Lesson 7: Change of Variable in Multiple Integrals

In Lessons 1,2,3,4,5,6, we discussed various tools that we can use to take multiple integrals including encoding our integrand

$$f : D \subseteq \overset{\substack{\text{vector-valued} \\ \text{input} \\ \downarrow}}{\mathbb{R}^n} \longrightarrow \underset{\substack{\uparrow \\ \text{real-valued output}}}{\mathbb{R}} \qquad \text{where } n = 2 \text{ or } 3$$

in cartesian coordinates, polar coordinates ($n=2$), cylindrical coordinates ($n=3$) or spherical coordinates ($n=3$).

In these discussions, we converted between different coordinate systems with the goal of "simplifying" the integration process. Recall u-substitution in Math 1B:

$$\int_{\textcircled{D}} f(u)\,\boxed{du} \overset{\substack{\text{area is equal} \\ \downarrow}}{=} \int_{\textcircled{\bar{D}}} f(u(x)) \cdot \boxed{u'(x)\,dx}$$

these regions may differ

differential form has an extra "cost" factor! this is related to Riemann sums & the size function idea.

Revisit change of variables for Single-variable Integrals

Example 13.7.0 p. 1034

Find the integral $\int_0^1 2\sqrt{2x+1}\, dx$

Solution: Let's begin by analyzing

1. The Integral symbols: $\int_D f \, d\omega$

A. The domain region $D \subseteq \mathbb{R}$

B. The integrand $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

(identify the encoding of input variables)

C. The differential form $d\omega$ used to assign sizes to "points" in domain region.

We start by drawing a diagram associated with this integral : To this end, let $f(x) = 2 \cdot \sqrt{2x+1}$

x	$f(x) = 2 \cdot \sqrt{2x+1}$
$-1/2$	$f(-1/2) = 0$
0	$f(0) = 2$
1	$f(1) = 2 \cdot \sqrt{3} \approx 3.464$
$3/2$	$f(3/2) = 4$
4	$f(4) = 6$

Side notes:

☐ To evaluate $\sqrt{2x+1}$, we need the radicand to be nonnegative

$\Rightarrow \quad 2x+1 \geq 0$

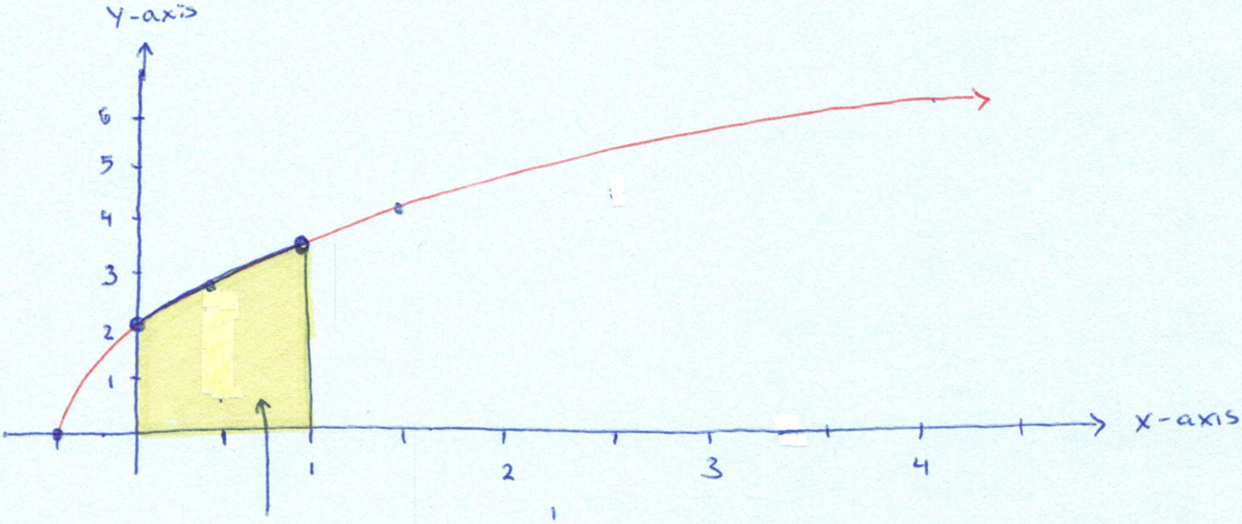
$\Rightarrow \quad 2x \geq -1$

$\Rightarrow \quad x \geq -1/2$

☐ Let's find perfect squares:

- $2x+1 = 4 \Rightarrow x = 3/2$
- $2x+1 = 9 \Rightarrow x = 4$

We can now graph this integrand in $\mathbb{R}^2$



this is the area we want to measure : $\int_0^1 2 \cdot \sqrt{2x+1}\, dx$

Side note: U-substitution

☐ Let $u(x) = 2x + 1$

$$\Rightarrow du = \underset{\uparrow}{d}[2x+1]$$

treat like a "differential" operator

$$= 2\,dx + 0$$

$$= 2 \cdot dx$$

☐ $x = 0 \Rightarrow u(0) = 2 \cdot 0 + 1 = 1$

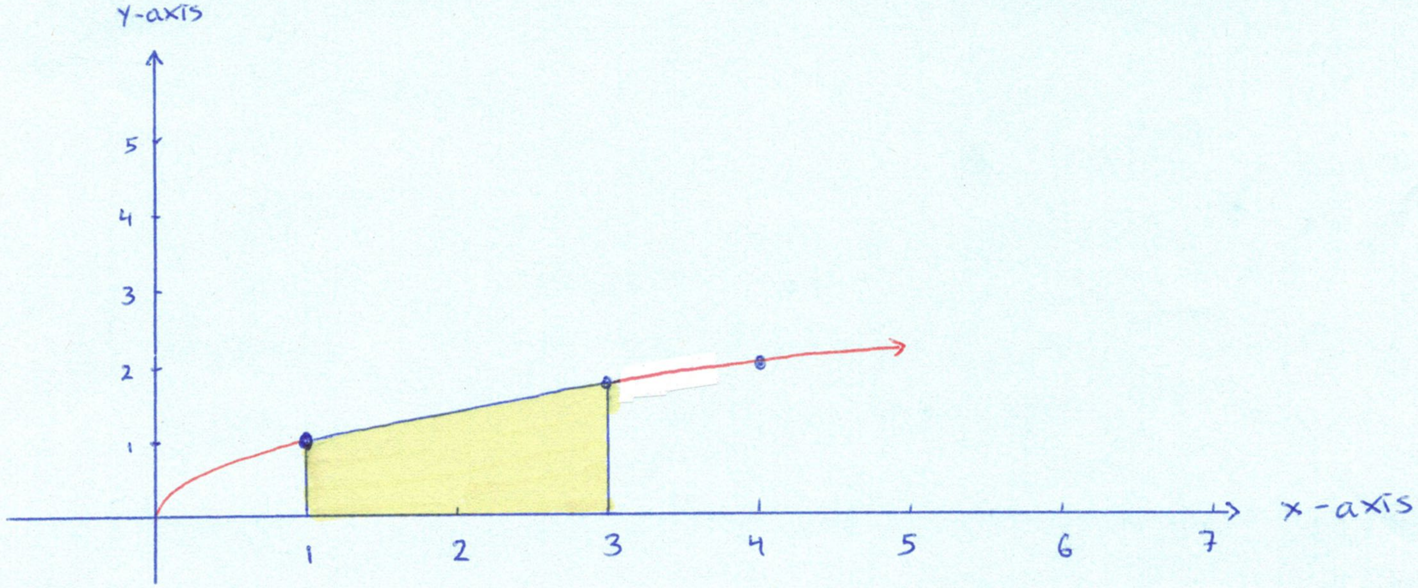
$x = 1 \Rightarrow u(1) = 2 \cdot 1 + 1 = 3$

$$\Rightarrow \int_0 f\,d\omega = \int_0^1 \sqrt{2x+1} \cdot 2 \cdot dx$$

$$= \int_1^3 \sqrt{u}\;du$$

x	u	$\sqrt{u}$
$-1/2$	0	0
	1	1
	2	$\sqrt{2} \approx$
	3	$\sqrt{3} \approx$
0	4	2

Now, we can redraw the diagram depicting the area with the new parameterization of the input doman:



This is the "equivalent" area we will measure : $\int_{1}^{3} \sqrt{u} \, du$

Let's analyze what is going on here:

$$f(x) = 2 \cdot \sqrt{\boxed{2x+1}}$$

one "point" in x-variable covers twice the distance in u (with a shift also)

$$= 2 \cdot \sqrt{u(x)} \qquad \text{where} \quad u(x) = 2x+1$$

$$= 2 \cdot \sqrt{\boxed{u}}$$

"pure" radicand: input into square root function is pure

$$= 2 \cdot g(u) \qquad \text{where} \quad g(u) = \sqrt{u}$$

$$\Rightarrow \int_D f \, dw = \int_D f(x) \, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} \underbrace{f(x_k^*)}_{\text{weight assigned to Kth interval through sampling process}} \cdot \overbrace{\Delta x_k}^{\text{Size of Kth interv}}$$

But, in this circumstance $f(x) = g(u(x)) = g \circ u(x)$

$$\Rightarrow \int_D f \, dw = \int_D g(u) \, dw$$

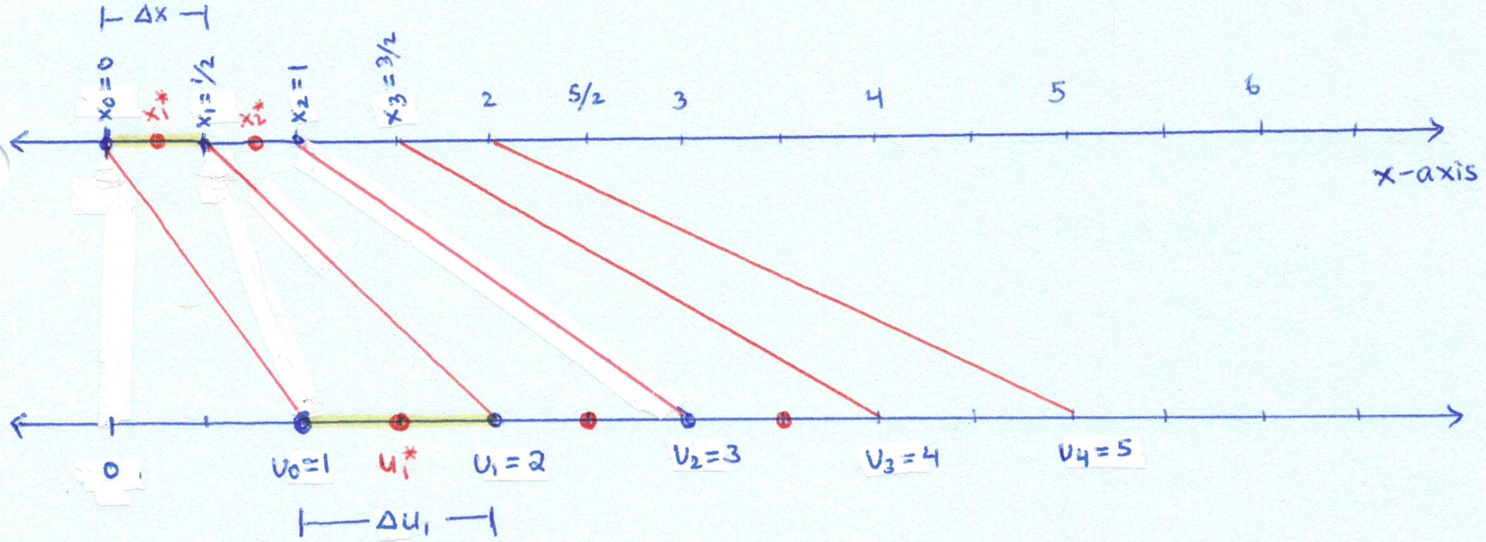
apriori: these may not be correlated

When we consider the finite sum:

$$\sum_{k=1}^{n} f(x_k^*)\cdot\underbrace{\Delta x_k}$$

this is supposed to represent size of the kth subregion

Let's partition our x-inputs: $\Delta x_k = \frac{1}{2}$



$$\Delta x_1 = \frac{1}{2} \Rightarrow \Delta u_1 = 1$$

$$\Delta x_2 = \frac{1}{2} \Rightarrow \Delta u_2 = 1$$

$$\vdots$$

$$\Delta x_k = 1/2 \Rightarrow \Delta u_k = 1 \Rightarrow \Delta u_k = 2\cdot\Delta x_k$$

$$\Rightarrow \quad D = \{ x \in \mathbb{R} : 0 \le x \le 1 \}$$

$$\Rightarrow \quad \int_D f \, d\omega = \int_{x \in [0,1]} 2 \cdot \sqrt{2x+1} \, dx$$

$$= \int_{\text{"}x=0\text{"}}^{\text{"}x=1\text{"}} 2 \cdot \sqrt{\boxed{2x+1}} \, dx$$

"sizes" measured w/r to variable x

the input to most important function for integrand is not encoded w/r to x values:

$$f(x) = 2 \cdot \sqrt{2x+1}$$

$$= 2 \cdot \sqrt{u} \longrightarrow \begin{cases} g(u) = \sqrt{u} \\ u(x) = 2x+1 \end{cases}$$

$$\Rightarrow \quad g(u(x)) = g \circ u(x)$$

As you might notice, the "simple" arithmetic technique known as u-substitution for integrals in Math 1B has a very nuanced effect when considering the differential forms

$$\int_0^1 2\cdot\sqrt{2x+1}\ \boxed{dx} = \int_1^3 \sqrt{u}\ \boxed{du}$$

used to measure sizes of subintervals.

In lesson 7 we will study analogous transformations where $f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and develop a theory of integration in which we can evaluate

$$\int_D f\, d\omega$$

when we change the method we use to encode region D.

Think of this as a "generalization" of u-substitution.

Let's begin by considering general geometric transformations between two regions

$$D = \{(x,y) : \text{proposition(s) } P(x,y) \quad \text{true}\} \subseteq \mathbb{R}^2$$

$$\bar{D} = \{(u,v) : \text{proposition(s) } P(u,v) \quad \text{true}\} \subseteq \mathbb{R}^2$$

To transform $\bar{D}$ into D, we will define a function $\vec{T} : \bar{D} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$\vec{T}(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$$

Notice this function takes "in" ordered pairs $(u,v) \in \bar{D}$ and outputs ordered pairs $(x,y) \in D$.

Example 13.7.1 p. 1035

Let $\bar{D} = \{(r, \theta) : 0 \le r \le 1$ and $0 \le \theta \le \pi/2\}$

Let's find "the image" of $\bar{D}$ under the transformation

$$\vec{T} : \bar{D} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

where $\vec{T}$ is defined by

$$\vec{T}(r, \theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(r, \theta) \\ y(r, \theta) \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\theta) \\ r \cdot \sin(\theta) \end{bmatrix}$$

☐ This is the famous transformat
that maps points in polar coordinates
to points in rectangular coordinates

Solution: Let's begin by considering a graphical/geometric representation of $\bar{D}$ in the polar plane.

To do so, we will consider a "naive" representation of $\bar{D} = \{(r, \theta) : 0 \le r \le 1$ and $0 \le \theta \le \pi/2\}$

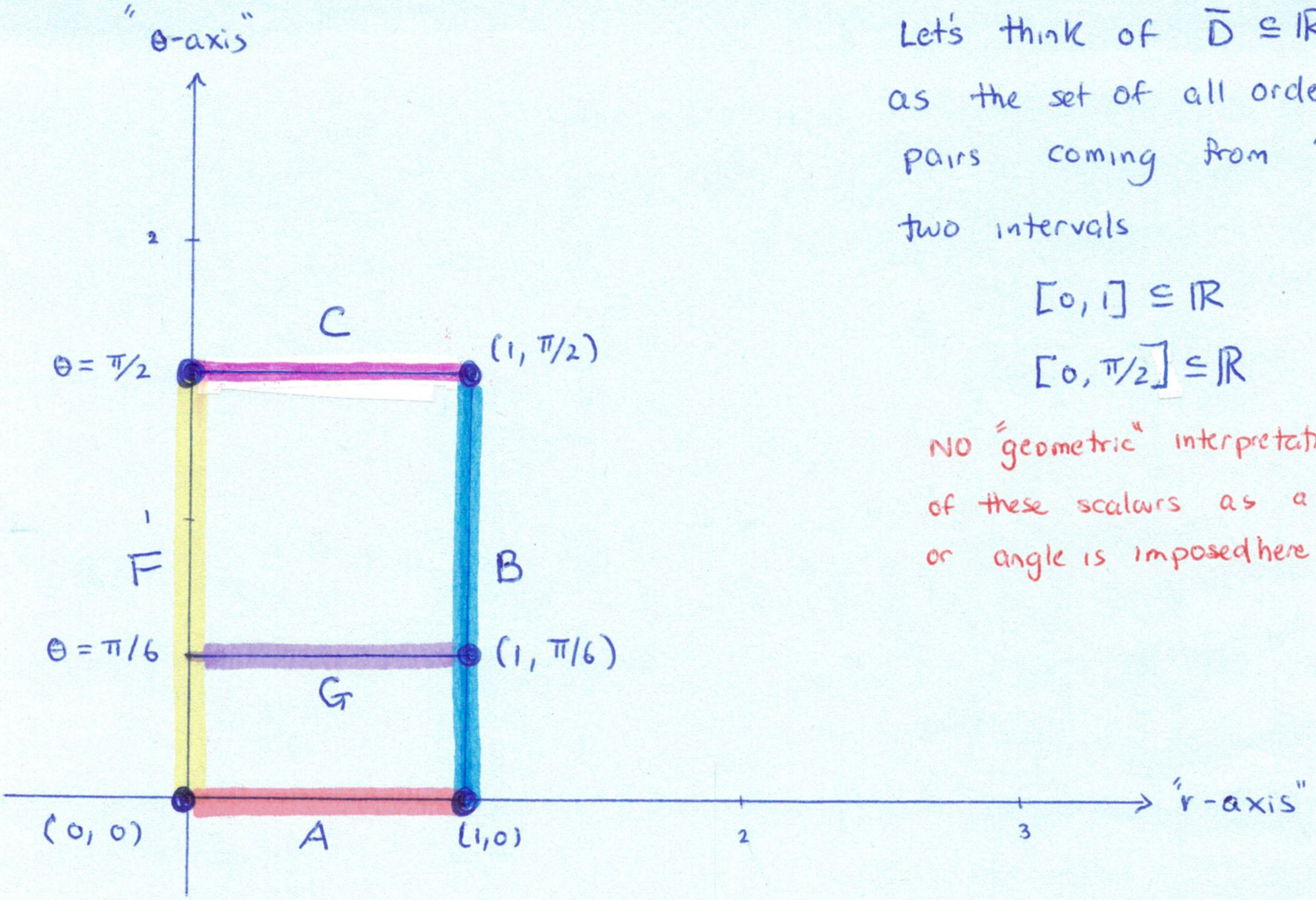
$$\Rightarrow \bar{D} = [0, 1] \underset{\uparrow}{\times} [0, \pi/2]$$

cross product of sets from Math 22
(not cross product of vectors from Math1C)

Let's think of $\bar{D} \subseteq \mathbb{R}^2$ as the set of all ordered pairs coming from "crossing" two intervals

$$[0, 1] \subseteq \mathbb{R}$$
$$[0, \pi/2] \subseteq \mathbb{R}$$

NO "geometric" interpretation of these scalars as a radius or angle is imposed here.



With this interpretation in mind, let's focus in on the boundary $\partial \bar{D}$ of $\bar{D}$ and focus on the following sets:

$$\boxed{A} = \{(r, \theta) : 0 \le r \le 1 , \theta = 0\}$$

$$\boxed{B} = \{(r, \theta) : r = 1 , 0 \le \theta \le \pi/2\}$$

$$\boxed{C} = \{(r, \theta) : 0 \le r \le 1 , \theta = \pi/2\}$$

$$\boxed{F} = \{(r, \theta) : r = 0 , 0 \le \theta \le \pi/2\}$$

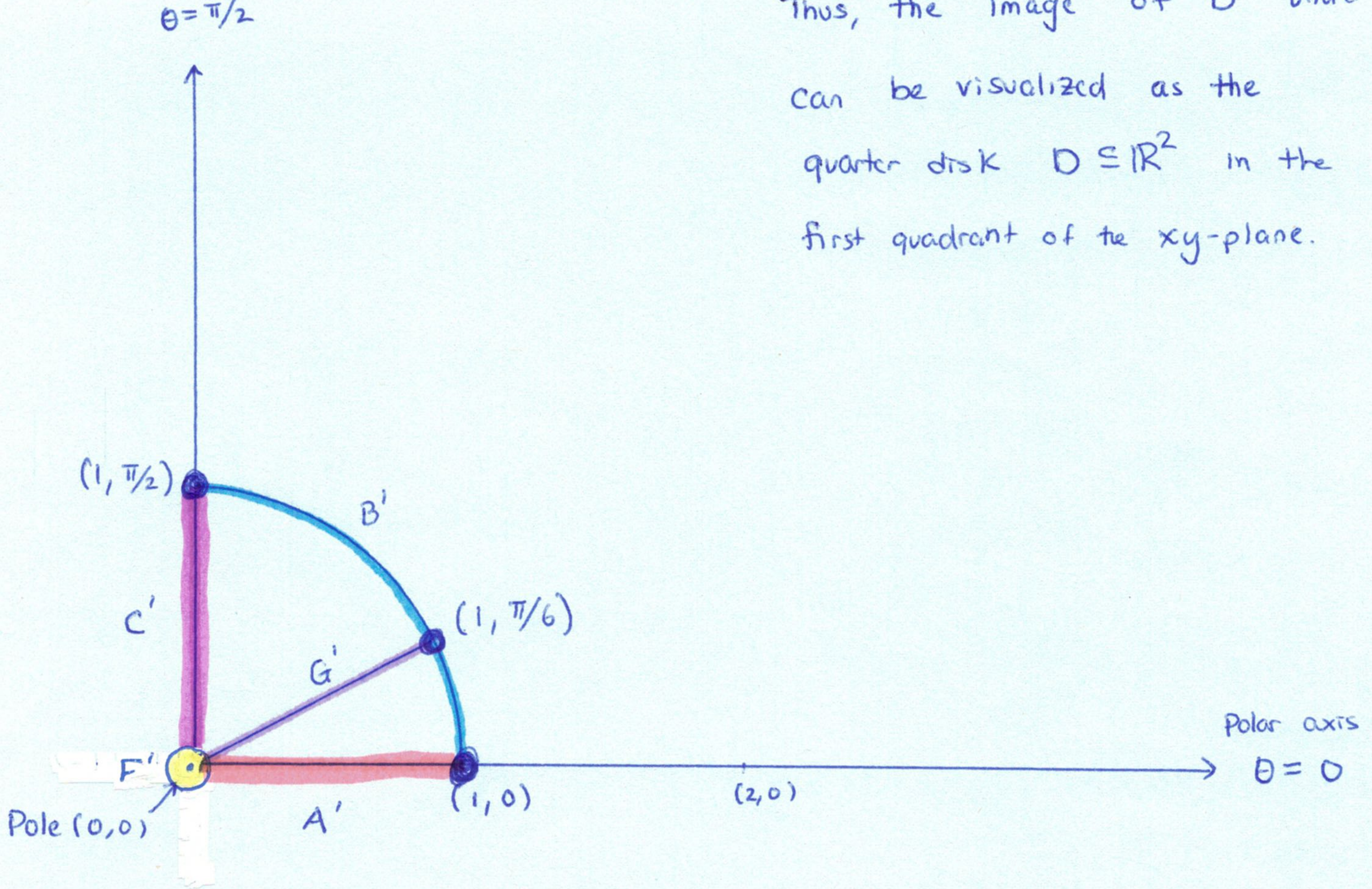
$$\boxed{G} = \{(r, \theta) : 0 \le r \le 1, \theta = \pi/6\}$$

Now let's look at the image of each of these sets under the transformation $\vec{T} : \bar{D} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Boundary set of $\bar{D}$ in $r\theta$ - plane	Transformation equations $x = r\cos(\theta)$ & $y = r\sin(\theta)$	Boundary of set D in the xy-plane
$\boxed{A} = \{(r,\theta) : 0 \le r \le 1 \text{ and } \theta = 0\}$	$\vec{T}(r,\theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$	$\boxed{A'} = \{(x,y) : 0 \le x \le 1, y = 0\}$
$\boxed{B} = \{(r,\theta) : r = 1, 0 \le \theta \le \pi/2\}$	$\vec{T}(r,\theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$	$\boxed{B'} = \{(x,y) : x^2 + y^2 = 1 \text{ and } x,y \ge 0\}$
$\boxed{C} = \{(r,\theta) : 0 \le r \le 1, \theta = \pi/2\}$	$\vec{T}(r,\theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$	$\boxed{C'} = \{(x,y) : x = 0 , 0 \le y \le 1\}$
$\boxed{F} = \{(r,\theta) : r = 0, 0 \le \theta \le \pi/2\}$	$\vec{T}(r,\theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\boxed{F'} = \{(x,y) : x = 0 \text{ \& } y = 0\}$
$\boxed{G} = \{(r,\theta) : 0 \le r \le 1, \theta = \pi/6$	$\vec{T}(r,\theta) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \cdot r \\ \frac{1}{2} \cdot r \end{bmatrix}$	$\boxed{G'} = \{(x,y) : x = \sqrt{3}/2\, t, y = \frac{t}{2}, 0 \le t$

Note: In this table we now impose a geometric interpretation of each point $(r,\theta) \in \bar{D}$ as representing a radius and angle. We then interpret the "meaning" of these transformations as visualized in the xy plane

Thus, the "image" of $\bar{D}$ under $\vec{T}$ can be visualized as the quarter disk $D \subseteq \mathbb{R}^2$ in the first quadrant of the xy-plane.



Notice: ☐ The "line" $F = \{(r,\theta) : r = 0 \ , \ 0 \le \theta \le \pi/2\} \subseteq \bar{D}$ collapses into a single point $F' = \{(x,y) : x = 0 \ \& \ y = 0\} \subseteq D$

The map $\vec{T}$ maps an infinite number of points from F in domain single point F' in codomain

☐ All other input domain points in $\bar{D} \subseteq \mathbb{R}^2$ have a "unique" image in the output codomain. In other words, as long as we don't consider input points (r,θ) along the $r=0$ axis, we can conclude that if $\vec{T}(r_1,\theta_1) = \vec{T}(r_2,\theta_2)$ then $(r_1,\theta_1) = (r_2,\theta_2)$. Such a map is called one-to-one.

Definition: Let $T : \bar{D} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a transformation.

from a region $\bar{D}$ to a region D. Let

$\vec{x}, \vec{y} \in \bar{D}$ be "points" in domain region $\bar{D}$.

We say $\vec{T}$ is one-to-one on $\bar{D}$ if and only if

the following proposition is true on $\bar{D}$:

If $\vec{T}(\vec{x}) = \vec{T}(\vec{y})$, then $\vec{x} = \vec{y}$.

□ In English, this condition says each point in the image has a "unique" preimage

Remark: The transformation $\vec{T} : \bar{D} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ from example 13.7.1 was <u>NOT</u> one-to-one on $\bar{D}$ since for $\vec{x} = (0,0)$ & $\vec{y} = (0, \pi/$

we had $\vec{0} = \vec{T}(0,0) = \vec{T}(0, \pi/2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and yet $\vec{x} \neq \vec{y}$.

In other words, two different input points $\vec{x}, \vec{y} \in \bar{D}$ w/ $\vec{x} \neq \vec{y}$ got mapped to same output.

Definition: The Jacobian Determinant of a Transformation $\vec{T}:\bar{D}\subseteq\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Let $\bar{D}\subseteq\mathbb{R}^2$ and define the transformation

$$\vec{T}:\bar{D}\subseteq\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

as follows:

$$\vec{T}(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$$

Then, the Jacobian determinant (also known as the Jacobian) of

is given by

$$J(u,v) = \underset{\uparrow}{\det}\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right)$$

determinant function from math 2B

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\leftarrow$ notation note: long vertical bars represent determinants

Example 13.7.2 p 1037

Let $\vec{T}(r,\theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$. Then compute the

Jacobian of $\vec{T}$.

Solution: Recall $J(r,\theta) = \det\left(\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right)$.

Then, we can consider our needed partials:

$$x = x(r,\theta) = r \cdot \cos(\theta) \Rightarrow \begin{cases} \frac{\partial x}{\partial r} = \cos(\theta) \\ \frac{\partial x}{\partial \theta} = -r\sin(\theta) \end{cases}$$

$$y = y(r,\theta) = r \cdot \sin(\theta) \Rightarrow \begin{cases} \frac{\partial y}{\partial r} = \sin(\theta) \\ \frac{\partial y}{\partial \theta} = r\cos(\theta) \end{cases}$$

$$\Rightarrow \quad J(r,\theta) = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} \begin{array}{l} \nearrow - \\ \\ \searrow + \end{array}$$

$$= r\cos^2(\theta) \quad - \quad -r\sin^2(\theta)$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

$$= r \cdot (\cos^2(\theta) + \sin^2(\theta)$$

$$= r \,.$$

Theorem 13.8 p. 1036 Change of variables in double integrals

Let $\bar{D} \subseteq \mathbb{R}^2$ be encoded w/r to variables u and v.

Let $\vec{T} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a transformation that maps closed and bounded regions $\bar{D} \subseteq \mathbb{R}^2$ in the uv-plane onto a region $D \subseteq \mathbb{R}^2$ in the xy-plane.

Assume $\vec{T}$ is one-to-one on the "interior" of $\bar{D}$ and $x = x(u,v)$ and $y = y(u,v)$ have continuous 1st partial derivatives. If $f : D \subseteq \mathbb{R}^2 \rightarrow$ is continuous, then

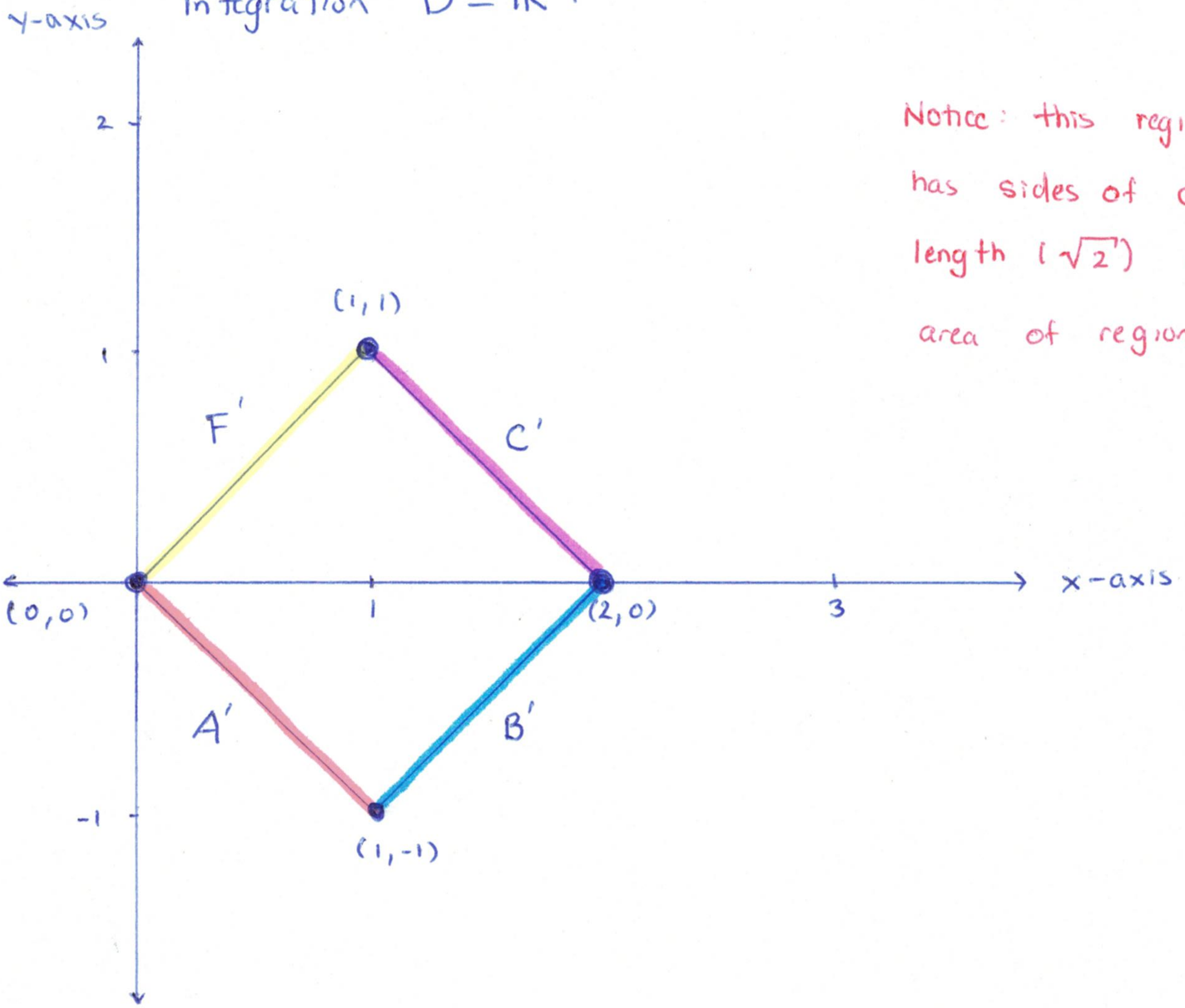
$$\iint_D f(x,y)\, dA = \iint_{\bar{D}} f(x(u,v), y(u,v))\, |J(u,v)|\, dA$$

Example 13.7.4 p. 1038)

Evaluate the double integral $\iint\limits_{\bar{D}} \sqrt{\frac{x-y}{x+y+1}}\, d\omega$

where $\bar{D}$ is the square with vertices $(0,0)$, $(1,-1)$, $(2,0)$, and $(1,1)$.

Solution: Let's begin by visualizing the region of integration $\bar{D} \subseteq \mathbb{R}^2$.



Notice: this region $\bar{D}$ has sides of equal length $(\sqrt{2})$ and total area of region in 2

We notice that when evaluating this integral over the given region $\bar{D} \subseteq \mathbb{R}^2$, we need to partition our region

$$\bar{D} = \bar{D}_1 \cup \bar{D}_2$$

where $\bar{D}_1 = \{ (x,y) : \quad 0 \leq x \leq 1 \quad \text{and} \quad -x \leq y \leq x \}$

$\bar{D}_2 = \{ (x,y) : \quad 1 \leq x \leq 2 \quad \text{and} \quad -2+x \leq y \leq 2-x \}$

Moreover, the integrand $f(x,y) = \sqrt{\dfrac{x-y}{x+y+1}}$ does not yield a simple antidervative.

With this in mind, we might ask ourselves:

A. Can we map $\bar{D}$ onto a new region D that is much simpler to integrate?

B. Can we choose new variables to simplify the integrand?

To this end, let's choose

$$u = u(x,y) = x - y \quad \text{and}$$

$$v = v(x,y) = x + y$$

Note: This transformation can be represented in matrix form as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a linear transformatio represention a rotation and a dialation.

Let's determine the image of $\bar{D}$ under this transformation

Boundary of $\bar{D}$ in the xy-plane	Transformation equation $u = x - y$ and $v = x + y$	Boundary of D in the uv-plane
A′: $0 \le x \le 1$ and $y = -x$	see next page	A: $0 \le u \le 2$ and $v = 0$
B′: $1 \le x \le 2$ and $y = x - 2$	see future page(s)	B: $u = 2$ and $0 \le v \le 2$
C′: $1 \le x \le 2$ and $y = 2 - x$	see future pages	C: $0 \le u \le 2$ and $v = 2$
F′: $0 \le x \le 1$ and $y = x$	See future pages	F: $u = 0$ and $0 \le v \le 2$

Transformation from A' to A:

$$u = x - y \quad \text{and} \quad v = x + y \quad \Rightarrow \quad u + v = 2x$$

$$\Rightarrow \quad x = \frac{u + v}{2} \qquad \boxed{\text{I}}$$

We also know that along A', we have

$$y = -x \quad \Rightarrow \quad x + y = 0$$

$$\Rightarrow \quad \boxed{v = 0} \qquad \boxed{\text{II}}$$

Combining $\boxed{\text{I}}$ and $\boxed{\text{II}}$, we see

$$x = \frac{u + v}{2} = \frac{u}{2} \quad \Rightarrow \quad 0 \le \frac{u}{2} \le 1$$

$$\Rightarrow \quad \boxed{0 \le u \le 2}$$

Transformation from B' to B:

From our equations defining B', we see

$$y = x - 2 \Rightarrow x - y = 2$$

$$\Rightarrow \boxed{u = 2}$$

We also know by the transformation equations

$$u + v = 2x \Rightarrow x = \frac{u+v}{2}$$

$$\Rightarrow x = \frac{2+v}{2}$$

Since $1 \leq x \leq 2$, we substitute this value of x in to find

$$1 \leq \frac{2+v}{2} \leq 2 \Rightarrow 2 \leq 2+v \leq 4$$

$$\Rightarrow \boxed{0 \leq v \leq 2}$$

Transformation from C' to C

On curve C', we know that

$$y = 2 - x \quad \Rightarrow \quad x + y = 2$$

$$\Rightarrow \quad \boxed{v = 2}$$

we also have

$$\frac{u+v}{2} = x \quad \Rightarrow \quad \frac{u+2}{2} = x$$

Since the curve C' bounds x with

$$1 \leq x \leq 2 \quad \Rightarrow \quad 1 \leq \frac{u+2}{2} \leq 2$$

$$\Rightarrow \quad 2 \leq u + 2 \leq 4$$

$$\Rightarrow \quad \boxed{0 \leq u \leq 2}$$

Transformation from F' to F

From our equation for curve F', we know

$$y = x \implies x - y = 0$$

$$\implies \boxed{u = 0}$$

Since we have $x = \frac{u+v}{2} = \frac{v}{2}$, we consider

$$0 \le x \le 1 \implies 0 \le \frac{v}{2} \le 1$$

$$\implies \boxed{0 \le v \le 2}$$

We can use this transformation to verify the vertices of the new region D :

vertex 1: $(x,y) = (0,0) \Rightarrow \begin{cases} u = 0-0 = 0 \ \checkmark \\ v = 0+0 = 0 \ \checkmark \end{cases}$

$$\Rightarrow \quad (u,v) = (0,0) \ \checkmark$$

vertex 2: $(x,y) = (1,-1) \Rightarrow \begin{cases} u = 1-(-1) = 2 \\ v = 1+(-1) = 0 \end{cases}$

$$\Rightarrow \quad (u,v) = (2,0)$$

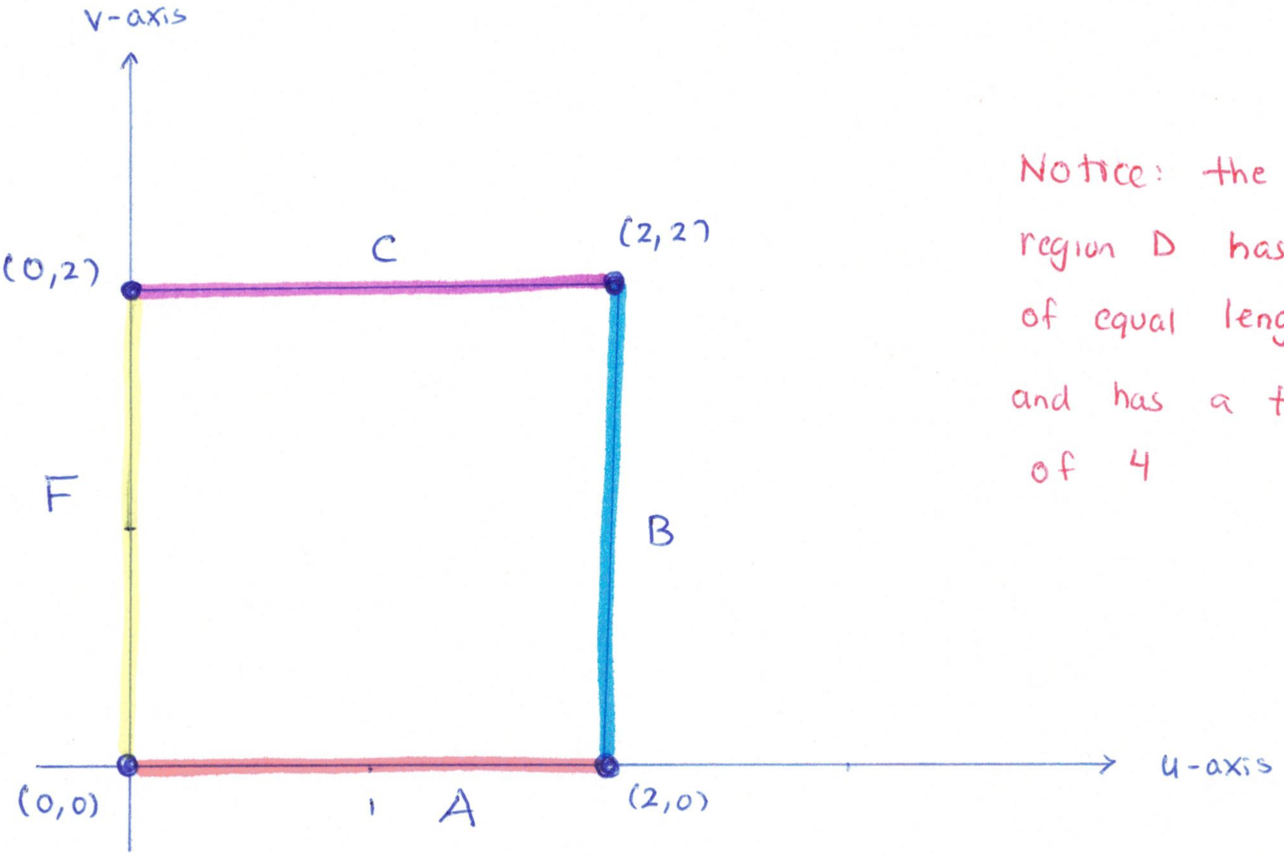
vertex 3: $(x,y) = (2,0) \Rightarrow \begin{cases} u = 2-0 = 2 \\ v = 2+0 = 2 \end{cases}$

$$\Rightarrow \quad (u,v) = (2,2)$$

vertex 4: $(x,y) = (1,1) \Rightarrow \begin{cases} u = 1-1 = 0 \\ v = 1+1 = 2 \end{cases}$

$$\Rightarrow \quad (u,v) = (0,2)$$

We now graph the boundary of our new, transformed region D under the chosen transformation:



Notice: the transformed region D has sides of equal length (2) and has a total area of 4

Now, we want to rewrite our integral using theorem 13.8

$$\iint_{\bar{D}} f(x,y)\, d\bar{\omega} = \iint_{D} f(u,v) \cdot |J(u,v)|\, d\omega$$

Recall: $\begin{cases} u = x - y \\ v = x + y \end{cases}$

To this end, we need to find

$$x = x(u,v) \qquad \text{and} \qquad y = y(u,v)$$

We already noted : $\begin{cases} u + v = 2x & \Rightarrow \quad x = \dfrac{u+v}{2} \\ v - u = 2y & \Rightarrow \quad y = \dfrac{v-u}{2} \end{cases}$

Then, $J(u,v) = \det\left(\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \right)$

$\dfrac{\partial x}{\partial u} = \dfrac{1}{2}$ $\qquad$ $\dfrac{\partial x}{\partial v} = \dfrac{1}{2}$

$\dfrac{\partial y}{\partial u} = -\dfrac{1}{2}$ $\qquad$ $\dfrac{\partial y}{\partial v} = \dfrac{1}{2}$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}^{\,-}_{\,+}$$

$$= \frac{1}{4} - \left(\frac{-1}{4}\right)$$

$$= \boxed{\frac{1}{2}.}$$

$$\iint_{\bar{D}} f(x,y)\, d\bar{\omega} = \iint_{\bar{D}} \sqrt{\frac{x-y}{x+y+1}}\, d\bar{\omega}$$

$$= \iint_{D} f(u,v)\ |J(u,v)|\ d\omega$$

$$= \iint_{D} \sqrt{\frac{u}{v+1}} \cdot \left|\frac{1}{2}\right| d\omega$$

$$= \int_0^2 \overset{\text{inner integral}}{\boxed{\int_0^2 \frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v+1}}\, du}}\, dv$$

Side note:

$$\int_0^2 \frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v+1}}\, du = \frac{1}{2\sqrt{v+1}} \int_0^2 \sqrt{u}\, du$$

$$= \frac{1}{2\sqrt{v+1}} \left(\frac{2}{3} \cdot u^{3/2} \Big|_0^2 \right)$$

$$= \frac{2\sqrt{2}}{3} \cdot \frac{1}{\sqrt{v+1}}$$

$$= \int_0^2 \frac{2\sqrt{2}}{3} \cdot (v+1)^{-1/2}\, dv$$

$$= \frac{2\sqrt{2}}{3} \cdot \frac{2}{1} \ (v+1)^{1/2} \Big|_0^2 = \boxed{\frac{4\sqrt{2}}{3} (\sqrt{3} - 1)}$$