 \end{bmatrix} = k \cdot \begin{bmatrix} 0.2 \\ 0.6 \\ 1.0 \\ 1.4 \end{bmatrix}$

E. $\begin{bmatrix} 0.0 \\ 4.9 \\ 9.8 \\ 14.7 \end{bmatrix} = k \cdot \begin{bmatrix} 0.0 \\ 0.4 \\ 0.8 \\ 1.2 \end{bmatrix}$

9. Consider the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ t \\ 0 \end{bmatrix},$$

Find the value of t so that the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

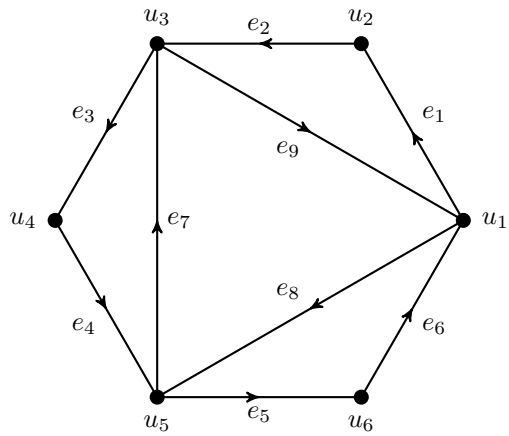
A. 1 B. 3 C. 2 D. -2 E. -5

10. Find the linearly independent set of vectors.

A. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ C. $\left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix. Use your entries for the incidence matrix to identify the correct answer for problem 12.



Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_5	u_6
e_1						
e_2						
e_3						
e_4						
e_5						
e_6						
e_7						
e_8						
e_9						

11. Then, let $\mathbf{a}_k$ represent the kth column of the incidence matrix $A \in \mathbb{R}^{9\times 6}$ associated with the directed graph above. Then, find the value of $\mathbf{a}_1 \cdot \mathbf{a}_6 + \mathbf{a}_2 \cdot \mathbf{a}_4$

A. -2 B. -1 C. 0 D. 1 E. 2

12. Define the following three vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

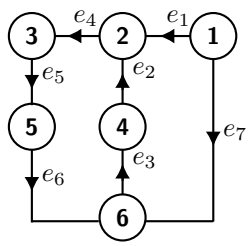
Choose the description below that best depicts $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

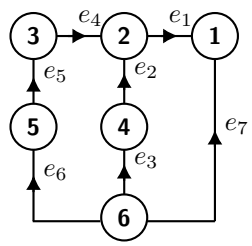
A. $\mathbb{R}^3$ B. a line C. $\mathbb{R}^2$ D. a plane E. three points

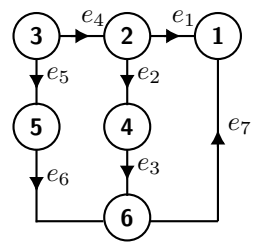
13. Let the following matrix $A \in \mathbb{R}^{7\times 6}$ be the incidence matrix for a directed graph:

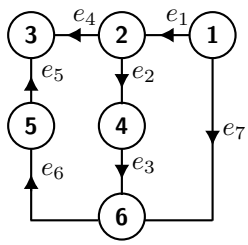
$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

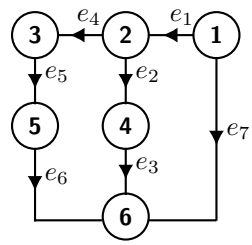
Again, the rows of A correspond to the graph's edges while the columns of A correspond to the nodes of the graph. Then, this is the incidence matrix for which of the following directed graphs:

A.


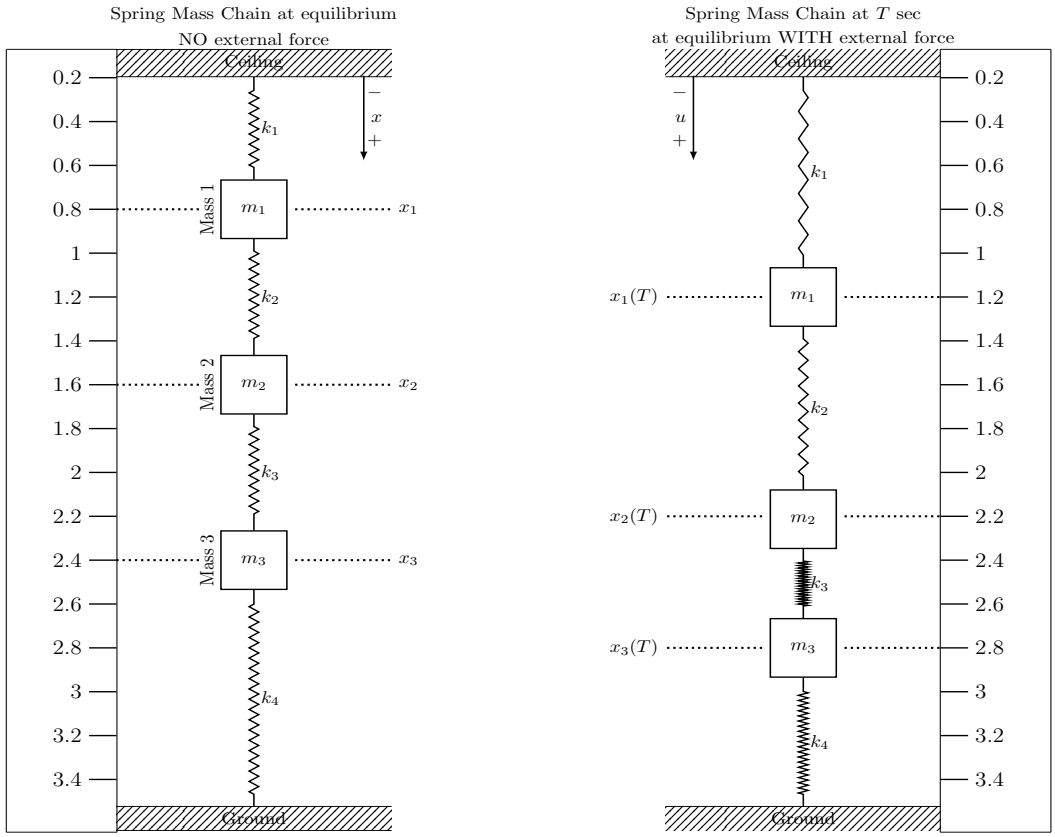
B.


C.


D.


E.


For the following problem, consider the following model for a 3-mass, 4-spring chain drawn below. Note that positive positions and positive displacements are marked in the downward direction. Assume the ruler gives position measurements in meters.



14. Use this diagram and the given position data, find the displacement vector $\mathbf{u}(T)$ that measures the change in position of each mass.

A. $\begin{bmatrix} 0.800 \\ 1.600 \\ 2.400 \end{bmatrix}$ B. $\begin{bmatrix} 1.200 \\ 2.200 \\ 2.800 \end{bmatrix}$ C. $\begin{bmatrix} 0.400 \\ 0.600 \\ 0.400 \end{bmatrix}$ D. $\begin{bmatrix} -0.400 \\ -0.600 \\ -0.400 \end{bmatrix}$ E. $\begin{bmatrix} 1.200 \\ 1.000 \\ 0.400 \end{bmatrix}$

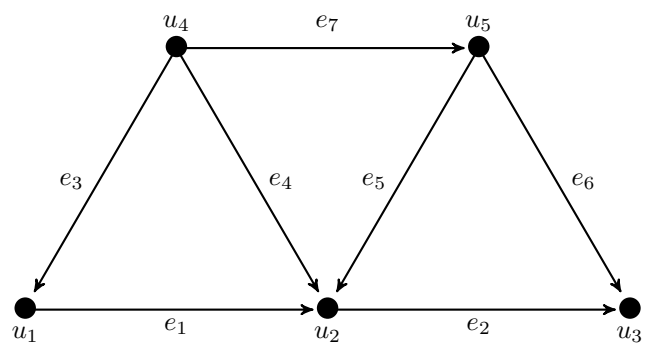
15. Let $n \in \mathbb{N}$ and let the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $\mathbf{x} \cdot \mathbf{y} = 0$. If $\|\mathbf{x}\|_2 = 15$ and $\|\mathbf{y}\|_2 = 16$, find

$$\|\mathbf{x} - \frac{1}{2}\mathbf{y}\|_2$$

A. 289 B. $\sqrt{161}$ C. 161 D. 17 E. Not enough information to solve

Free Response

$\boxed{10}$ 16. Consider the truss system drawn below



A. Create the incidence matrix associated with this directed graph.

B. Suppose that this directed graph corresponds to a bridge. Suppose you impose a coordinate system on this graph so that the position of node u_1 is at vertex

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If each edge is exactly 2 units long and each triangle is equilateral, create a vertex matrix $V \in \mathbb{R}^{2\times 5}$ to model the locations of each node in this coordinate system.

$\boxed{10}$ 17. Let $n \in \mathbb{N}$. Suppose that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. Recall that the inner product between vectors $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

Prove that $(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} = a\,\mathbf{x} \cdot \mathbf{z} + b\,\mathbf{y} \cdot \mathbf{z}$. In other words, prove that the inner product is linear in the first argument. Please show each step of your proof explicitly and explain your work.

$\boxed{10}$ 18. Consider the list set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 6 \end{bmatrix}$$

A. Determine whether or not these vectors are linear dependent. If they are linearly dependent, explicitly demonstrate the linear dependent relationship between these vectors. Regardless, fully justify your answers.

B. Determine whether you can find a set of scalars x_1, x_2, x_3 that are not all equal to zero such that

$$\sum_{k=1}^{3} x_k \cdot \mathbf{a}_k = \mathbf{0}.$$

If such scalars exist, provide specific values that satisfy this equation. If not, explain why such scalars cannot exist.

C. Find a vector in $\mathbb{R}^4$ that is not in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

$\boxed{10}$ 19. Let $n \in \mathbb{N}$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Show that the following holds:

$$\cos(\theta) = \frac{\|\mathbf{x}+\mathbf{y}\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2}{4 \cdot \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2}$$

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem) Suppose that $a, b, c \in \mathbb{R}$ are positive numbers such that $a > b > c$. Then, consider the function

$$R(x_1, x_2, x_3) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

If we require that $x_1^2 + x_2^2 + x_3^2 = 1$, what is the range of the function R?

Use for Scratch Work