Name : ______________ **Class Number**: ______

Exam 1, Version 2A
Math 2B: Linear Algebra

What are the rules of this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note card), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones or put your cell phones into airplane mode during this exam. Please place your cell phones inside your bag. No cell phones will be allowed on your desk.
- Close your bag and put it under your seat.

How long is this exam?

- This exam is scheduled for a 110 minute class period.
- Make sure you have 7 sheets of paper (14 pages front and back) including this cover page.
- There are a total of 20 separate questions (100 points) on this exam including:
  - 5 True/False Questions (10 points)
  - 10 Multiple Choice Questions (50 points)
  - 4 Free-Response Questions (40 points)
  - 1 Optional, Extra Credit Challenge Problem (10 points)

What can I use on this exam?

- You may use one note card that is no larger than 8.5 inches by 11 inches This note card must be handwritten. You may write on both sides of this note card. PLEASE SUBMIT YOUR NOTECARD WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.

How will I be graded on the Free-Response Questions?

- Read the directions carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score on this exam. Neatness and organization are IMPORTANT!
- You will be graded on proper use of vector notation and matrix notation on this exam.
- You will be graded on proper use of theorems and definitions from in-class discussions and homework.

True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

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1. T F The vectors $\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are equal.

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2. T F A linear combination of vectors is the same thing as the span of these vectors.

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3. T F Let $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{x} \in \mathbb{R}^n$. If $f(\mathbf{x}) = A \cdot \mathbf{x}$, then

$$\text{Codomain}(f) = \mathbb{R}^n.$$

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4. T F Let $n \in \mathbb{N}$ and suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^n$. If $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$, then we know that the set of vector $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is linearly dependent.

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5. T F Let $n \in \mathbb{N}$ and let $\mathbf{x} \in \mathbb{R}^n$. Suppose that $\alpha \in \mathbb{R}$ with $\alpha < 0$. Then,

$$\|\alpha \mathbf{x}\|_2 = -\alpha \|\mathbf{x}\|_2$$

Multiple Choice

(50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Consider the set of vectors given by

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

Which of the following vectors sets is equivalent to the span of these three vectors?

A. $\left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$

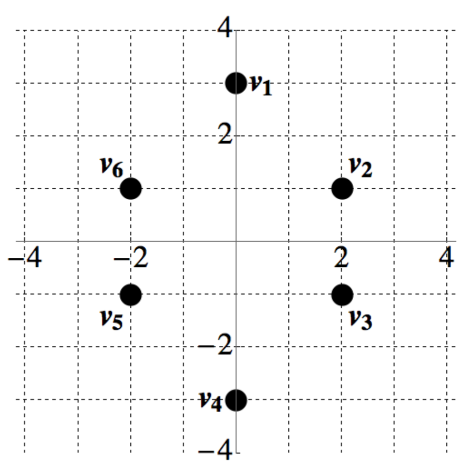
B. $\mathbb{R}^6$

C. $\left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_2 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2, 3 \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2, 3 \right\}$

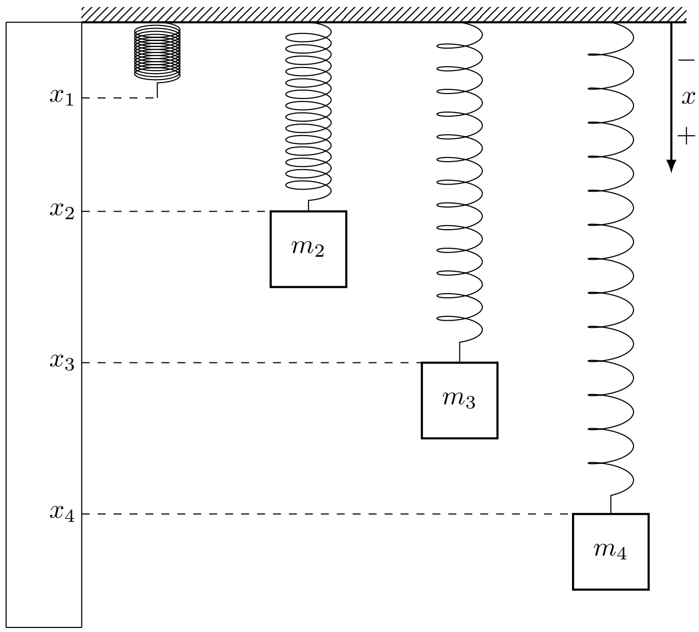
7. Consider the following diagram in $\mathbb{R}^2$:



This figure depicts the set of vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \in \mathbb{R}^2$. Assume that V is the vertex matrix that encode the coordinates of each vertex with $V(:, k) = \mathbf{v}_k$. Set $\mathbf{x} = (V(1, :))^T$ and $\mathbf{y} = (V(2, :))^T$. Which of the following gives $\mathbf{x} \cdot \mathbf{y}$?

A. -3 B. -1 C. 0 D. 1 E. 3

8. Consider the experiment below. Suppose we hang three masses on the same spring and record the position data for that spring. Assume the spring constant is known to be $k = 20$ N/m. Assume also that the acceleration due to earth's gravity is $g = 9.8N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfy Hooke's law exactly.



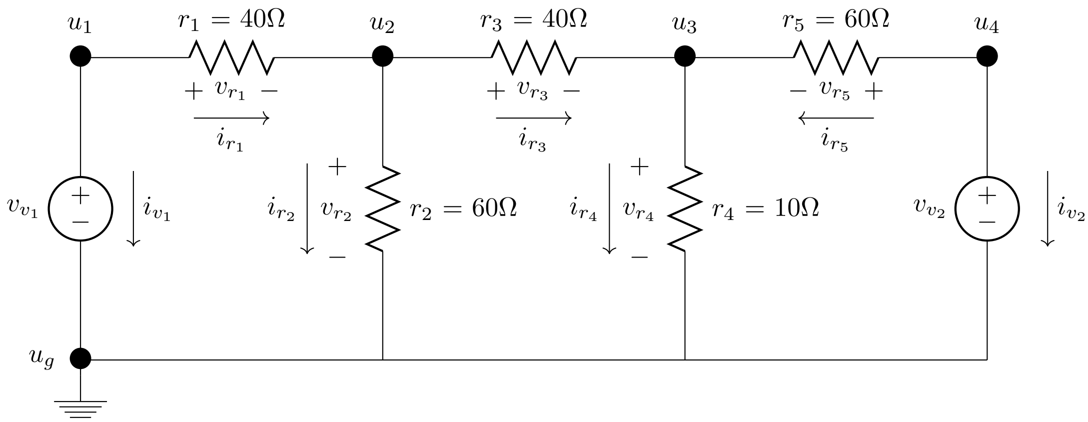
In order to model the relationship between the displacement of the movable end of the spring and the internal force stored in the spring, we introduce two 4×1 vectors given by

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.20 \\ 0.40 \\ 0.60 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Each entry m_i is measured in kg. The entries of the position vector x_i, measured in meters. We know $x_1 = 0.08$m and the other entries $x_2, x_3, x_4 \in \mathbb{R}$ can be calculated from our knowledge of vector $\mathbf{m}$ and Hooke's Law. Which of the following gives the vector $\mathbf{x}$ in this situation?

A. $\begin{bmatrix} 0.000 \\ 0.098 \\ 0.196 \\ 0.294 \end{bmatrix}$ B. $\begin{bmatrix} 0.00 \\ 1.96 \\ 3.92 \\ 5.88 \end{bmatrix}$ C. $\begin{bmatrix} 0.080 \\ 0.178 \\ 0.276 \\ 0.374 \end{bmatrix}$ D. $\begin{bmatrix} 0.08 \\ 2.04 \\ 4.00 \\ 5.96 \end{bmatrix}$ E. $\begin{bmatrix} 0.08 \\ 0.09 \\ 0.10 \\ 0.11 \end{bmatrix}$

Consider the following circuit for problems



Suppose that you measure the vector of node potentials using a voltmeter and you find

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_g \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \\ 4 \\ 10 \\ 0 \end{bmatrix}$$

Use this information and the he conventions we defined in class, to answer questions 9 and 10 below.

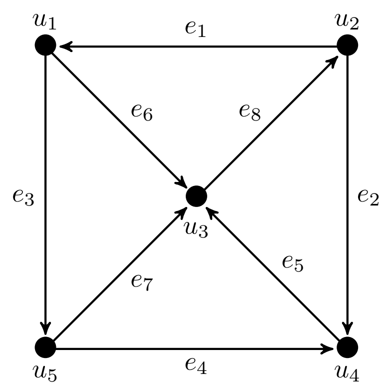
9. Which of the following gives vector **v** of voltage drops across each element?

A. $\begin{bmatrix} 8 \\ 12 \\ 8 \\ 4 \\ 6 \\ 20 \\ 10 \end{bmatrix}$ B. $\begin{bmatrix} -8 \\ -12 \\ -8 \\ -4 \\ -6 \\ -20 \\ -10 \end{bmatrix}$ C. $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ 20 \\ 10 \end{bmatrix}$ D. $\begin{bmatrix} 20 \\ 10 \\ 8 \\ 12 \\ 6 \\ 4 \\ 6 \end{bmatrix}$ E. $\begin{bmatrix} 8 \\ 12 \\ 8 \\ 4 \\ -6 \\ 20 \\ 10 \end{bmatrix}$

10. Which of the following gives vector **i** of currents through each element?

A. $\begin{bmatrix} 320 \\ 840 \\ 320 \\ 40 \\ 360 \\ -320 \\ -360 \end{bmatrix}$ B. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}$ D. $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ -0.2 \\ -0.1 \end{bmatrix}$ E. Not enough information to calculate

Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix. Use your entries for the incidence matrix to identify the correct answer for problem 12.



Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_5
e_1					
e_2					
e_3					
e_4					
e_5					
e_6					
e_7					
e_8					

11. Let $A \in \mathbb{R}^{8\times 5}$ be the incidence matrix you found above. Recall that the rows of A correspond to the graph's edges while the columns of A correspond to the nodes of this graph. Suppose

$$\mathbf{x} = A(:,1) \qquad\qquad \mathbf{y} = \Big[A(:,2) + A(:,3)\Big]$$

Find the inner product $\mathbf{x}\cdot\mathbf{y}$:

A. -2 B. -1 C. 0 D. 1 E. 2

12. Which of the following sets of vectors spans $\mathbb{R}^3$

A. $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix} \right\}$

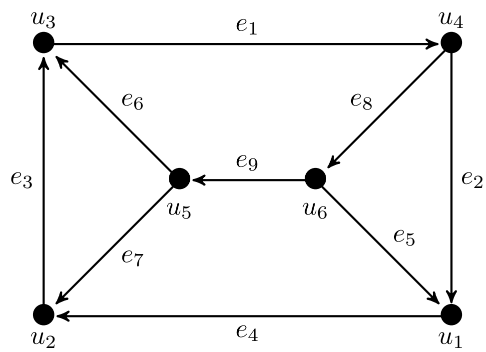
D. $\left\{ \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}4\\2\\4\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}$

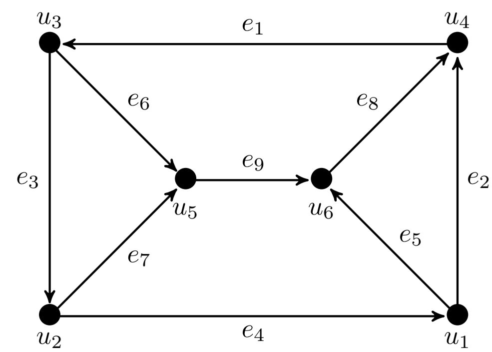
13. Let the following matrix $A \in \mathbb{R}^{9\times 6}$ be the incidence matrix for a directed graph:

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

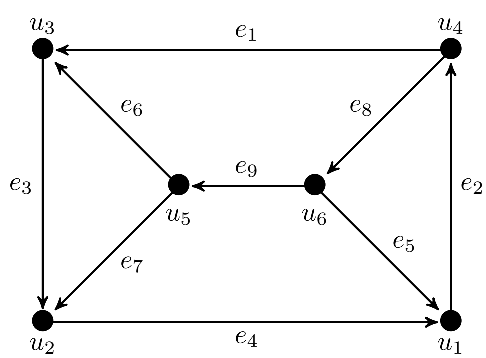
This is the incidence matrix for which of the following directed graphs:



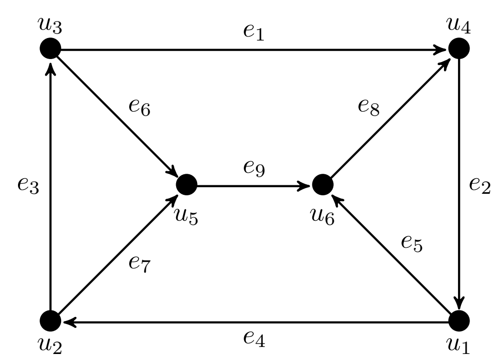
A.



B.



C.



D.

14. Suppose that we are testing the relationship between the voltage across a resistor and the current running through that resistor. In other words, suppose we set up an experiment to verify Ohm's law. During our experiment, we measure the 7 different voltage values across our resistor, given in the vector

$$\mathbf{v} = \begin{bmatrix} 0.0 \\ 3.0 \\ 6.0 \\ 9.0 \\ 12.0 \\ 15.0 \\ 18.0 \end{bmatrix}$$

Each entry of the vector $\mathbf{v}$ is measured in volts (V). We also know that our resistor has a resistance value of $R = 10\Omega$. Given this information, which of the following do we expect our current vector $\mathbf{i} \in \mathbb{R}^7$, measured in amperes (A), to be close to?

A. $\mathbf{i} = \begin{bmatrix} 0.0 \\ 30.0 \\ 60.0 \\ 90.0 \\ 120.0 \\ 150.0 \\ 180.0 \end{bmatrix}$ B. $\mathbf{i} = \begin{bmatrix} 0.000 \\ 0.030 \\ 0.060 \\ 0.090 \\ 0.120 \\ 0.150 \\ 0.180 \end{bmatrix}$ C. $\mathbf{i} = \begin{bmatrix} 0.00 \\ 0.30 \\ 0.60 \\ 0.90 \\ 1.20 \\ 1.50 \\ 1.80 \end{bmatrix}$ D. $\mathbf{i} = \begin{bmatrix} 10.0 \\ 13.0 \\ 16.0 \\ 19.0 \\ 22.0 \\ 25.0 \\ 28.0 \end{bmatrix}$

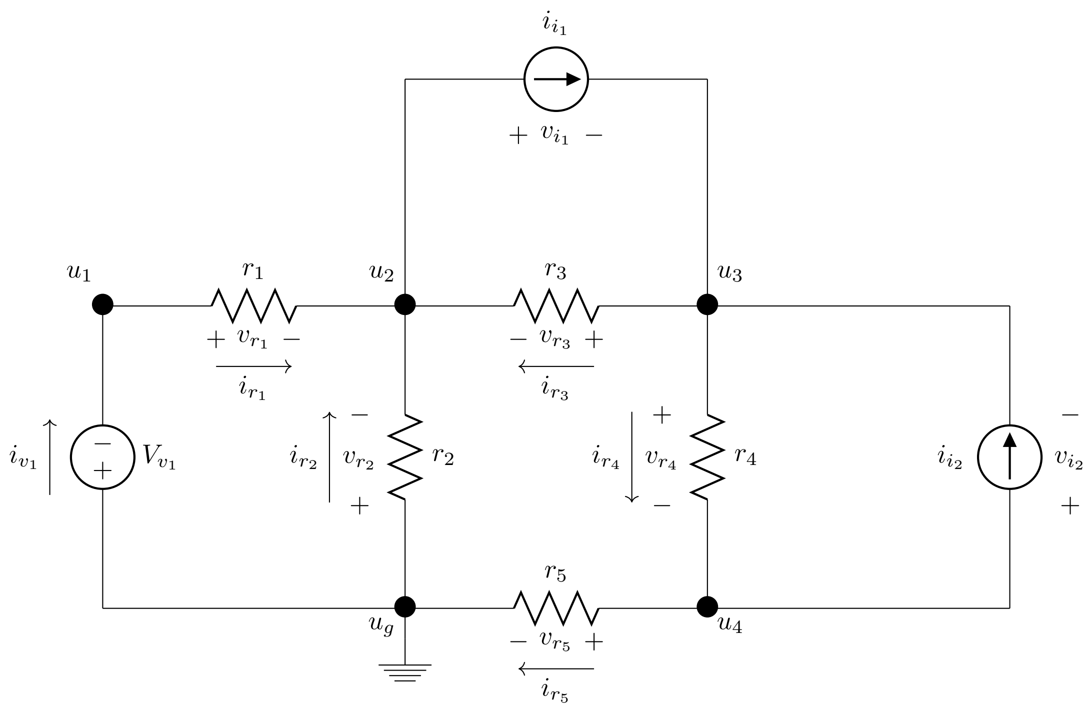
15. Let $n \in \mathbb{N}$ and let the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $\mathbf{x} \cdot \mathbf{y} = 0$. If $\|\mathbf{x}\|_2 = 12$ and $\|\mathbf{y}\|_2 = 10$, find

$$\| -2\mathbf{x} + \mathbf{y}\|_2$$

A. 676 B. 26 C. $\sqrt{576}$ D. 576 E. Not enough information to solve

Free Response

$\boxed{10}$ 16. Consider the circuit diagram below.



A. Create vector models for $\mathbf{v}, \mathbf{i}$ and $\mathbf{u}$. Specifically identify the dimensions of each of these vectors.

B. Show how to calculate the voltage drop across each element as the difference between node voltage potentials. Write the voltage drop calculations for the entire system as a linear combination of vectors.

C. Write the KCL equations for the entire system as a linear combination of vectors.

$\boxed{10}$ 17. Let $A = \{x \in \mathbb{Z} : x^2 \leq 30\}$ and $B = \{x \in \mathbb{Z} : |x| \leq 5\}$. Prove that $A = B$.

$\boxed{10}$ 18. Consider the list set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

A. Show that these vectors are linear dependent by demonstrating that you can write one of these vectors as a linear combination of the other two.

B. Find a set of scalars x_1, x_2, x_3 that are not all equal to zero such that $\sum_{k=1}^{3} x_k \cdot \mathbf{a}_k = \mathbf{0}$.

C. Is this set of scalars you found below unique? Explain your reasoning.

$\boxed{10}$ 19. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then use the algebraic properties of the inner product and 2-norm to prove

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 = 2\left(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\right)$$

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem) Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n \in \mathbb{R}^m$ be a set of vectors with $\mathbf{u}_k \neq \mathbf{0}$ for all $k \in \{1, 2, ..., n\}$. Suppose that $\mathbf{u}_i \cdot \mathbf{u}_k = 0$ for all $1 \leq i, k, \leq n$ with $i \neq k$. Then prove that these vectors are linearly independent.

