
True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

---

1. T **F** The vectors $\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are equal.

---

2. T **F** A linear combination of vectors is the same thing as the span of these vectors.

---

3. T **F** Let $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{x} \in \mathbb{R}^n$. If $f(\mathbf{x}) = A \cdot \mathbf{x}$, then

$$\text{Codomain}(f) = \mathbb{R}^n.$$

---

4. **T** F Let $n \in \mathbb{N}$ and suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^n$. If $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$, then we know that the set of vector $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is linearly dependent.

---

5. **T** F Let $n \in \mathbb{N}$ and let $\mathbf{x} \in \mathbb{R}^n$. Suppose that $\alpha \in \mathbb{R}$ with $\alpha < 0$. Then,

$$\|\alpha \mathbf{x}\|_2 = -\alpha \|\mathbf{x}\|_2$$

Multiple Choice

(50 points: 5 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Consider the set of vectors given by

$$\mathbf{a}_1 = \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix},$$

Which of the following vectors sets is equivalent to the span of these three vectors?

A. $\left\{ \begin{bmatrix}x_1\\x_1\\x_1\\x_1\\x_1\\x_1\end{bmatrix} : x_1 \in \mathbb{R} \right\}$

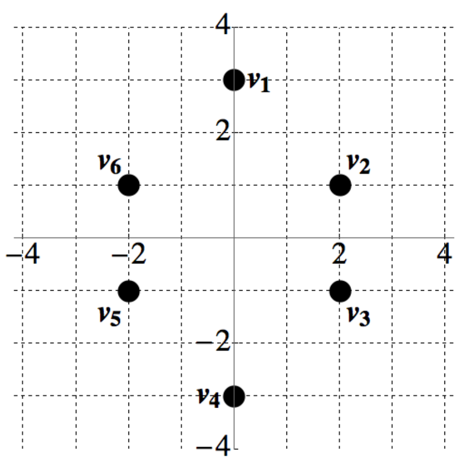
B. $\mathbb{R}^6$

C. $\left\{ \begin{bmatrix}x_1\\x_1\\x_2\\x_2\\x_3\\x_3\end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1,2,3 \right\}$

D. $\left\{ \begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix}x_1\\x_2\\x_3\\x_1\\x_2\\x_3\end{bmatrix} : x_i \in \mathbb{R} \textbf{ for } i = 1,2,3 \right\}$

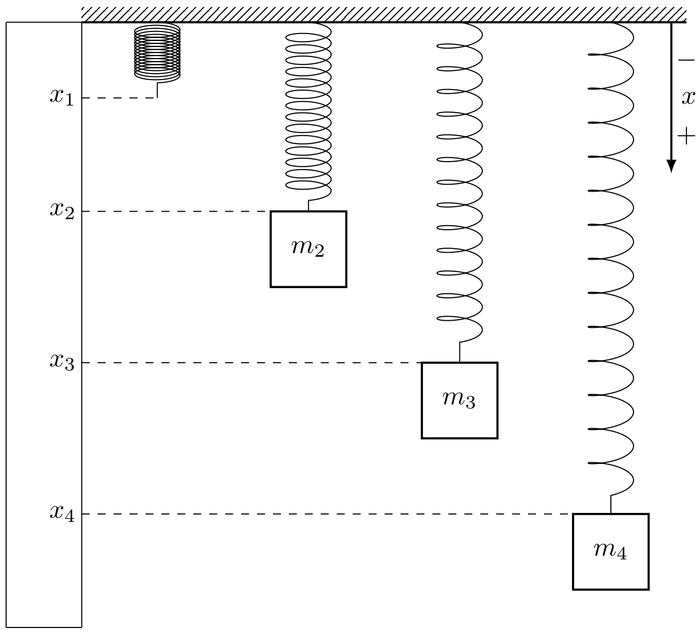
7. Consider the following diagram in $\mathbb{R}^2$:



This figure depicts the set of vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \in \mathbb{R}^2$. Assume that V is the vertex matrix that encode the coordinates of each vertex with $V(:,k) = \mathbf{v}_k$. Set $\mathbf{x} = (V(1,:))^T$ and $\mathbf{y} = (V(2,:))^T$. Which of the following gives $\mathbf{x} \cdot \mathbf{y}$?

A. -3 B. -1 **C.** 0 D. 1 E. 3

8. Consider the experiment below. Suppose we hang three masses on the same spring and record the position data for that spring. Assume the spring constant is known to be $k = 20$ N/m. Assume also that the acceleration due to earth's gravity is $g = 9.8N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfy Hooke's law exactly.



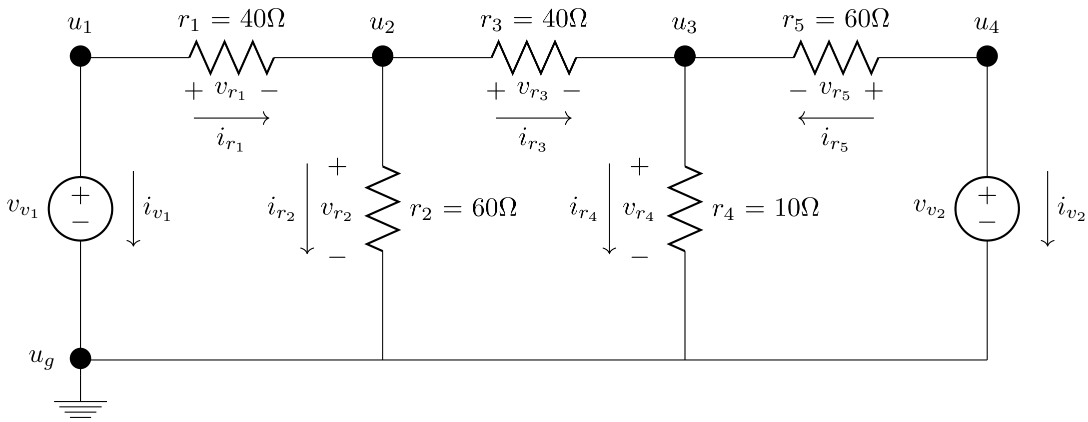
In order to model the relationship between the displacement of the movable end of the spring and the internal force stored in the spring, we introduce two 4×1 vectors given by

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.20 \\ 0.40 \\ 0.60 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Each entry m_i is measured in kg. The entries of the position vector x_i, measured in meters. We know $x_1 = 0.08$m and the other entries $x_2, x_3, x_4 \in \mathbb{R}$ can be calculated from our knowledge of vector $\mathbf{m}$ and Hooke's Law. Which of the following gives the vector $\mathbf{x}$ in this situation?

A. $\begin{bmatrix} 0.000 \\ 0.098 \\ 0.196 \\ 0.294 \end{bmatrix}$ B. $\begin{bmatrix} 0.00 \\ 1.96 \\ 3.92 \\ 5.88 \end{bmatrix}$ **C.** $\begin{bmatrix} 0.080 \\ 0.178 \\ 0.276 \\ 0.374 \end{bmatrix}$ D. $\begin{bmatrix} 0.08 \\ 2.04 \\ 4.00 \\ 5.96 \end{bmatrix}$ E. $\begin{bmatrix} 0.08 \\ 0.09 \\ 0.10 \\ 0.11 \end{bmatrix}$

Consider the following circuit for problems



Suppose that you measure the vector of node potentials using a voltmeter and you find

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_g \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \\ 4 \\ 10 \\ 0 \end{bmatrix}$$

Use this information and the he conventions we defined in class, to answer questions 9 and 10 below.

---

9. Which of the following gives vector $\mathbf{v}$ of voltage drops across each element?

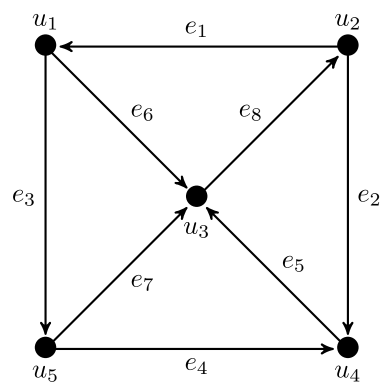
A. $\begin{bmatrix} 8 \\ 12 \\ 8 \\ 4 \\ 6 \\ 20 \\ 10 \end{bmatrix}$ B. $\begin{bmatrix} -8 \\ -12 \\ -8 \\ -4 \\ -6 \\ -20 \\ -10 \end{bmatrix}$ C. $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ 20 \\ 10 \end{bmatrix}$ D. $\begin{bmatrix} 20 \\ 10 \\ 8 \\ 12 \\ 6 \\ 4 \\ 6 \end{bmatrix}$ E. $\begin{bmatrix} 8 \\ 12 \\ 8 \\ 4 \\ -6 \\ 20 \\ 10 \end{bmatrix}$

---

10. Which of the following gives vector $\mathbf{i}$ of currents through each element?

A. $\begin{bmatrix} 320 \\ 840 \\ 320 \\ 40 \\ 360 \\ -320 \\ -360 \end{bmatrix}$ B. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}$ **D.** $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.1 \\ -0.2 \\ -0.1 \end{bmatrix}$ E. Not enough information to calculate

Consider the directed graph given below. Use this graph to fill in the corresponding incidence matrix. Use your entries for the incidence matrix to identify the correct answer for problem 12.



Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_5
e_1					
e_2					
e_3					
e_4					
e_5					
e_6					
e_7					
e_8					

11. Let $A \in \mathbb{R}^{8\times 5}$ be the incidence matrix you found above. Recall that the rows of A correspond to the graph's edges while the columns of A correspond to the nodes of this graph. Suppose

$$\mathbf{x} = A(:,1) \qquad\qquad \mathbf{y} = \Big[A(:,2) + A(:,3)\Big]$$

Find the inner product $\mathbf{x}\cdot\mathbf{y}$:

A. -2 B. -1 C. 0 D. 1 E. 2

12. Which of the following sets of vectors spans $\mathbb{R}^3$

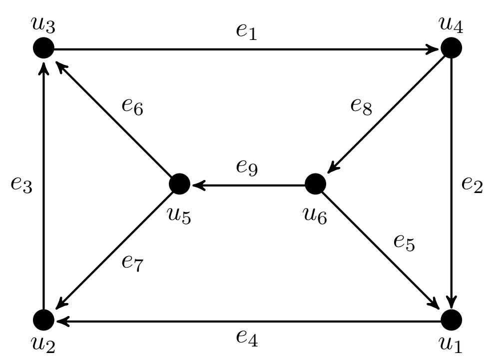
A. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ E. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

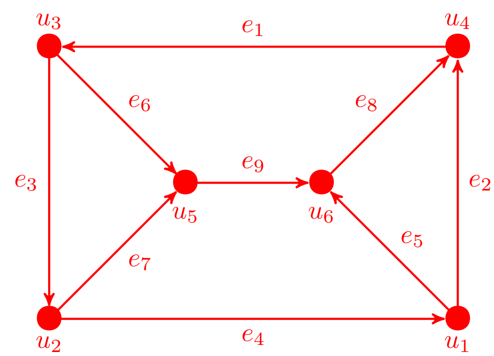
13. Let the following matrix $A \in \mathbb{R}^{9\times 6}$ be the incidence matrix for a directed graph:

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

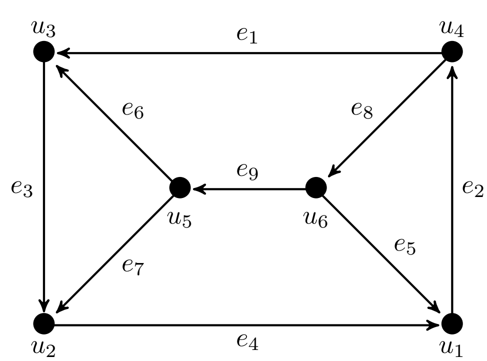
This is the incidence matrix for which of the following directed graphs:



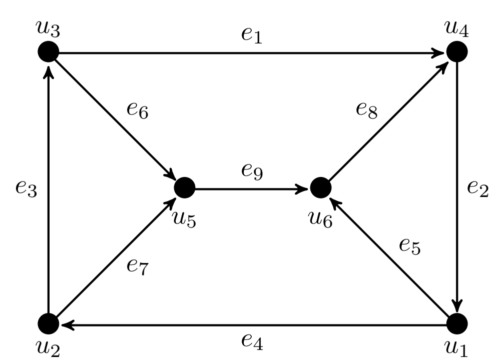
A.



B.



C.



D.

14. Suppose that we are testing the relationship between the voltage across a resistor and the current running through that resistor. In other words, suppose we set up an experiment to verify Ohm's law. During our experiment, we measure the 7 different voltage values across our resistor, given in the vector

$$\mathbf{v} = \begin{bmatrix} 0.0 \\ 3.0 \\ 6.0 \\ 9.0 \\ 12.0 \\ 15.0 \\ 18.0 \end{bmatrix}$$

Each entry of the vector $\mathbf{v}$ is measured in volts (V). We also know that our resistor has a resistance value of $R = 10\Omega$. Given this information, which of the following do we expect our current vector $\mathbf{i} \in \mathbb{R}^7$, measured in amperes (A), to be close to?

A. $\mathbf{i} = \begin{bmatrix} 0.0 \\ 30.0 \\ 60.0 \\ 90.0 \\ 120.0 \\ 150.0 \\ 180.0 \end{bmatrix}$

B. $\mathbf{i} = \begin{bmatrix} 0.000 \\ 0.030 \\ 0.060 \\ 0.090 \\ 0.120 \\ 0.150 \\ 0.180 \end{bmatrix}$

C. $\mathbf{i} = \begin{bmatrix} 0.00 \\ 0.30 \\ 0.60 \\ 0.90 \\ 1.20 \\ 1.50 \\ 1.80 \end{bmatrix}$

D. $\mathbf{i} = \begin{bmatrix} 10.0 \\ 13.0 \\ 16.0 \\ 19.0 \\ 22.0 \\ 25.0 \\ 28.0 \end{bmatrix}$

---

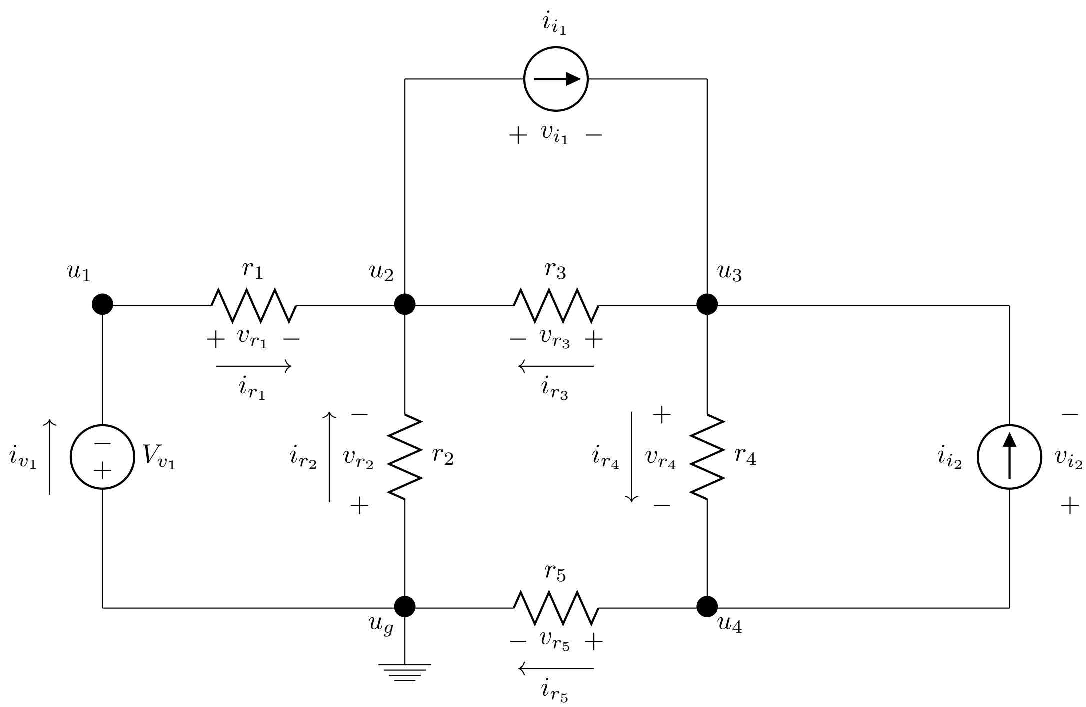
15. Let $n \in \mathbb{N}$ and let the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $\mathbf{x} \cdot \mathbf{y} = 0$. If $\|\mathbf{x}\|_2 = 12$ and $\|\mathbf{y}\|_2 = 10$, find

$$\| -2\mathbf{x} + \mathbf{y}\|_2$$

A. 676 **B.** 26 C. $\sqrt{576}$ D. 576 E. Not enough information to solve

Free Response

$\boxed{10}$ 16. Consider the circuit diagram below.



A. Create vector models for $\mathbf{v}, \mathbf{i}$ and $\mathbf{u}$. Specifically identify the dimensions of each of these vectors.

Solution: We begin our modeling process by creating our vector $\mathbf{u} \in \mathbb{R}^5$ given by

$$\mathbf{u}_g = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_g \end{bmatrix} = \begin{bmatrix} \text{the voltage potential of node 1 (relative to the ground node)} \\ \text{the voltage potential of node 2 (relative to the ground node)} \\ \text{the voltage potential of node 3 (relative to the ground node)} \\ \text{the voltage potential of node 4 (relative to the ground node)} \\ \text{the voltage potential of the ground node (assumed to be zero in practice)} \end{bmatrix}$$

We know this vector is a 5×1 since there are 5 distinct nodes in this circuit.

If we were to measure the values of these variables on a physical circuit, we hold the black probe of the voltmeter to the ground node. Then we apply the red probe to each of the other nodes on the circuit. We would read the associated potentials using our voltmeter and record the data in a 5×1 vector. The last entry of this vector is assumed to be zero since this the reference voltage value against which all other potentials are measured.

Solution: Next, we create our vector $\mathbf{v} \in \mathbb{R}^7$ that stores the voltage drop across each element. In this case, we have

$$\mathbf{v} = \left[\begin{array}{c} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{array}\right] = \left[\begin{array}{c} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ \hdashline v_{v_1} \\ \hdashline v_{i_1} \\ v_{i_2} \end{array}\right] = \left[\begin{array}{l} \text{the voltage drop across resistor 1} \\ \text{the voltage drop across resistor 2} \\ \text{the voltage drop across resistor 3} \\ \text{the voltage drop across resistor 4} \\ \text{the voltage drop across resistor 5} \\ \hdashline \text{the voltage drop across voltage source 1} \\ \hdashline \text{the voltage drop across current source 1} \\ \text{the voltage drop across current source 2} \end{array}\right]$$

Since we calculate one voltage drop across each element in our circuit and since there are precisely 8 elements in our circuit, we know there are exactly 8 entries in this vector. In each case, the sub-block vectors $\mathbf{v}_r \in \mathbb{R}^5, \mathbf{v}_i \in \mathbb{R}^2$ are unknown. The sub-block $\mathbf{v}_v \in \mathbb{R}^1$ is given by the values of the voltage source. We can identify the individual entries in each of these vectors as follows:

$$\mathbf{v}_r = \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \end{bmatrix}, \qquad \mathbf{v}_v = \begin{bmatrix} v_{v_1} \end{bmatrix}, \qquad \mathbf{v}_i = \begin{bmatrix} v_{i_1} \\ v_{i_2} \end{bmatrix}.$$

To measure each entry of this vector on a physical circuit, we hold the black probe of the voltmeter to the lead with the negative voltage reference sign. Then we apply the red probe to the other lead lead with the negative voltage reference sign. We would read the associated voltage drop across each element and store this measured date in our 8×1 vector.

Solution:
Finally, we create our vector $\mathbf{i} \in \mathbb{R}^8$ that stores the current running through each element. In this case, we have

$$\mathbf{i} = \begin{bmatrix} \mathbf{i}_r \\ \hdashline \mathbf{i}_v \\ \hdashline \mathbf{i}_i \end{bmatrix} = \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ \hdashline i_{v_1} \\ \hdashline i_{i_1} \\ i_{i_2} \end{bmatrix} = \begin{bmatrix} \text{the current through resistor 1} \\ \text{the current through resistor 2} \\ \text{the current through resistor 3} \\ \text{the current through resistor 4} \\ \text{the current through resistor 5} \\ \hdashline \text{the current through voltage source 1} \\ \hdashline \text{the current through current source 1} \\ \text{the current through current source 2} \end{bmatrix}$$

Since we calculate one voltage drop across each element in our circuit and since there are precisely 8 elements in our circuit, we know there are exactly 8 entries in this vector. in each case, the sub-block vectors $\mathbf{i}_r \in \mathbb{R}^5, \mathbf{i}_v \in \mathbb{R}^1$ are unknown. The sub-block $\mathbf{i}_i \in \mathbb{R}^2$ is given by the values of the current sources. We can identify the individual entries in each of these vectors as follows:

$$\mathbf{i}_r = \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \end{bmatrix}, \qquad \mathbf{i}_v = \begin{bmatrix} i_{v_1} \end{bmatrix}, \qquad \mathbf{i}_i = \begin{bmatrix} i_{i_1} \\ i_{i_2} \end{bmatrix}.$$

To measure each entry of this vector on a physical circuit, we hold the black probe of the ammeter to the lead with the negative voltage reference sign. Then we apply the red probe to the other lead lead with the negative voltage reference sign. We would read the associated current running through each element and store this measured date in our 8×1 vector.

B. Show how to calculate the voltage drop across each element as the difference between node voltage potentials. Write the voltage drop calculations for the entire system as a linear combination of vectors.

Solution: To calculate the voltage drop across each element in the circuit, we take the difference between the voltage potentials from the nodes of the two leads of each elements. In this difference, we assign a positive value to the node which has a positive voltage sign and a negative value to the node with a negative voltage sign. With this in mind, we calculate:

$$\mathbf{v} = \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ \hdashline v_{v_1} \\ \hdashline v_{i_1} \\ v_{i_2} \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ -u_2 + u_g \\ -u_2 + u_3 \\ u_3 - u_4 \\ u_4 - u_g \\ \hdashline -u_1 + u_g \\ \hdashline u_2 - u_3 \\ -u_3 + u_4 \end{bmatrix} = u_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + u_2 \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} + u_4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + u_g \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

As we will see, we can write this linear combination as a matrix-vector product.

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ \hdashline v_{v_1} \\ \hdashline v_{i_1} \\ v_{i_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ \hdashline -1 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_g \end{bmatrix}$$

Notice, this matrix is the same matrix as is given by the incidence matrix corresponding to the digraph for this circuit:

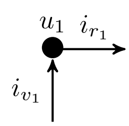
Directed Graph
Incident Matrix Table

	u_1	u_2	u_3	u_4	u_g
e_{r_1}	1	-1	0	0	0
e_{r_2}	0	-1	0	0	1
e_{r_3}	0	-1	1	0	0
e_{r_4}	0	0	1	-1	0
e_{r_5}	0	0	0	1	-1
e_{v_1}	-1	0	0	0	1
e_{i_1}	0	1	-1	0	0
e_{i_2}	0	0	-1	1	0

C. Write the KCL equations for the entire system as a linear combination of vectors.

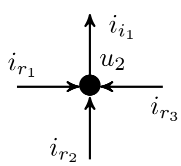
Solution: Recall the Kirchoff's Current Law states that the sum or difference of all currents entering or existing a single node must be equal to zero. We use the current reference direction to determine the sign of the current variables in this sum. An element whose current reference direction enters a node has a negative sign assigned to its current variable. On the other hand, we assign a positive sign to be assigned to a current variable with a current reference direction that leaves a node.

Solution: Node 1:. The Kirchoff's Current law equation for node 1 is:



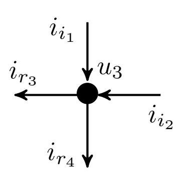
$$0 = i_{r_1} - i_{v_1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_{v_1} \\ i_{i_1} \\ i_{i_2} \end{bmatrix}$$

Node 2:. The Kirchoff's Current law equation for node 2 is:



$$0 = -i_{r_1} - i_{r_2} - i_{r_3} + i_{i_1} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_{v_1} \\ i_{i_1} \\ i_{i_2} \end{bmatrix}$$

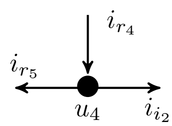
Node 3:. The Kirchoff's Current law equation for node 3 is:



$$0 = i_{r_3} + i_{r_4} - i_{i_1} - i_{i_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_{v_1} \\ i_{i_1} \\ i_{i_2} \end{bmatrix}$$

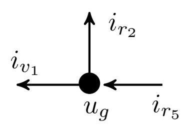
Solution:

Node 4:. The Kirchoff's Current law equation for node 4 is



$$0 = -i_{r_4} + i_{r_5} + i_{i_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_{v_1} \\ i_{i_1} \\ i_{i_2} \end{bmatrix}$$

Ground Node:. The Kirchoff's Current law equation for the ground node is:



$$0 = i_{r_2} - i_{r_5} + i_{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_{v_1} \\ i_{i_1} \\ i_{i_2} \end{bmatrix}$$

We can write the entire set of KCL equations as a linear combination as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = i_{r_1} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + i_{r_2} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i_{r_3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i_{r_4} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + i_{r_5} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + i_{v_1} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i_{i_1} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + i_{i_2} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

We can also write the combined Kirchoff's Current law equations using a matrix-vector product.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ \hdashline i_{v_1} \\ \hdashline i_{i_1} \\ i_{i_2} \end{bmatrix}$$

$\boxed{10}$ 17. Let $A = \{x \in \mathbb{Z} : x^2 \leq 30\}$ and $B = \{x \in \mathbb{Z} : |x| \leq 5\}$. Prove that $A = B$.

Solution: We know from Lesson 1 that in order to prove these two sets are equal, we need to show:

i. $A \subseteq B$

ii. $B \subseteq A$

Case i: $A \subseteq B$

Proof. Let $x \in A$. Then, by definition, we have x is an integer, meaning $x \in \{0, -1, 1, -2, 2, ...\}$. We also know

$$
\begin{aligned}
x \in A \quad &\Longrightarrow \quad x^2 \leq 30 \\
&\Longrightarrow \quad x^2 \leq 25 \text{ since the closest perfect square is } 25 \\
&\Longrightarrow \quad \sqrt{x^2} \leq \sqrt{25} \\
&\Longrightarrow \quad |x| \leq 5 \\
&\Longrightarrow \quad x \in B
\end{aligned}
$$

Thus, we see that $A \subseteq B$. □

Case ii: $B \subseteq A$

Proof. Let $x \in B$. Then, by definition, we have x is an integer. Moreover, we also know

$$
\begin{aligned}
x \in B \quad &\Longrightarrow \quad |x| \leq 5 \\
&\Longrightarrow \quad -5 \leq x \leq 5 \\
&\Longrightarrow \quad -5 \leq x \leq 0 \text{ and } 0 \leq x \leq 5 \\
&\Longrightarrow \quad 25 \geq x^2 \geq 0 \text{ and } 0 \leq x^2 \leq 25 \\
&\Longrightarrow \quad 0 \leq x^2 \leq 30 \\
&\Longrightarrow \quad x \in A
\end{aligned}
$$

Thus, we see that $B \subseteq A$. □

Since we have proved both $A \subseteq B$ and $B \subseteq A$, we conclude that $A = B$.

$\boxed{10}$ 18. Consider the list set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

A. Show that these vectors are linear dependent by demonstrating that you can write one of these vectors as a linear combination of the other two.

Solution: We see that we can write $\mathbf{a}_3$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ as follows:

$$\mathbf{a}_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + -2 \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \end{bmatrix} = 4 \cdot \mathbf{a}_1 + -2 \cdot \mathbf{a}_2$$

B. Find a set of scalars x_1, x_2, x_3 that are not all equal to zero such that $\sum_{k=1}^{3} x_k \cdot \mathbf{a}_k = \mathbf{0}$.

Solution: We can rewrite our linear combination equation from above to set the right hand side equal to zero:

$$\mathbf{a}_3 = 4 \cdot \mathbf{a}_1 + -2 \cdot \mathbf{a}_2 \qquad \Longrightarrow \qquad -4 \cdot \mathbf{a}_1 + 2 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3 = \mathbf{0}$$

Then, the nonzero scalars

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$$

have the property required in this problem.

C. Is this set of scalars you found below unique? Explain your reasoning.

Solution: No, the scalars we found above are not unique. We can take any scalar multiple of the vector $\mathbf{x}$ and the result will still produce a linear combination equal to zero.

$\boxed{10}$ 19. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then use the algebraic properties of the inner product and 2-norm to prove

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 = 2\left(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\right)$$

Solution: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider:

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x}+\mathbf{y}) + \mathbf{y} \cdot (\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|_2^2 + 2\ \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|_2^2\end{aligned}$$

Using the same reasoning, we can find

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|_2^2 &= (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x}-\mathbf{y}) - \mathbf{y} \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|_2^2 - 2\ \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|_2^2\end{aligned}$$

Then, combining these two results, we see

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 &= \|\mathbf{x}\|^2 + 2\ \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 - 2\ \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|_2^2 \\ &= 2\|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2^2 \\ &= 2\Big(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\Big)\end{aligned}$$

This is exactly what we wanted to show.

Challenge Problem

20. (Optional, Extra Credit, Challenge Problem) Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n \in \mathbb{R}^m$ be a set of vectors with $\mathbf{u}_k \neq \mathbf{0}$ for all $k \in \{1, 2, ..., n\}$. Suppose that $\mathbf{u}_i \cdot \mathbf{u}_k = 0$ for all $1 \leq i, k, \leq n$ with $i \neq k$. Then prove that these vectors are linearly independent.

