INSTRUCTOR NOTE: This problem was designed to test students on proof set up for direct subset proof since that proof set up is crucial to Lesson 22.

1. (6 points) Let $X = \{x \in \mathbb{N} : x^2 < 6\}$ and $Y = \{\ y : y = 2^n$ for $n \in \{0,1\}\}$. Show $X = Y$.

To show $X = Y$ we need to show

i. $X \subseteq Y$

ii. $Y \subseteq X$

i. Let's start by showing i. $X \subseteq Y$. To this end:

Assume $x \in X \Rightarrow x \in \mathbb{N}$ and $x^2 < 6$

$\Rightarrow x \in \{1,2,3,4,\dots\}$ and $x^2 < 6$

$\Rightarrow x = 1$ or $x = 2$

$\Rightarrow x = 2^0$ or $x = 2^1$

$\Rightarrow x = 2^n$ for $n \in \{0,1\}$

$\Rightarrow x \in Y$

$\Rightarrow X \subseteq Y$

ii we continue by showing ii $Y \subseteq X$. To this end:

Suppose $y \in Y \Rightarrow y = 2^n$ for $n = 0,1$

$\Rightarrow y = 2^0$ or $y = 2^1$

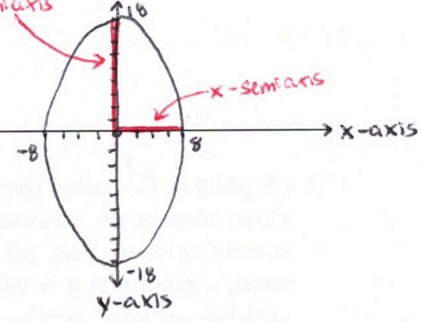
$\Rightarrow y = 1$ or $y = 2$

$\Rightarrow y^2 = 1$ or $y^2 = 4$

$\Rightarrow y \in \mathbb{N}$ and $y^2 < 6$

$\Rightarrow y \in X$

$\Rightarrow Y \subseteq X \Rightarrow X = Y$ □



2. (6 points) Let $S \subseteq \mathbb{R} \times \mathbb{R}$ be the following relation:

$$S = \left\{ (x, y) : \underbrace{\frac{x^2}{64} + \frac{y^2}{324} \leq 1} \right\}$$

This is an equation for an ellipse with x-semiaxis length $a = 8$ and y-semiaxis length $b = 18$. We can graph this ellipse on axis above to get idea of domain and range.

Find each of the following

i. The domain space of S
ii. The domain of S
iii. The codomain of S
iv. The range of S

Justify each of your answers using the precise set-theoretic definition of a relation and make sure you discuss the relationship between your answers and the cross product of sets from which S is chosen.

Doman Space$(S) = \mathbb{R}$ since $S \subseteq \mathbb{R} \times \mathbb{R}$

$$\text{Doman}(S) = \{ x \in \mathbb{R} : \text{there is a } y \in \mathbb{R} \text{ with } (x,y) \in S \}$$

$$= \{ x \in \mathbb{R} : \frac{x^2}{64} \leq 1 \}$$

☐ Note: Here we set $y=0$ since $y^2 \geq 0$ and thus if $y > 0$ we are on the interior of the domain. With $y=0$, we get to the boundary points

$$= \{ x \in \mathbb{R} : x^2 \leq 64 \}$$

$$= \{ x \in \mathbb{R} : -8 \leq x \leq 8 \}$$

$$= [-8, 8]$$

Codomain$(S) = \mathbb{R}$ since $S \subseteq \mathbb{R} \times \mathbb{R}$

$$\text{Rng}(S) = \{ y \in \mathbb{R} : \text{there is a } x \in \mathbb{R} \text{ with } (x,y) \in S \}$$

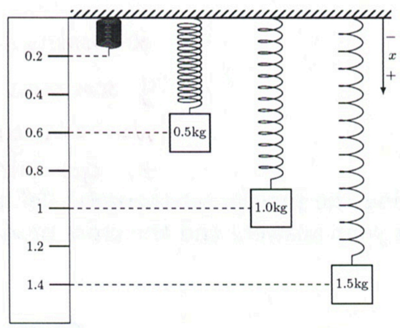
$$= \{ y \in \mathbb{R} : \frac{y^2}{324} \leq 1 \}$$

☐ Note: Here we set $x=0$ since $x > 0 \Rightarrow x^2 > 0$ corresponding to interior point in our rng. To get boundary we consider the max possible y values when $x=0$.

$$= \{ y \in \mathbb{R} : y^2 \leq 324 \}$$

$$= \{ y \in \mathbb{R} : |y| \leq 18 \} = [-18, 18]$$

3. (6 points) Consider the ideal version of a Hooke's law experiment depicted below. Suppose we hang three masses on the same ideal extension spring and record the position data for that spring using a metric ruler so that all positions are measured in meters. Assume also that the acceleration due to earth's gravity is $g = 9.8N/kg$. Finally, suppose that the mass of the spring is zero and that this spring satisfies an ideal version of Hooke's law.



Create a vector model that describes Hooke's law by forming each of the following vectors

i. Mass vector $\mathbf{m}$

ii Raw position vector $\mathbf{x}$

iii. Spring force vector $\mathbf{f}_s$

iv. Displacement vector $\mathbf{u}$

Explain your work and demonstrate how to use scalar-vector multiplication and vector-vector addition in this modeling exercise.

Let $\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.5 \\ 1.0 \\ 1.5 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.6 \\ 1.0 \\ 1.4 \end{bmatrix}$

$\vec{f}_s = \begin{bmatrix} f_{s_1} \\ f_{s_2} \\ f_{s_3} \\ f_{s_4} \end{bmatrix} = -9.8 \cdot \vec{m} = \begin{bmatrix} 0.0 \\ -4.9 \\ -9.8 \\ -14.7 \end{bmatrix}$

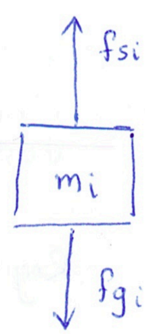
$\vec{u} = \vec{x} + -x_0 \cdot \vec{1}_{4\times1}$

$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + -x_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ x_3 - x_0 \\ x_4 - x_0 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.4 \\ 0.8 \\ 1.2 \end{bmatrix}$

Note on internal spring force vector

To calculate the force stored in the spring, we use a free-body diagram



By Newton's second law, we know

$\sum F = f_{gi} + f_{si} = m \cdot a = 0$

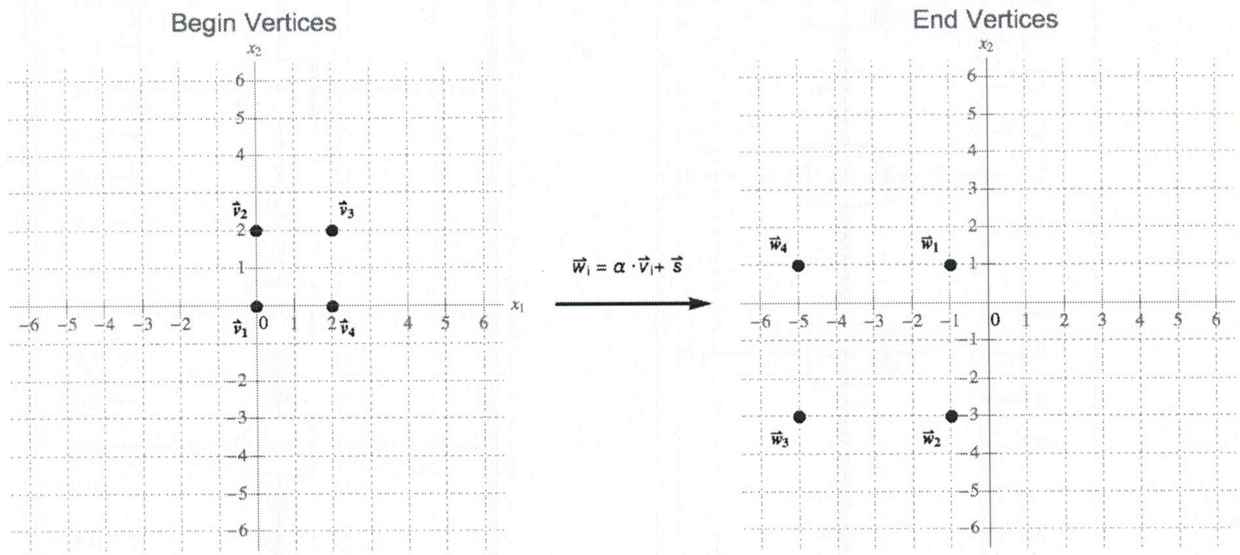
Since masses are at equilibrium

$\Rightarrow f_{si} = -f_{gi} = -9.8 m_i$

4. (6 points) Suppose we model a square using a set of four *begin vertices* given by the four vertices

$$\mathbf{v}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}0\\2\end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix}2\\0\end{bmatrix}.$$

Define a set of four *end vertices* with the property that $\mathbf{w}_i = \alpha \cdot \mathbf{v}_i + \mathbf{s}$ for a special scalar $\alpha \in \mathbb{R}$ and special vector $\mathbf{s} \in \mathbb{R}^2$, where $i \in [4]$. Under these assumptions, we graph both the begin and end vertices for this problem in the figure below.



Using this information, set up a system of equations that might help you figure out the specific values of α and $\mathbf{s}$ used in this problem. Then, find these values and explain your work.

$$\vec{w}_1 = \alpha \cdot \vec{v}_1 + \vec{s} \Rightarrow \begin{bmatrix} w_{11}\\ w_{21}\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} = \alpha\begin{bmatrix}0\\0\end{bmatrix} + \vec{s} \Rightarrow \boxed{\vec{s} = \begin{bmatrix}-1\\1\end{bmatrix}}$$

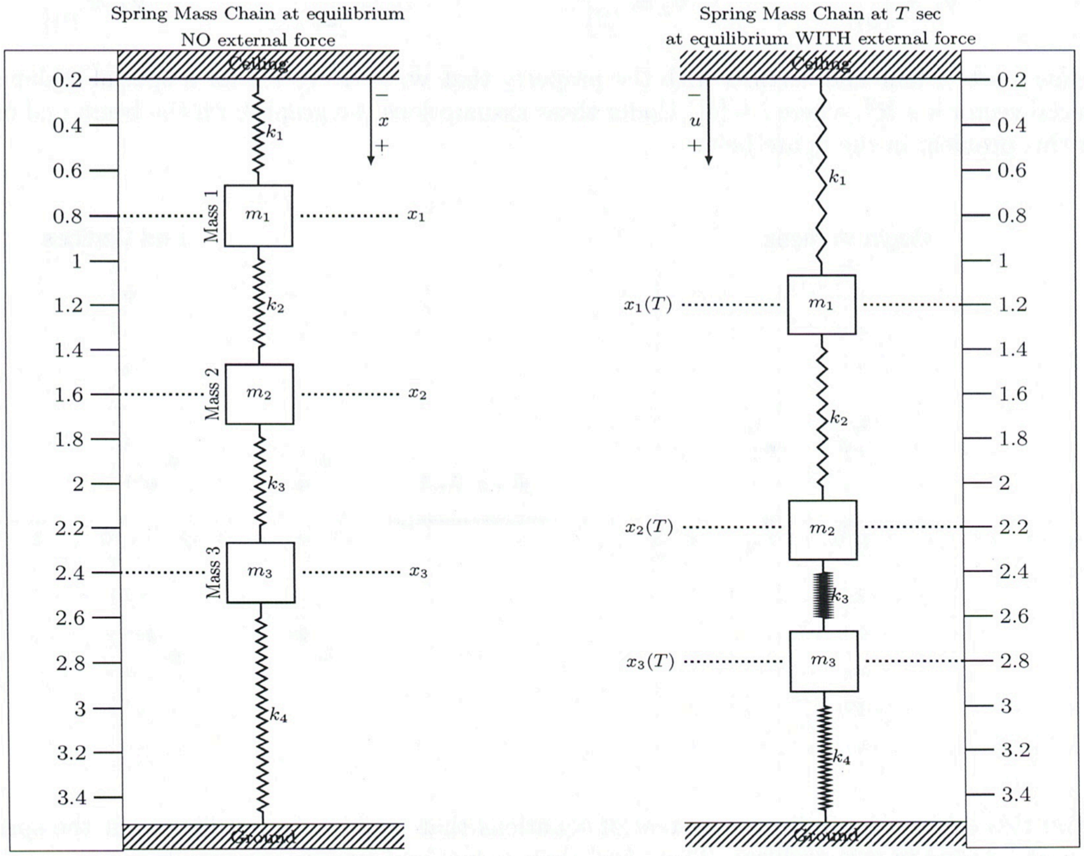
$$\vec{w}_2 = \alpha \cdot \vec{v}_2 + \vec{s} \Rightarrow \begin{bmatrix} w_{12}\\ w_{22}\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix} = \alpha\begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\Rightarrow \alpha\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow \alpha\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\-4\end{bmatrix}$$

$$\Rightarrow 2\alpha = -4 \Rightarrow \alpha = \frac{-4}{2} \Rightarrow \boxed{\alpha = -2}$$

5. (8 points) Consider the following model for a 3-mass, 4-spring chain drawn below. Note that positive positions and positive displacements are marked in the downward direction. Assume the ruler gives position measurements in meters.



Use this diagram and the given position data, set up the vectors $\mathbf{x}_0, \mathbf{x}(T), \mathbf{u}(T)$ that we discussed in class. Describe the significance of each vector. Also, show how we can use scalar-vector multiplication and vector-vector addition to form $\mathbf{u}(T)$ from the vectors $\mathbf{x}_0$ and $\mathbf{x}(T)$.

$$\vec{x}_0 = \begin{bmatrix} 0.8 \\ 1.6 \\ 2.4 \end{bmatrix}$$

☐ Students should use sentences to explain set up and meaning of these vectors...

$$\vec{x}(T) = \begin{bmatrix} 1.2 \\ 2.2 \\ 2.8 \end{bmatrix}$$

$$\vec{u}(T) = \vec{x}(T) - \vec{x}_0 = \begin{bmatrix} 1.2 \\ 2.2 \\ 2.8 \end{bmatrix} - \begin{bmatrix} 0.8 \\ 1.6 \\ 2.4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \\ 0.4 \end{bmatrix}$$

For problems 6 - 8, describe each of the given problems in detail. In order to earn full credit on each problem, your answer should accurately address each of the following :

i. Identify the problem statement
ii. Identify the given and unknown quantities (explicitly identify relevant dimensions)
iii. Identify the function description of this problem (explicitly discuss domain, codomain and range)
iv. Describe in as much detail as you can how each problem is similar to and different from the other two problems in the list below.

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6. (6 points) Problem 1: The Matrix-Vector Multiplication Problem

Solution: The matrix-vector multiplication problem is as follows:

Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, calculate unknown vector $\mathbb{b} \in \mathbb{R}^m$ such that

$$A\mathbf{x} = \mathbb{b}$$

The matrix-vector multiplication is a "forward problem." In particular, let's define the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad f(\mathbf{x}) = A\mathbf{x}$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is a subset set of the codomain $\mathbb{R}^m$. We know from our discussion in class that, for some matrices, the range of f may be a strict subset of the codomain (meaning there may be some elements of $\mathbb{R}^m$ that are not used as output by the function f)

Matrix-vector multiplication is a forward problem because we start with the function description (defined by matrix A) and we are given one specific input value $\mathbf{x}$ in the domain. From this information, we are asked to find the corresponding output value $\mathbb{b}$ in the range of function $f(\mathbf{x})$. When solving the matrix-vector multiplication problem, we map from the domain forward into the range. Hence, we call this a forward problem.

The matrix-vector multiplication problem is intimately connected with the general linear-system problem. Matrix-vector multiplication is the forward problem while the general linear-systems problem represents the backward problem (also known as inverse problem). As discussed below, when solving general linear-systems problems, we start with a $\mathbf{b} \in \text{Rng}(f)$ and produce all $\mathbb{x} \in \text{Dom}(f)$ such that $A(\mathbb{x}) = \mathbf{b}$.

7. (6 points) Problem 2A: The Nonsingular Linear-Systems Problem

Solution: The nonsingular linear-systems multiplication problem is as follows:

> Given a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$, find the unknown and desired input vector $\mathrm{x} \in \mathbb{R}^n$ such that
>
> $$A\mathrm{x} = \mathbf{b}$$

Just like the matrix-vector multiplication problem, we can describe the linear-systems using the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad f(\mathbf{x})$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^n$.
- the range of f is $\mathbb{R}^n$ since nonsingular matrices have very special properties. In particular, in class Jeff claimed that nonsingular matrices create matrix-vector multiplication functions that are both one-to-one and onto (also known as bijective).

The nonsingular linear-systems problem is a backward problem because we start with the function description (defined by nonsingular matrix A) and we are given one specific output value $\mathbf{b}$ in the range of $f(\mathbf{x})$. From this information, we are asked to the unique input values x in the domain of our function such that

$$f(\mathrm{x}) = \mathbf{b}.$$

When solving the nonsingular linear-systems problem, we begin in the range and work our way backwards to the domain. Hence, we call this a backward problem. The nonsingular linear-systems problem is a special case of the general linear-systems problem in that the matrices in the NLSP must be both square (the same number of rows and columns) and nonsingular. In the GLSP, the given matrices can be rectangular (different number of rows and columns).

8. (6 points) Problem 2B: The General Linear-Systems Problem

Solution: The general linear-systems multiplication problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, find all unknown vectors $\mathrm{x} \in \mathbb{R}^n$ such that
> $$A\mathrm{x} = \mathbf{b}.$$

Just like the matrix-vector multiplication problem, we can describe the linear-systems using the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad f(\mathbf{x}) = A \cdot \mathbf{x}$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is a subset of the codomain $\mathbb{R}^m$ and may not be the entire codomain (depending on the structure of matrix A)

The linear-systems problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the range of $f(\mathbf{x})$. From this information, we are asked to find all possible input values x in the domain of our function such that

$$f(\mathrm{x}) = \mathbf{b}.$$

When solving the linear-systems problem, we begin in the range and work our way backwards to the domain. Hence, we call this a backward problem.

Remarks (for students who want to earn the highest marks): In addition to the comments above, here are some other remarks about this problem

- The solution to a linear-systems problem may not exist. If it does exist, it may not be unique. A great analogy comes from solving backward problems for the nonlinear function $f(x) = x^2$. Let's look at three backward problems:

  A. Unique solution: $f(x) = x^2 = 0$
  B. Multiple solutions: $f(x) = x^2 = 4$
  C. No solutions: $f(x) = x^2 = -4$

Although the theory behind solving linear systems is much different than the theory for solving quadratic equations, analogies about the existence and uniqueness of solutions abound. Linear systems problems may have:

A. Unique solution: $\mathbf{b} \in \mathrm{Rng}(f)$ and A has full column rank.

B. Non-unique (multiple) solutions: $\mathbf{b} \in \mathrm{Rng}(f)$ and A is rank-deficient.

C. No Solution: $\mathbf{b} \notin \mathrm{Rng}(f)$ in which case we can minize the difference between $\mathbf{b}$ and the range of f: This minimization problem is known as the full-rank, least-squares problem when matrix A has full column rank.

Challenge Problem

9. (5 points) Optional, Extra Credit, Challenge Problem: Suppose that $a, b, c \in \mathbb{R}$ are positive numbers such that $a > b > c$. Then, consider the function

$$R(x_1, x_2, x_3) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}$$

If we require that $x_1^2 + x_2^2 + x_3^2 = 1$, what is the range of the function R?

Note if $a > b > c > 0 \Rightarrow a^2 > b^2 > c^2 > 0$

$\Rightarrow 0 < \frac{1}{a^2} < \frac{1}{b^2} < \frac{1}{c^2}$

$\Rightarrow$ □ If $\vec{x} \in \mathbb{R}^3$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and

$\|\vec{x}\|_2 = 1$, then max $R(x_1, x_2, x_3)$ happens at $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with

$R(0,0,1) = \frac{1}{c^2}$

□ by similar reasoning, we see min of $R(x_1, x_2, x_3)$ happens at $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with

$R(1,0,0) = \frac{1}{a^2}$

$\Rightarrow \operatorname{Rng}(R) = \left[\frac{1}{a^2}, \frac{1}{c^2}\right]$

Lemma: suppose $y_1, y_2, y_3 \in \mathbb{R}$ s.t. $y_i \geq 0$

and $y_1 + y_2 + y_3 = 1$. suppose $\alpha < \beta < \gamma$

Then, To maximize $\alpha y_1 + \beta y_2 + \gamma y_3$ we set

$y_1 = y_2 = 0$ and $y_3 = 1$

$\Rightarrow$ max value is γ

Lagrange multpliers $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = 1$

$$\nabla R = \lambda \cdot \nabla g$$

$$\Rightarrow \begin{bmatrix} 2x_1/a^2 \\ 2x_2/b^2 \\ 2x_3/c^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$\Rightarrow$ Equation 1: $\frac{2x_1}{a^2} = \lambda 2x_1$

Equation 2: $\frac{2x_2}{b^2} = \lambda 2x_2$

Equation 3: $\frac{2x_3}{c^2} = \lambda 2x_3$

Equation 4: $x_1^2 + x_2^2 + x_3^2 = 1$

$$\frac{2x_1}{a^2} = 2x_1 \cdot \lambda \Rightarrow \frac{2x_1}{a^2} - 2x_1\lambda = 0$$

$$\Rightarrow 2x_1\left(\frac{1}{a^2} - \lambda\right) = 0$$

$$\Rightarrow x_1 = 0 \quad \text{or} \quad \frac{1}{a^2} - \lambda = 0$$

$$\Rightarrow x_1 = 0 \quad \text{or} \quad \lambda = \frac{1}{a^2}$$

$$\Rightarrow \frac{2x_2}{b^2} = 2x_2 \cdot \lambda \Rightarrow 2x_2\left(\frac{1}{b^2} - \lambda\right) = 0$$

$$\Rightarrow x_2 = 0 \quad \text{or} \quad \lambda = \frac{1}{b^2}$$

$$\Rightarrow \frac{2x_3}{c^2} = 2x_3 \cdot \lambda \Rightarrow x_3 = 0 \quad \text{or} \quad \lambda = \frac{1}{c^2}$$

$$\Rightarrow x_1 = x_2 = 0 \text{ and } x_3 = 1$$

$$\Rightarrow \boxed{\text{max value at } R(0,0,1) = \frac{1}{c^2}}$$

$$\lambda = \frac{1}{a^2} \Rightarrow x_2 = x_3 = 0$$

$$\Rightarrow x_1 = 1$$

$$\Rightarrow \boxed{\text{min value at } R(1,0,0) = \frac{1}{a^2}}$$