True/False (10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T **F** Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then the matrix-vector product $A \cdot \mathbf{x}$ represents a linear combination of the rows of A with scalar multiples defined by the entries of $\mathbf{x}$.

2. **T** F All functions are relations.

3. **T** F Any set of vectors that contains the zero vector must be linearly dependent.

4. T **F** Any two matrices that are conformable for matrix multiplication must have the same number of rows.

5. T **F** Since all entries of the vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are zero, these vectors are equal.

Multiple Choice (60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen, mark your answer on your Scantron form.

6. Define vectors

$$\mathbf{x} = \begin{bmatrix} t \\ -4 \\ 2 \\ t \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -t \\ t \\ 5 \\ 1 \end{bmatrix}$$

Find <u>all</u> values of scalar t so that the inner product $\mathbf{x} \cdot \mathbf{y} = 0$

A. $t = -2$ B. $t = 5$ and $t = -2$ C. $t = 5$ and $t = 2$ **D. $t = -5$ and $t = 2$** E. $t = 5$

7. Suppose that $\mathbf{e}_k \in \mathbb{R}^3$ is the 3×1 elementary basis vector with $\mathbf{e}_k = I_3(:,k)$ for $k = 1,2,3$. Let

$$A = -2 \cdot \mathbf{e}_3 \cdot \mathbf{e}_1^T + 4 \cdot \mathbf{e}_2 \cdot \mathbf{e}_2^T + 3 \cdot \mathbf{e}_3 \cdot \mathbf{e}_3^T - \mathbf{e}_1 \cdot \mathbf{e}_2^T$$

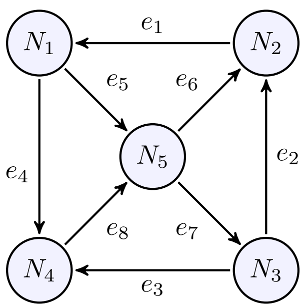
Then, which of the following gives $A(:,2) \cdot A(1,:)$?

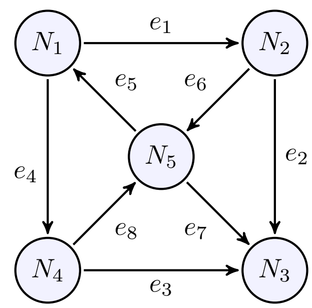
A. $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ B. 4 C. 1 **D.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

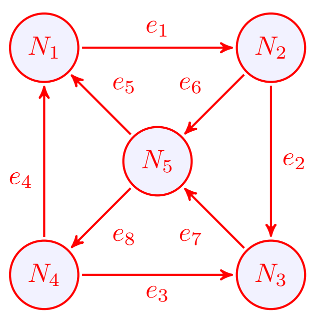
8. Let the following matrix $A \in \mathbb{R}^{8\times 5}$ be the incidence matrix for a directed graph:

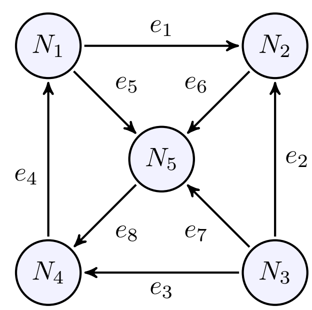
$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Then, this is the incidence matrix for which of the following directed graphs:

A. 

B. 

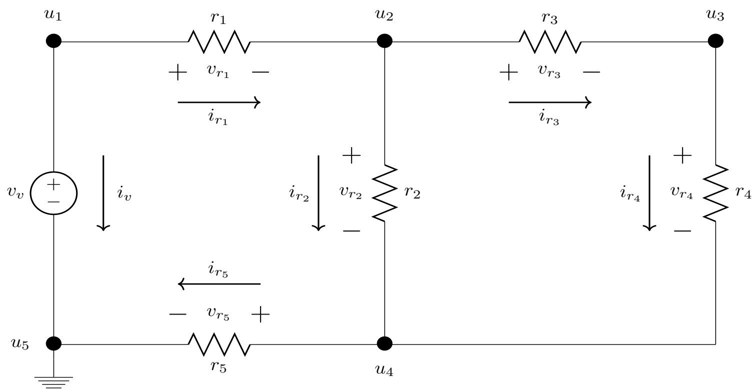
C. 

D. 

9. Suppose that we define ellipse $E = \left\{(x,y) : \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1\right\}$. Find the range, $\mathrm{Rng}(E)$, of this relation.

A. $(-5,5)$ B. $[-5,5]$ **C.** $[-3,3]$ D. $(-3,3)$ E. $\mathbb{R}$

Consider the following ideal circuit diagram. Use this figure to answer questions 10, 11, and 12 below.



10. Which of the following matrix-vector products is used to calculate the voltage across each circuit element.

A. $$\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ v_v \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

B. $$\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ v_v \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

C. $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ v_v \end{bmatrix}$$

D. $$\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ v_v \end{bmatrix} = \begin{bmatrix} \frac{1}{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ v_v \end{bmatrix}$$

11. Which matrix-vector multiplication problems gives Kirchoff's Current Laws for the entire circuit?

A. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

B. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \\ v_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

C. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

D. $$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \\ i_v \end{bmatrix}$$

12. Which of the following matrix-vector multiplication problems can be used to state Ohm's Law for each resistor in the circuit?

A. $\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \end{bmatrix} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$

B. $\begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \end{bmatrix} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix} \cdot \begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \end{bmatrix}$

C. $\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \end{bmatrix} = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \end{bmatrix}$

D. $\begin{bmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ v_{r_4} \\ v_{r_5} \end{bmatrix} = \begin{bmatrix} \frac{1}{r_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r_4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r_5} \end{bmatrix} \cdot \begin{bmatrix} i_{r_1} \\ i_{r_2} \\ i_{r_3} \\ i_{r_4} \\ i_{r_5} \end{bmatrix}$

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13. Let 10110110 be an 8−bit binary integer. What is the decimal representation of this number?

A. 109 B. 364 C. 218 D. 80880880 **E.** 182

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14. Define three vectors in $\mathbb{R}^4$ as

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 10 \\ 15 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 25 \\ 100 \\ 225 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 5 \\ -45 \\ -45 \\ 5 \end{bmatrix}$$

We can confirm that $\mathbf{a}_4 = 5 \cdot \mathbf{a}_1 - 15 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$. Choose the vector $\mathbf{x} \in \mathbb{R}^4$ such that

$$\left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4\ \right] \cdot \mathbf{x} = \mathbf{0}$$

A. $\begin{bmatrix} 45 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ B. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} -5 \\ 15 \\ -1 \\ -1 \end{bmatrix}$ **D.** $\begin{bmatrix} 5 \\ -15 \\ 1 \\ -1 \end{bmatrix}$ E. The product will never be zero

15. Let $A \in \mathbb{R}^{8\times 4}$, $B \in \mathbb{R}^{4\times 7}$, and $C \in \mathbb{R}^{7\times 5}$. Let the matrix D be formed by the product

$$D = (A \cdot B \cdot C)^T$$

What are the dimensions of the matrix $[D(:,4)]^T$?

A. 8×5 B. 5×8 **C.** 1×5 D. 1×8 E. 5×1

16. Consider the set of vectors given by

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

Which of the following vectors sets is equivalent to the span of these three vectors?

A. $\mathbb{R}^4$

B. $\left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$

C. $\left\{ \begin{bmatrix} x_1 \\ x_1 \\ x_2 \\ x_2 \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2 \right\}$

D. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{bmatrix} : x_i \in \mathbb{R} \textbf{ for } i = 1, 2 \right\}$

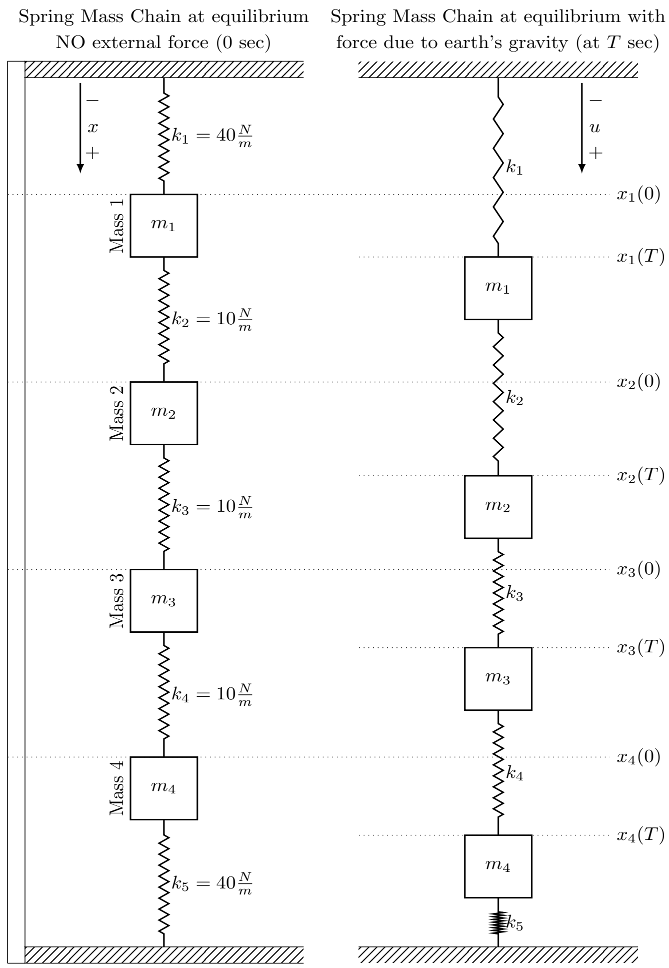
17. Define the matrix $B \in \mathbb{R}^{3\times 3}$ as a sum of elementary matrices given by

$$B = D_1(2) + S_{21}(2) + S_{31}(3) - S_{13}(-4).$$

Which of the following matrices is equivalent to B?

A. $\begin{bmatrix} 3 & 0 & 4 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 0 & 4 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 2 & 0 & -4 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 4 & 0 & 4 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ E. $\begin{bmatrix} 3 & 0 & -4 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

For problems 17 and 18, consider the following 4-mass, 5-spring chain presented below. Notice that positive positions and positive displacements are marked in the downward direction. Assume the acceleration due to earth's gravity is $g = 9.8 m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy Hooke's law exactly.



18. Recall that the initial position vector $\mathbf{x}_0$ and the final position vector $\mathbf{x}(T)$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and when $t = T$ respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.400 \\ 0.600 \\ 0.800 \end{bmatrix} \qquad \mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \\ x_4(T) \end{bmatrix} = \begin{bmatrix} 0.249 \\ 0.498 \\ 0.698 \\ 0.849 \end{bmatrix}$$

where each entry is given in meters. Using this information, which of the following vectors gives the force vector $\mathbf{f}_s$ that encodes the forces stored in each spring in this system?

A. $\begin{bmatrix} 1.960 \\ 0.490 \\ 0.000 \\ -0.490 \\ -1.960 \end{bmatrix}$ B. $\begin{bmatrix} 1.960 \\ 0.980 \\ 0.980 \\ 1.960 \end{bmatrix}$ C. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.000 \\ 0.490 \\ 1.470 \end{bmatrix}$ D. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.490 \\ 1.470 \end{bmatrix}$ E. $\begin{bmatrix} 0.049 \\ 0.049 \\ 0.000 \\ -0.049 \\ -0.049 \end{bmatrix}$

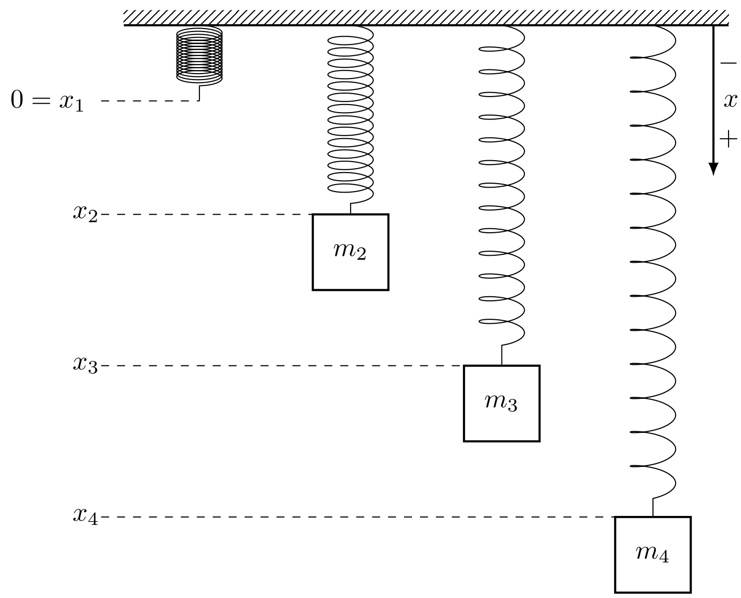
19. Under the same assumptions as the problem above, which of the following gives the mass vector

$$\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix}^T$$

measured in kg, used to produce this position data?

A. $\begin{bmatrix} 0.200 \\ 0.100 \\ 0.100 \\ 0.200 \end{bmatrix}$ B. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.490 \\ 1.470 \end{bmatrix}$ C. $\begin{bmatrix} 0.250 \\ 0.200 \\ 0.200 \\ 0.250 \end{bmatrix}$ D. $\begin{bmatrix} 2.450 \\ 1.960 \\ 1.960 \\ 2.450 \end{bmatrix}$ **E.** $\begin{bmatrix} 0.150 \\ 0.050 \\ 0.050 \\ 0.150 \end{bmatrix}$

20. Consider the experiment below. Suppose we hang three masses on the same spring and record the position data for that spring. Assume the spring constant is known to be $k = 5$ N/m. Assume also that the acceleration due to earth's gravity is $g = 9.8N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfy Hooke's law exactly.



In order to model the relationship between the displacement of the movable end of the spring and the internal force stored in the spring, we introduce two 4×1 vectors given by

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Each entry m_i is measured in kg. The entries of the position vector x_i, measured in meters. We know $x_1 = 0$m and the other entries $x_2, x_3, x_4 \in \mathbb{R}$ can be calculated from our knowledge of vector $\mathbf{m}$ and Hooke's Law. Which of the following gives the vector $\mathbf{x}$ in this situation?

A. $\begin{bmatrix} 0.0 \\ 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$ **B.** $\begin{bmatrix} 0.000 \\ 0.196 \\ 0.392 \\ 0.588 \end{bmatrix}$ C. $\begin{bmatrix} 0.00 \\ 0.02 \\ 0.04 \\ 0.06 \end{bmatrix}$ D. $\begin{bmatrix} 0.0 \\ 0.5 \\ 1.0 \\ 1.5 \end{bmatrix}$ E. $\begin{bmatrix} 0.0 \\ 4.9 \\ 9.8 \\ 14.7 \end{bmatrix}$

Free Response

$\boxed{10}$ 21. Suppose you are enrolled in a math course in which your final percent score is calculated as a weighted average. Below is a table that describes the important details of this class's grading scheme:

Grade Category on Syllabus	Total Points Available	Percentage Weight
Homework	200	10%
Projects	500	15%
Exam 1	100	20%
Exam 2	100	20%
Final Exam	100	35%

Suppose the teacher of this class does NOT have a grade replacement policy for your exam scores. With this in mind, respond to the following three questions.

A. Set up a vector model $\mathbf{g} \in \mathbb{R}^5$ that encodes all aspects of your course grade. Define each entry of $\mathbf{g}$ and describe your choices.

Solution: To create this vector model, we will define a 5×1 vector given by

$$\mathbf{g} = \begin{bmatrix} \dfrac{h}{200} \\[2ex] \dfrac{p}{500} \\[2ex] \dfrac{e_1}{100} \\[2ex] \dfrac{e_2}{100} \\[2ex] \dfrac{e_3}{100} \end{bmatrix}$$

In this case, we will set

$$h = \text{total points earned in homework grade category}$$

$$p = \text{total points earned in project grade category}$$

$$e_1 = \text{total points earned on exam 1}$$

$$e_2 = \text{total points earned on exam 2}$$

$$e_3 = \text{total points earned on the final exam}$$

This grade vector stores the percent score earned in each grade category for this course.

B. Demonstrate how to use the inner-product operation to calculate your final grade in this class.

Solution: In order to calculate the final percent score in this class, we consider the following inner product:

$$\mathbf{g} \cdot \mathbf{c} = \begin{bmatrix} \frac{h}{200} \\ \frac{p}{500} \\ \frac{e_1}{100} \\ \frac{e_2}{100} \\ \frac{e_3}{100} \end{bmatrix} \cdot \begin{bmatrix} 0.10 \\ 0.15 \\ 0.20 \\ 0.20 \\ 0.35 \end{bmatrix} = \frac{h}{200} \cdot 0.10 + \frac{p}{500} \cdot 0.15 + \frac{e_1}{100} \cdot 0.20 + \frac{e_2}{100} \cdot 0.20 + \frac{e_3}{100} \cdot 0.35$$

C. Suppose on the night before the final, you know you've earned the following scores:

Grade Category on Syllabus	Points You Earned
Homework	186
Projects	420
Exam 1	82
Exam 2	90

Assuming you want to get above a 85% in this class, determine the minimum percent score you will need to earn on the final exam to achieve your goal. Show your work.

Solution: In this case, we are given

$$h = 186, \qquad p = 420, \qquad e_1 = 82, \qquad e_2 = 90$$

and we want to find e_3 such that In order to calculate the final percent score in this class, we consider the following inner product:

$$\left(\frac{186}{200} \cdot 0.10 + \frac{420}{500} \cdot 0.15 + \frac{82}{100} \cdot 0.20 + \frac{90}{100} \cdot 0.20 + \frac{e_3}{100} \cdot 0.35 \right) \geq 0.85.$$

We can isolate e_3 in this inequality to find that

$$e_3 \geq 82.$$

To earn a minimum of 85% in this class, we need to earn a minimum of 82 points on the final exam.

$\boxed{10}$ 22. Describe, in detail, each of the following problems. For each problem, your should:

i. Identify the problem statement
ii. Identify the given and unknown quantities (explicitly identify relevant dimensions)
iii. Identify the function description of this problem (explicitly discuss domain, codomain and range)
iv. Describe how each problem is similar to and different from the other problem.

A. The Matrix-Vector Multiplication Problem

Solution: The matrix-vector multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, calculate unknown vector $\mathbf{b} \in \mathbb{R}^m$ such that
>
> $$A\mathbf{x} = \mathbf{b}$$

The matrix-vector multiplication is a "forward problem." In particular, let's define the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad f(\mathbf{x}) = A\mathbf{x} = \sum_{k=1}^{n} x_k\, A(:,k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is $\text{Span}\{A(:,k)\}_{k=1}^{n}$

Matrix-vector multiplication is a forward problem because we start with the function description (defined by matrix A) and we are given one specific input value $\mathbf{x}$ in the domain. From this information, we are asked to find the corresponding output value $\mathbf{b}$ in the range of function $f(\mathbf{x})$. When solving the matrix-vector multiplication problem, we map from the domain forward into the range. Hence, we call this a forward problem.

The matrix-vector multiplication problem is intimately connected with the linear system problem. Matrix-vector multiplication is the forward problem while the linear systems problem represents the backward problem (also known as inverse problem). As discussed below, when solving linear-systems problems, we start with a $\mathbf{b} \in \text{Rng}(f)$ and produce all $\mathbf{x} \in \text{Dom}(f)$ such that $A(\mathbf{x}) = \mathbf{b}$

Remarks (for students who want to earn above a 90%): In addition to the comments above, here are some other remarks about this problem

- For a matrix-vector multiplication problem $\mathbf{b} = A\mathbf{x}$ with $A^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$, solving this problem requires a total of $m \cdot (2n-1)$ operations between scalars.
- The solution to a matrix vector multiplication is unique. Each output vector $\mathbf{b}$ is given as a linear combination of the columns of A with scalar weights defined by the coefficient entries of $\mathbf{x}$.

B. The Square Linear-Systems Problem

Solution: The linear-systems multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$, find all unknown vectors $\mathbf{x} \in \mathbb{R}^n$ such that
>
> $$A \cdot \mathbf{x} = \mathbf{b}$$

Just like the matrix-vector multiplication problem, we can describe the linear-systems using the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad f(\mathbf{x}) = A \cdot \mathbf{x} = \sum_{k=1}^{n} x_k \, A(:, k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^n$.
- the range of f is $\text{Span}\{A(:, k)\}_{k=1}^{n}$

The linear-systems problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the range of $f(\mathbf{x})$. From this information, we are asked to find all possible input values $\mathbf{x}$ in the domain of our function such that

$$f(\mathbf{x}) = \mathbf{b}.$$

When solving the linear-systems problem, we begin in the range and work our way backwards to the domain. Hence, we call this a backward problem.

Remarks (for students who want to earn above a 90%): In addition to the comments above, here are some other remarks about this problem

- The solution to a linear-systems problem may not exist. If it does exist, it may not be unique. A great analogy comes from solving backward problems for the nonlinear function $f(x) = x^2$. Let's look at three backward problems:
  A. No solutions: $f(x) = x^2 = -4$
  B. Unique solution: $f(x) = x^2 = 0$
  B. Multiple solutions: $f(x) = x^2 = 4$

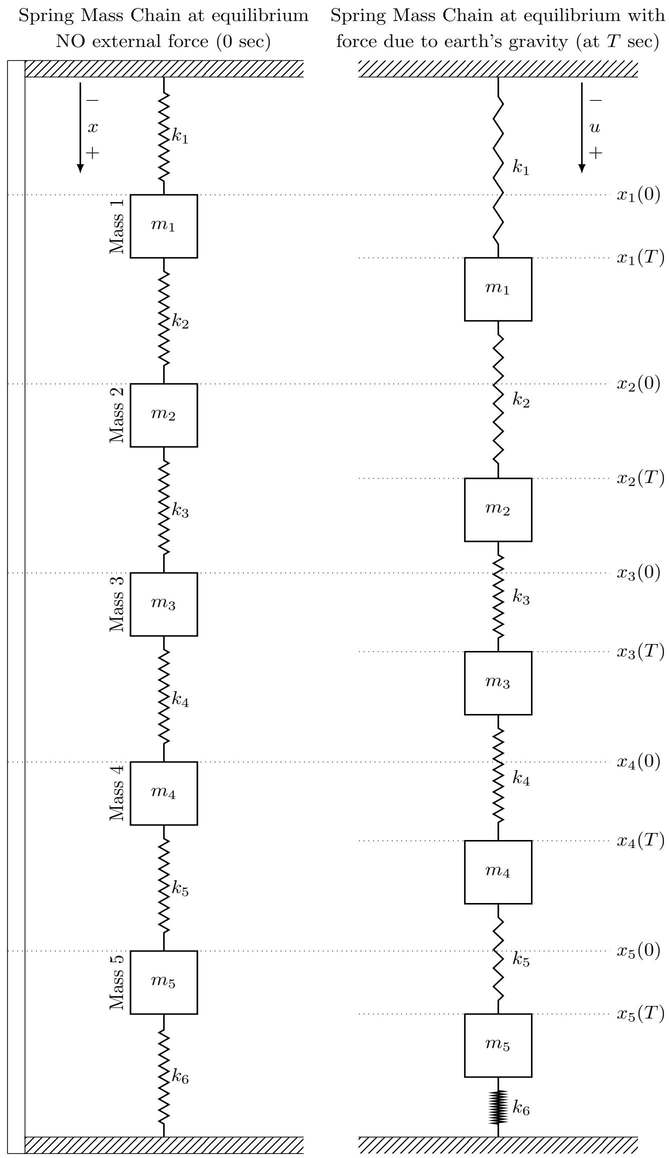
Although the theory behind solving linear systems is much different than the theory for solving quadratic equations, analogies about the existence and uniqueness of solutions abound. Linear systems problems may have:

A. No Solution: $\mathbf{b} \notin \text{Rng}(f) = \text{Span}\{A(:, k)\}_{k=1}^{n}$ (known as least-squares problem)

B. Unique solution: $\mathbf{b} \in \text{Rng}(f) = \text{Span}\{A(:, k)\}_{k=1}^{n}$ and A has linearly independent columns

C. Non-unique (multiple) solutions: $\mathbf{b} \in \text{Rng}(f)$ and A has linearly dependent columns

$\boxed{10}$ 23. Mass-spring problem



A. Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(T), \text{ and } \mathbf{u}$$

where these vectors represent the initial position vector, the final position vector, and the displacement vector, respectively (as discussed in class and in our lesson notes).

Solution: Recall from our in-class discussion, we have

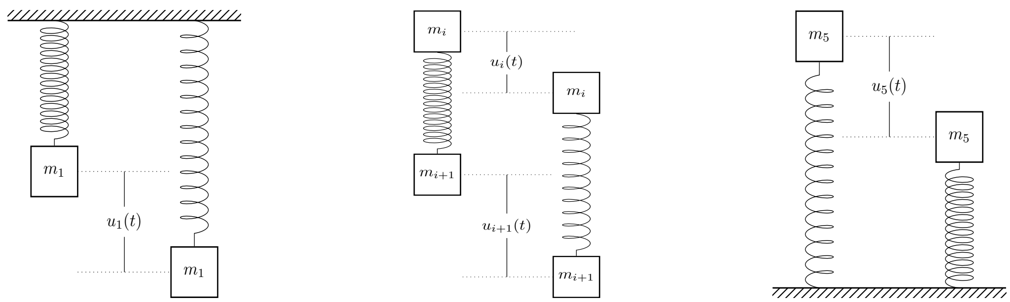
$$\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \end{bmatrix}, \qquad \mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \\ x_4(T) \\ x_5(T) \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{x}(T) - \mathbf{x}_0 = \begin{bmatrix} x_1(T) - x_1(0) \\ x_2(T) - x_2(0) \\ x_3(T) - x_3(0) \\ x_4(T) - x_4(0) \\ x_5(T) - x_5(0) \end{bmatrix}.$$

B. Show how to calculate the elongation vector $\mathbf{e}$ as a matrix-vector product

$$\mathbf{e} = A\mathbf{u}$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient e_i in this vector. Your answer should include specific references to the diagram of the 5-mass, 6-spring chain above.

Solution: The elongation vector $\mathbf{e}$ is a 6×1 vector with entry e_i representing the elongation of spring i. We can find the elongation of each spring by considering each mass separately. We will consider the first mass, the inner masses (masses 2, 3, and 4) and mass 5 separately. To this end, consider the diagrams:



Using these diagrams, we see that our desired elongation vector is given by

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ u_4 - u_3 \\ u_5 - u_4 \\ -u_5 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

We can write this linear combination as a matrix-vector product as following

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = A\mathbf{u}$$

where $\mathbf{u}$ is the 5×1 displacement vector from part (A) above. In this case, the matrix $A \in \mathbb{R}^{6\times 5}$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We write

$$\boxed{\mathbf{e}(t) = A\mathbf{u}(t)} \tag{1}$$

C. Show how to calculate the spring force vector $\mathbf{f}_s$ as a matrix-vector product

$$\mathbf{f}_s = C\mathbf{e}$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

Solution: Recall that Hooke's law states that the the force stored inside a spring is directly proportional to the elongation of the spring. In other words, for a spring with spring constant k_i, Hooke's law states that

$$f_{s_i} = k_i e_i$$

Thus, we can create a force vector that stores the forces in each of the five springs in our system due to the elongations discussed in part (B) above. To this end, we see

$$\mathbf{f}_s = \begin{bmatrix} f_{s_1} \\ f_{s_2} \\ f_{s_3} \\ f_{s_4} \\ f_{s_5} \\ f_{s_6} \end{bmatrix} = \begin{bmatrix} k_1\, e_1 \\ k_2\, e_2 \\ k_3\, e_3 \\ k_4\, e_4 \\ k_5\, e_5 \\ k_6\, e_6 \end{bmatrix} = e_1 \begin{bmatrix} k_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_2 \begin{bmatrix} 0 \\ k_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 \\ 0 \\ k_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_4 \\ 0 \\ 0 \end{bmatrix} + e_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_5 \\ 0 \end{bmatrix} + e_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k_6 \end{bmatrix}$$

Again, we can write the force vector $\mathbf{f}_s$ as the matrix-vector product

$$\mathbf{f}_s = \begin{bmatrix} f_{s_1} \\ f_{s_2} \\ f_{s_3} \\ f_{s_4} \\ f_{s_5} \\ f_{s_6} \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} = C\mathbf{e}$$

where $\mathbf{e}$ is our elongation vector from above. The diagonal matrix $C \in \mathbb{R}^{6\times 6}$ is defined as

$$C = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}$$

We write

$$\boxed{\mathbf{f}_s(t) = C\mathbf{e}(t)} \tag{2}$$

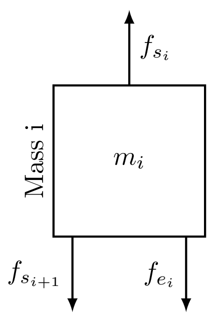
D. Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y} = -A^T \mathbf{f}_s$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}$ with equations from parts B and C to form the stiffness matrix K. Note, you do not have to find the entry-by-entry definition of K.

Solution:

We now introduce the vector $\mathbf{y}(t)$ to store the difference between the forces between the springs attached to each mass. To find the entries of $\mathbf{y}(t)$, consider the free body diagrams for mass i, for $i = 1, ..., 5$.



When writing the individual entries of $\mathbf{y}(t)$ we will assume that positive forces result in positive displacements. Since we've oriented positive displacement in the downward direction, we also orient positive force in the downward direction.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix} = \begin{bmatrix} f_{s_2}(t) - f_{s_1}(t) \\ f_{s_3}(t) - f_{s_2}(t) \\ f_{s_4}(t) - f_{s_3}(t) \\ f_{s_5}(t) - f_{s_4}(t) \\ f_{s_6}(t) - f_{s_5}(t) \end{bmatrix} = - \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \\ f_{s_6}(t) \end{bmatrix}$$

We transform this into a matrix-vector product

$$\boxed{\mathbf{y}(t) = -A^T \mathbf{f}_s(t)} \tag{3}$$

where $\mathbf{f}_s \in \mathbb{R}^6$ is the force vector from part (C) above. We see A^T is the transpose of the matrix A from part (A) above.

Solution: In this problem, we will using equations (1), (2), and (3) to create stiffness matrix K. To this end, note

$$\begin{aligned}
\mathbf{y}(t) &= -A^T\mathbf{f}_s(t) && \text{by equation (3)}\\
&= -A^T C\mathbf{e}(t) && \text{by equation (2)}\\
&= -A^T CA\mathbf{u}(t) && \text{by equation (1)}\\
&= -K\mathbf{u}(t)
\end{aligned}$$

If we let $K = A^T CA$, we can then write

$$\boxed{\mathbf{y}(t) = -K\mathbf{u}(t)} \tag{4}$$

We can form our stiffness matrix K explicitly using matrix-matrix multiplication with

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0\\ 0 & k_2 & 0 & 0 & 0 & 0\\ 0 & 0 & k_3 & 0 & 0 & 0\\ 0 & 0 & 0 & k_4 & 0 & 0\\ 0 & 0 & 0 & 0 & k_5 & 0\\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0\\ -k_2 & k_2+k_3 & -k_3 & 0 & 0\\ 0 & -k_3 & k_3+k_4 & -k_4 & 0\\ 0 & 0 & -k_4 & k_4+k_5 & -k_5\\ 0 & 0 & 0 & -k_5 & k_5+k_6 \end{bmatrix}$$

This is a tridiagonal, symmetric matrix.

E. Use Newton's second law to derive the matrix equation

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{f}_e$$

where $\mathbf{f}_e$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

Solution: From Newton's second law, we know that

$$\text{Net Force} \;=\; \text{Mass} \;\times\; \text{Acceleration}$$

We can apply this law to each mass individually to create a differential equation that describes our system, given by

$$\Sigma\mathbf{F} = \begin{bmatrix} \Sigma F_1 \\ \Sigma F_2 \\ \Sigma F_3 \\ \Sigma F_4 \end{bmatrix} = \begin{bmatrix} m_1\,\ddot{u}_1(t) \\ m_2\,\ddot{u}_2(t) \\ m_3\,\ddot{u}_3(t) \\ m_4\,\ddot{u}_4(t) \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \\ \ddot{u}_5(t) \end{bmatrix}$$

where ΣF_i represents the net force on mass i and $\ddot{u}_i(t) = \dfrac{d^2}{dt^2}\Big[u_i(t)\Big]$ for $i \in \{1, 2, 3, 4, 5\}$. We write the matrix-vector multiplication

$$\boxed{\Sigma\mathbf{F} = M\ddot{\mathbf{u}}(t)} \qquad\qquad \text{where } M = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \tag{5}$$

Further, since all forces are assumed to be positive in the downward direction we see

$$\begin{bmatrix} \Sigma F_1 \\ \Sigma F_2 \\ \Sigma F_3 \\ \Sigma F_4 \\ \Sigma F_5 \end{bmatrix} = \begin{bmatrix} f_{e_1}(t) + f_{s_2}(t) - f_{s_1}(t) \\ f_{e_2}(t) + f_{s_3}(t) - f_{s_2}(t) \\ f_{e_3}(t) + f_{s_4}(t) - f_{s_3}(t) \\ f_{e_4}(t) + f_{s_5}(t) - f_{s_4}(t) \\ f_{e_5}(t) + f_{s_6}(t) - f_{s_5}(t) \end{bmatrix} = \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \\ f_{e_3}(t) \\ f_{e_4}(t) \\ f_{e_5}(t) \end{bmatrix} + \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix}$$

Thus, we can write

$$\boxed{\Sigma\mathbf{F} = \mathbf{f}_e(t) + \mathbf{y}(t)} \tag{6}$$

Solution: By combing equations (4), (5), and (6), we see

$$M\ddot{\mathbf{u}}(t) = \mathbf{f}_e(t) + \mathbf{y}(t)$$

$$\Longrightarrow \qquad M\ddot{\mathbf{u}}(t) = \mathbf{f}_e(t) + -K\mathbf{u}(t)$$

By moving $-K$ onto the other side of the equation, we have

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}_e(t) \tag{7}$$

Since we have assume that we study the system at equilibrium for $t = T$, we know $\ddot{\mathbf{u}}(T) = \mathbf{0}$ and we have

$$K\,\mathbf{u}(T) = \mathbf{f}_e(T)$$

Remark (for students who want to earn above a 100%):

- In this derivation, we've used a very general approach to allow $t \in (0, T]$. Only at the very end of our work, did we substitute the value of $t = T$ to represent the case that our masses have settled down to equilibrium. As we will see, this general approach will come in very useful during our discussion of the eigenvalue-eigenvector problem.
- In fact, we have derived a coupled ordinary differential equation in the work above. For those of you that have taken (or will take) Math 2A at Foothill, you may notice that equation (7) is a vector version of the 2nd order differential equation for a harmonic oscillator with no damping and general forcing function.

Challenge Problem

24. (Optional, Extra Credit, Challenge Problem)

Let $\mathbf{x} \in \mathbb{R}^n$ be a column vector. Recall that we defined the 2-norm of $\mathbf{x}$ to be

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

This is one example of a much larger class of vector norms, known as $p-$norms. To create a $p-$norm, we choose a real number $p \geq 1$ and set

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

Using this definition, we can set $p = \infty$ and define the $\infty-$norm (read "infinity norm"), using the following definition

$$\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$$

Prove $\lim\limits_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$

Solution: Let $n \in \mathbb{N}$ and suppose $p \in \mathbb{R}$ with $p \geq 1$. Let $\mathbf{x} \in \mathbb{R}^n$. We begin our proof by establishing lower and upper bounds on $\|\mathbf{x}\|_p$ in terms of $\|\mathbf{x}\|_\infty$. These inequalities are given as follows:

- Lower bound: $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p$
- Upper bound: $\|\mathbf{x}\|_p \leq \sqrt[p]{n} \cdot \|\mathbf{x}\|_\infty$

After we establish these inequalities, we will take our limit as $p \to \infty$ and use the sandwich theorem to conclude our desired result.

Lower bound: Let's being by establishing the lower bound. To this end, let

$$j = \arg\max_{1\leq i\leq n} \{|x_i|\}$$

Based on the definition of the p-norm, we know

$$\sum_{i=1}^{n} |x_i|^p = |x_j|^p + \sum_{\substack{i=0 \\ i\neq j}}^{n} |x_i|^p = \|\mathbf{x}\|_\infty^p + \sum_{\substack{i=0 \\ i\neq j}}^{n} |x_i|^p$$

Since $|x_i| \geq 0$ for all $i = 1, 2, ..., n$, we know that $+\sum\limits_{\substack{i=0 \\ i\neq j}}^{n} |x_i|^p \geq 0$. We immediately conclude

$$\|\mathbf{x}\|_\infty^p \leq \|\mathbf{x}\|_p^p$$

Taking the square root of both sides of this produces the desired lower-bound inequality.

Upper bound: We move onto our desired upper bound. Again, we start with the definition of the $p-$norm

$$\|\mathbf{x}\|_p^p = \sum_{i=0}^{n} |x_i|^p \leq \sum_{i=0}^{n} \left(\max_{1\leq j\leq n} |x_j|\right)^p = \sum_{i=0}^{n} \|\mathbf{x}\|_\infty^p = n \cdot \|\mathbf{x}\|_\infty^p$$

Taking the square root of both sides produces the desired upper-bound inequality.

Limiting process: With both of these inequalities in hand, we have the following

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq \sqrt[p]{n} \cdot \|\mathbf{x}\|_\infty$$

Taking the limit as $p \to \infty$ of this chain inequality produces the bound

$$\|\mathbf{x}\|_\infty \leq \lim_{p\to\infty} \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_\infty$$

With this we conclude that $\|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$

Important lemmas: The above analysis relies on two important lemmas:

A. If $0 \leq a \leq b$, then $0 \leq \sqrt[p]{a} \leq \sqrt[p]{b}$ for any real $p \geq 1$.

Proof. To prove this lemma, we will revert back to calculus. Consider a real number $p \geq 1$. Define the function

$$f(x) = x^{1/p}$$

for $x > 0$. If we can show that $f'(x) > 0$, we conclude that $f(x)$ is increasing on the interval $(0, \infty)$. To this end, recall

$$\frac{1}{p} \cdot x^{(1/p-1)}$$

Since $x > 0$, we know $x^{(1/p-1)} > 0$ and we conclude that $f(x)$ is increasing. □

B. If $n \in \mathbb{N}$, then $\lim\limits_{p\to\infty} \sqrt[p]{n} = 1$

Proof. Suppose $L = \lim\limits_{p\to\infty} \sqrt[p]{n} \geq 0$. Then, taking the logarithm of each side, we see

$$\log(L) = \log\left(\lim_{p\to\infty} \sqrt[p]{n}\right) = \lim_{p\to\infty} \log\left(\sqrt[p]{n}\right)$$

Since $\sqrt[p]{n} = n^{1/p}$, we can use the power rule of logarithms to conclude

$$\log(L) = \lim_{p\to\infty} \frac{1}{p} \cdot \log(n) = \log(n) \cdot \lim_{p\to\infty} \frac{1}{p} = 0.$$

Hence, we conclude that

$$10^{\log(L)} = 10^0$$

which proves $L = 1$. □