1. (6 points) Let

$$A = \{a \in \mathbb{Z} : a^3 - 4a^2 + 3a = 0\} \text{ and } B = \{ b : b+1 = 2^n \text{ for } n \in \{0,1,2\} \}.$$

Show $A = B$.

To show $A = B$, we need to show: $\begin{cases} \text{Step i: } A \subseteq B \\ \text{Step ii. } B \subseteq A \end{cases}$

Step i) We begin by establishing $A \subseteq B$ via direct proof. To this end:

Assume $a \in A \Rightarrow a^3 - 4a^2 + 3a = 0$

$\Rightarrow a(a^2 - 4a + 3) = 0$

$\Rightarrow a(a-1) \cdot (a-3) = 0$

$\Rightarrow a = 0 \quad \text{or} \quad a = 1 \quad \text{or} \quad a = 3$

$\Rightarrow a+1 = 2^0 \quad \text{or} \quad a+1 = 2^1 \quad \text{or} \quad a+1 = 2^2$

$\Rightarrow a+1 = 2^n \quad \text{for} \quad n \in \{0,1,2\}$

$\Rightarrow a \in B$

$\Rightarrow A \subseteq B$ by definition ☑

Step ii) We continue by showing $B \subseteq A$ via direct proof. Thus,

Suppose $b \in B \Rightarrow b+1 = 2^n \quad \text{for} \quad n \in \{0,1,2\}$

$\Rightarrow b+1 = 2^0 \quad \text{or} \quad b+1 = 2^1 \quad \text{or} \quad b+1 = 2^2$

$\Rightarrow b = 0 \quad \text{or} \quad b = 1 \quad \text{or} \quad b = 3$

$\Rightarrow b = 0 \quad \text{or} \quad b-1 = 0 \quad \text{or} \quad b-3 = 0$

$\Rightarrow b \cdot (b-1) \cdot (b-3) = 0$

$\Rightarrow b \cdot (b^2 - 4b + 3) = 0$

$\Rightarrow b^3 - 4b^2 + 3b = 0$

$\Rightarrow b \in A$

$\Rightarrow B \subseteq A$ by definition

Thus, since $A \subseteq B$ and $B \subseteq A$, we know $A = B$, as was to be shown. ①

2. (6 points) Consider the following four sets:

$$S = \{(x,y) : y = x^2\} \subseteq \mathbb{R} \times \mathbb{R}.$$
$$R = \{(y,x) : y = x^2\} \subseteq \mathbb{R} \times \mathbb{R}.$$
$$R_+ = \{(y,x) : y = x^2 \text{ with } x \geq 0\} \subseteq \mathbb{R} \times \mathbb{R}.$$
$$R_- = \{(y,x) : y = x^2 \text{ with } x \leq 0\} \subseteq [0,\infty) \times \mathbb{R}.$$

For each these sets, determine the following

   i. The domain space

   ii. The domain

   iii. The codomain

   iv. The range

Then, classify each set as either a relation or function. Justify your decisions using the appropriate formal, set-theoretic definitions from Lesson 2.

We begin by recalling the formal, set-theoretic definitions of i. – iv. above. To this end, we suppose X is a relation from set A to set B. By definition, we know

$$X \subseteq A \times B$$

and domain space$(X) = A$, codomain$(X) = B$, $\text{dom}(X) = \{a \in A : \text{ there is a } b \in B \text{ with } (a,b) \in X\}$

$\text{Rng}(X) = \{b \in B : \text{ there is a } a \in A \text{ s.t. } (a,b) \in X\}$

Using these definitions, we see :

The Set S — Function

  domain space $(S) = \mathbb{R}$

  domain $(S) = \mathbb{R}$

  codomain $(S) = \mathbb{R}$

  rng $(S) = [0,\infty)$

The Set R+ — Relation

  domain space $(R_+) = \mathbb{R}$

  dom $(R_+) = [0,\infty)$

  codomain $(R_+) = \mathbb{R}$

  Rng $(R_+) = [0,\infty)$

The set R — Relation

  domain space $(R) = \mathbb{R}$

  domain $(R) = [0,\infty)$

  codomain $(R) = \mathbb{R}$

  Rng $(R) = \mathbb{R}$

The set R- — Function

  domain space $(R_-) = [0,\infty)$

  dom $(R_-) = [0,\infty)$

  codomain $(R_-) = \mathbb{R}$

  Rng $(R_-) = (-\infty, 0]$

②

3. (6 points) Consider the ideal version of a Hooke's law experiment depicted below. Suppose we hang three masses on the same ideal extension spring and record the position data for that spring using a metric ruler so that all positions are measured in meters. Assume also that the acceleration due to earth's gravity is $g = 9.8 N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfies an ideal version of Hooke's law.

Let's define xo to be position of movable end of spring with no mass attached. In this case, we see $x_0 = 0.1$
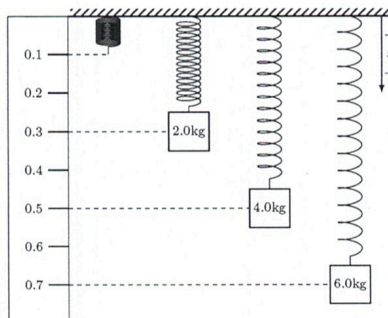
Note: Let's assume these are all point masses with neglible dimensions

Create a vector model that describes Hooke's law by forming each of the following vectors

   i. Mass vector **m**

  ii Raw position vector **x**

  iii. Spring force vector $\mathbf{f}_s$

  iv. Displacement vector **u**

Explain your work and demonstrate how to use scalar-vector multiplication and vector-vector addition in this modeling exercise.

$$\text{Let } \vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 2.0 \\ 4.0 \\ 6.0 \end{bmatrix} \quad \left(\begin{array}{c}\text{Known as the}\\ \text{mass vector}\end{array}\right)$$

Note on internal force stored in spring

□ sec Exam 1 V4A solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.5 \\ 0.7 \end{bmatrix} \quad \left(\begin{array}{c}\text{Known as}\\ \text{position vector}\\ \text{and tracks position}\\ \text{of movable end}\\ \text{of spring}\end{array}\right)$$

$$\vec{f}_s = \begin{bmatrix} f_{s_1} \\ f_{s_2} \\ f_{s_3} \\ f_{s_4} \end{bmatrix} = -9.8 \cdot \vec{m} = \begin{bmatrix} -0.0 \\ -19.6 \\ -39.2 \\ -58.8 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \vec{x} - x_0 \cdot \vec{1}_{4\times1}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + -x_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ x_3 - x_0 \\ x_4 - x_0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.2 \\ 0.4 \\ 0.6 \end{bmatrix} \quad \left(\begin{array}{c}\text{Known as}\\ \text{the displacement}\\ \text{vector that stores}\\ \text{change in position}\\ \text{yielded by each mass}\end{array}\right)$$

③

4. (6 points) Suppose we model a square using a set of four *begin vertices* given by the four vertices

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Define a set of four *end vertices* with the property that $\mathbf{w}_i = \alpha \cdot \mathbf{v}_i + \mathbf{s}$ for a special scalar $\alpha \in \mathbb{R}$ and special vector $\mathbf{s} \in \mathbb{R}^2$, where $i \in [4]$. Under these assumptions, we graph both the begin and end vertices for this problem in the figure below.

Begin Vertices

End Vertices

$$\vec{w}_i = \alpha \cdot \vec{v}_i + \vec{s}$$
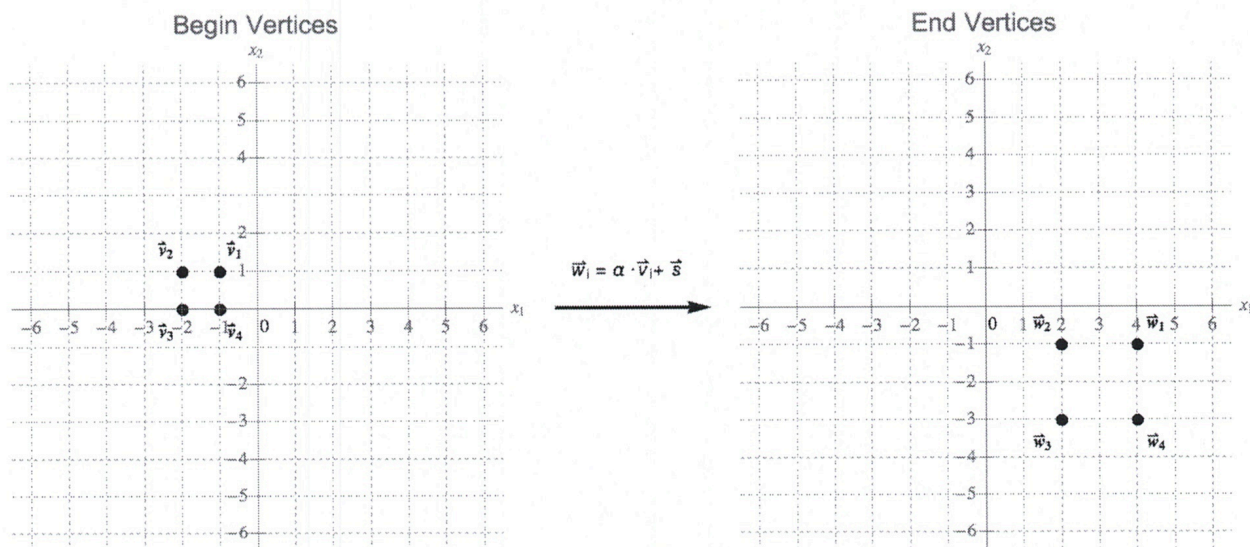
Using this information, set up a system of equations that might help you figure out the specific values of α and $\mathbf{s}$ used in this problem. Then, find these values and explain your work.

$$\vec{W}_1 = \alpha \cdot \vec{V}_1 + \vec{S} \Rightarrow \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \vec{S} \Rightarrow \vec{S} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{W}_2 = \alpha \cdot \vec{V}_2 + \vec{S} = \begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \vec{S} \Rightarrow \vec{S} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Then, we see $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
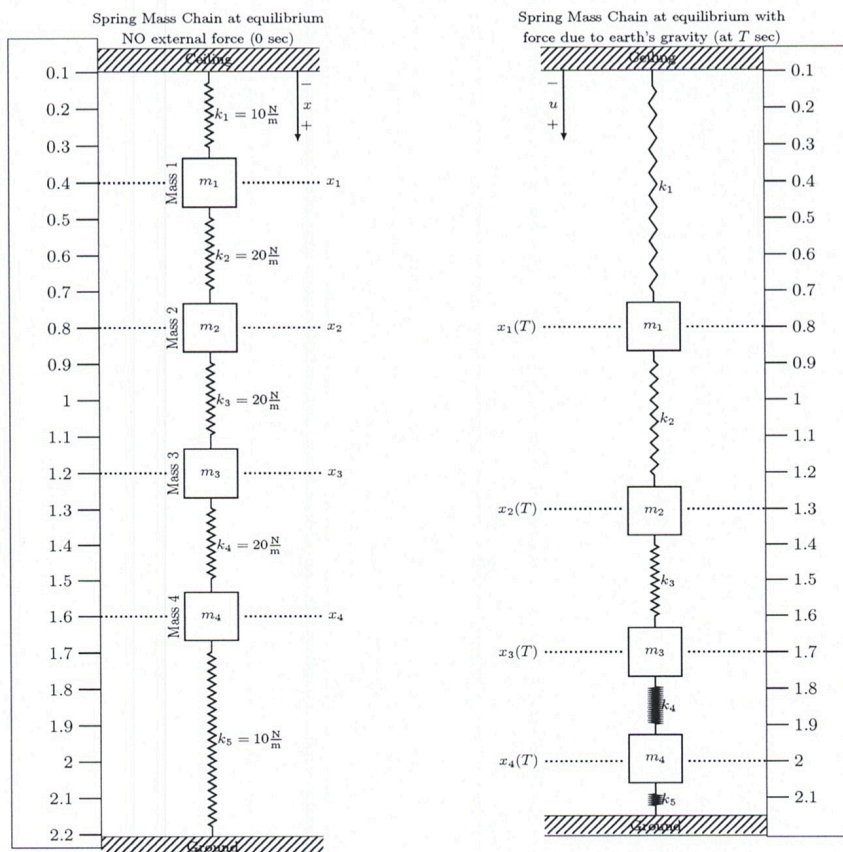
$$\Rightarrow \alpha \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \boxed{\alpha = 2}$$

$$\Rightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \vec{S} \Rightarrow \vec{S} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 6 \\ -3 \end{bmatrix} = \vec{S}}$$

$\checkmark$ check: $\vec{W}_4 = 2 \cdot \vec{V}_4 + \vec{S}$

$$= 2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

5. (8 points) Consider the following model for a 4-mass, 5-spring chain drawn below. Note that positive positions and positive displacements are marked in the downward direction. Assume the ruler gives position measurements in meters.

Spring Mass Chain at equilibrium NO external force (0 sec)

Spring Mass Chain at equilibrium with force due to earth's gravity (at T sec)

Use this diagram and the given position data, set up the vectors $\mathbf{x}_0, \mathbf{x}(T), \mathbf{u}(T)$ that we discussed in class. Describe the significance of each vector. Also, show how we can use scalar-vector multiplication and vector-vector addition to form $\mathbf{u}(T)$ from the vectors $\mathbf{x}_0$ and $\mathbf{x}(T)$.

$$\vec{x}_0 = \vec{x}(0) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.8 \\ 1.2 \\ 1.6 \end{bmatrix}$$

$$\vec{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \\ x_4(T) \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1.3 \\ 1.7 \\ 2.0 \end{bmatrix}$$

$$\vec{u}(T) = \vec{x}(T) + -1 \cdot \vec{x}$$

$$= \begin{bmatrix} 0.8 \\ 1.3 \\ 1.7 \\ 2.0 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0.4 \\ 0.8 \\ 1.2 \\ 1.6 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.5 \\ 0.5 \\ 0.4 \end{bmatrix}$$

☐ Students should explain set up in detail.

(use full sentences and describe what you're doing and why...

Simple language to display at least three dimensions in concept image)

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⑤

6. (6 points) Describe as much as you can about Problem 0: The Applied Math Modeling Process. In order to earn full credit on this problem, your answer should

    i. Identify the stages of the Applied Math Modeling Process (including a diagram with descriptions)

    ii. Explain where and how vector and matrix modeling arises in this process

    iii. Describe how Problem 0 is related to our use of the word "given" in Problems 1, 2A, 2B, 3, and 4

    iv. Be written in your own voice with as much unique thought as possible.

□ Please see Lesson 0, video 1. Remember, the point of this problem is for you to develop your own concept image and articulate this in your own language. Recreate the arguments and ideas for yourself!

For problems 6 - 8, describe each of the given problems in detail. In order to earn full credit on each problem, your answer should accurately address each of the following :

   i. Identify the problem statement

   ii. Identify the given and unknown quantities (explicitly identify relevant dimensions)

   iii. Identify the function description of this problem (explicitly discuss domain, codomain and range)

   iv. Describe in as much detail as you can how each problem is similar to and different from the other two problems in the list below.

---

6. (6 points) Problem 1: The Matrix-Vector Multiplication Problem

Solution: The matrix-vector multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, calculate unknown vector $\mathbf{b} \in \mathbb{R}^m$ such that
>
> $$A\mathbf{x} = \mathbf{b}$$

The matrix-vector multiplication is a "forward problem." In particular, let's define the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad\qquad f(\mathbf{x}) = A\mathbf{x}$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$

- the codomain f is $\mathbb{R}^m$.

- the range of f is a subset set of the codomain $\mathbb{R}^m$. We know from our discussion in class that, for some matrices, the range of f may be a strict subset of the codomain (meaning there may be some elements of $\mathbb{R}^m$ that are not used as output by the function f)

Matrix-vector multiplication is a forward problem because we start with the function description (defined by matrix A) and we are given one specific input value $\mathbf{x}$ in the domain. From this information, we are asked to find the corresponding output value $\mathbf{b}$ in the range of function $f(\mathbf{x})$. When solving the matrix-vector multiplication problem, we map from the domain forward into the range. Hence, we call this a forward problem.

The matrix-vector multiplication problem is intimately connected with the general linear-system problem. Matrix-vector multiplication is the forward problem while the general linear-systems problem represents the backward problem (also known as inverse problem). As discussed below, when solving general linear-systems problems, we start with a $\mathbf{b} \in \text{Rng}(f)$ and produce all $\mathbf{x} \in \text{Dom}(f)$ such that $A(\mathbf{x}) = \mathbf{b}$.

8. (6 points) Problem 2B: The General Linear-Systems Problem

Solution: The general linear-systems multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, find all unknown vectors $\mathbf{x} \in \mathbb{R}^n$ such that
>
> $$A\mathbf{x} = \mathbf{b}.$$

Just like the matrix-vector multiplication problem, we can describe the linear-systems using the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad\qquad f(\mathbf{x}) = A \cdot \mathbf{x}$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$

- the codomain f is $\mathbb{R}^m$.

- the range of f is a subset of the codomain $\mathbb{R}^m$ and may not be the entire codomain (depending on the structure of matrix A)

The linear-systems problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the range of $f(\mathbf{x})$. From this information, we are asked to find all possible input values $\mathbf{x}$ in the domain of our function such that

$$f(\mathbf{x}) = \mathbf{b}.$$

When solving the linear-systems problem, we begin in the range and work our way backwards to the domain. Hence, we call this a backward problem.

Remarks (for students who want to earn the highest marks): In addition to the comments above, here are some other remarks about this problem

- The solution to a linear-systems problem may not exist. If it does exist, it may not be unique. A great analogy comes from solving backward problems for the nonlinear function $f(x) = x^2$. Let's look at three backward problems:

  A. Unique solution: $f(x) = x^2 = 0$
  B. Multiple solutions: $f(x) = x^2 = 4$
  C. No solutions: $f(x) = x^2 = -4$

Although the theory behind solving linear systems is much different than the theory for solving quadratic equations, analogies about the existence and uniqueness of solutions abound. Linear systems problems may have:

A. Unique solution: $\mathbf{b} \in \mathrm{Rng}(f)$ and A has full column rank.

B. Non-unique (multiple) solutions: $\mathbf{b} \in \mathrm{Rng}(f)$ and A is rank-deficient.

C. No Solution: $\mathbf{b} \notin \mathrm{Rng}(f)$ in which case we can minize the difference between $\mathbf{b}$ and the range of f: This minimization problem is known as the full-rank, least-squares problem when matrix A has full column rank.

9. (5 points) Optional, Extra Credit, Challenge Problem: Suppose $n \in \mathbb{N}$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $x_i, y_i \in [26]$ for all $i \in [n]$. Prove or disprove the following conjecture:

Conjecture: If $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$, then the entries of $\mathbf{y}$ are identical to the entries of $\mathbf{x}$ through some permutation $\pi : [n] \longrightarrow [n]$ with $y_i = x_{\pi(i)}$ for all $i \in [n]$.

None available :)