Name : ____________________ **Class Number**: ______

Math 1D: Lesson 6 Suggested Problems

Theoretic Problems: Discussed in-class

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1. Construct the cylindrical coordinate system in $\mathbb{R}^3$ from first principles. In particular:

   A. Explain how to create a cylindrical coordinate system by extending the polar coordinate system from $\mathbb{R}^2$ into $\mathbb{R}^3$. What are the similarities and differences between polar and cylindrical coordinates.

   B. Starting from first principles and without looking back at your notes, create a description of each of the following objects in cylindrical coordinates: cylinder, cylindrical shell, vertical half plane, horizontal plane, cone.

   C. Derive each of the formulas to convert from cartesian coordinates in $\mathbb{R}^3$ to cylindrical coordinates.

   E. Derive each of the formulas to convert from cylindrical coordinates to cartesian coordinates in $\mathbb{R}^3$.

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2. Let $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function on a region D in $\mathbb{R}^3$ where D will be translated between cartesian coordinates and cylindrical coordinates.

   A. Explain how to visualize the region

   $$D = \Big\{(r, \theta, z) : \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta),\ G\Big(r\cos(\theta), r\sin(\theta)\Big) \le z \le H\Big(r\cos(\theta), r\sin(\theta)\Big)\Big\}$$

   in general. Specifically describe the meaning and effect of the functions $g(\theta), h(\theta), G\Big(r\cos(\theta), r\sin(\theta)\Big)$ and $H\Big(r\cos(\theta), r\sin(\theta)\Big)$

   B. Explain how to set up the uniform discretization (a regular partition) of D and how to enumerate subregions from $k = 1, 2, ..., n$.

   C. Explain how to choose a sample point $(r_k^*, \theta_k^*, z_k^*)$ from the kth subregion of the partition using the midpoint rule for the radial component.

   D. Explain how to translate the Riemann sum

   $$\sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\ \Delta V_k$$

   into the integral by taking a limit with respect to Δ where Δ is the maximum value of Δr, $\Delta\theta$ and Δz.

   E. Derive the formula for the volume of the kth sector of our partition as $\Delta V_k = r_k^* \cdot \Delta r \cdot \Delta\theta\Delta z$

   F. Using parts A - E above, explain why the differential form dV used to measure "sizes" of points encoded in cylindrical coordinates has a factor of r in it where $dV = r \cdot dr \cdot d\theta \cdot dz$

   G. Now, explain why we define of the triple integral of our function f:

   $$\iiint_D f(x, y, z)\ dV = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} \int_{G}^{H} f(r, \theta, z)\ \ dz\ r\ dr\ \ d\theta$$

3. Construct the spherical coordinate system in $\mathbb{R}^3$ from first principles. In particular:

   A. Explain how to create a spherical coordinate system in $\mathbb{R}^3$. What are the similarities and differences between spherical and cylindrical coordinates.

   B. Starting from first principles and without looking back at your notes, create a description of each of the following objects in spherical coordinates: sphere with radius ρ, a cone, vertical half plane, horizontal plane, cylinder.

   C. Derive each of the formulas to convert from cartesian coordinates in $\mathbb{R}^3$ to spherical coordinates.

   E. Derive each of the formulas to convert from spherical coordinates to cartesian coordinates in $\mathbb{R}^3$.

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4. Let $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function on a region D in $\mathbb{R}^3$ where D will be translated between cartesian coordinates and spherical coordinates.

   A. Explain how to visualize the region

   $$D = \{(\rho, \phi, \theta) : \alpha \leq \theta \leq \beta,\ a \leq \phi \leq b,\ g(\phi, \theta) \leq \rho \leq h(\phi, \theta)\}$$

   in general. Specifically describe the meaning and effect of the functions $g(\phi, \theta)$ and $h(\phi, \theta)$.

   B. Explain how to set up the uniform discretization (a regular partition) of D and how to enumerate subregions from $k = 1, 2, ..., n$.

   C. Explain how to choose a sample point $(\rho_k^*, \phi_k^*, \theta_k^*)$ from the kth subregion of the partition using the midpoint rule for the radial component.

   D. Explain how to translate the Riemann sum

   $$\sum_{k=1}^{n} f(\rho_k^*, \phi_k^*, \theta_k^*)\ \Delta V_k$$

   into the integral by taking a limit with respect to Δ where Δ is the maximum of $\Delta\rho$, $\Delta\phi$ and $\Delta\theta$.

   E. Derive the formula for the volume of the kth sector of our partition as

   $$\Delta V_k \approx (\rho_k^*)^2 \cdot \sin(\phi_k)\ \Delta\rho\ \Delta\phi\ \Delta\theta$$

   F. Using parts A - E above, explain why the differential form dV used to measure "sizes" of points encoded in spherical coordinates has a factor of $\rho^2 \sin(\phi)$ in it where $dV = \rho^2 \cdot \sin(\phi)\, d\rho \cdot d\phi \cdot d\theta$

   G. Now, explain why we define of the triple integral of our function f:

   $$\iiint_D f(x, y, z)\ dV = \int_{\alpha}^{\beta} \int_{a}^{b} \int_{g(\phi,\theta)}^{h(\phi,\theta)} f(\rho, \phi, \theta)\ \rho^2 \cdot \sin(\phi)\, d\rho \cdot d\phi \cdot d\theta$$

Problems Solved in Jeff's Handwritten Notes

3. Example 13.5.1 p.1008
4. Example 13.5.3 p. 1011 - 1012
5. Example 13.5.4 p. 1012 - 1013
6. Example 13.5.6 p. 1017 - 1018

Suggested Problems: Answers in Book

3. Exercise 13.5.9 p. 1019
4. Exercise 13.5.10 p. 1019
5. Exercise 13.5.21 p. 1019
6. Example 13.5.5 p. 1014
7. Exercise 13.5.39 p. 1020

Optional Challenge Problems

3. Exercise 13.5.75 p. 1022
4. Exercise 13.5.77 p. 1023
5. Exercise 13.5.78 p. 1023
6. Exercise 13.5.79 p. 1023
7. Exercise 13.5.80 p. 1023