Lesson 6: Triple Integrals in Cylindrical & Spherical Coordinates

Recall the main idea behind "triple" integrals is

to create an integral

$$\int_D f \, dw$$

for a three variable, real-valued function

$$f: D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$$

As we saw previously, the coordinate system we use

to encode D makes a large impact on how we

measure sizes of subregions in the domain and also effects

what types of regions can be "easily" encoded.

We begin this lession by introducing the cylindrical coordinate system to encode $D \subseteq \mathbb{R}^3$ with

for a region in cylindrical coordinates, we will choose appropriate subsets I_r, I_θ, I_z to encode our region

$$D = \{ \underbrace{(r, \theta, z)} : r \in I_r, \theta \in I_\theta, z \in I_z \text{ and } I_r, I_\theta, I_z \subseteq \mathbb{R}$$

this coordinate system is an "extension" of polar coordinates where the third coordinate represents height

In this case, notice that the ordered triplet

$$P(r, \theta, z)$$

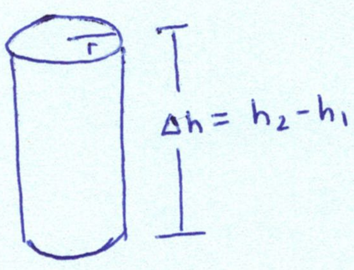
has the first two coordinates r and θ as the polar coordinates for point P^* which is the orthogonal projection of P onto the xy-plane.

Let's create some regions in $\mathbb{R}^3$ using cylindrical coordinates:

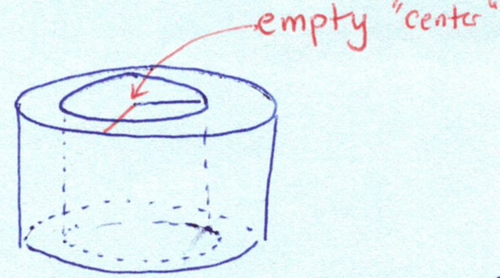
$$\mathbb{R}^3 = \{ (r, \theta, z) : 0 \leq r , \quad 0 \leq \theta < 2\pi, \quad z \in \mathbb{R} \}$$

Cylinder centered at $(0,0)$ with finite height
(just the boundary, not the filling)

$$C = \{ (r, \theta, z) : \quad r = a \in \mathbb{R} \text{ with } a > 0, \quad h_1 \leq z \leq h_2 , h_1, h_2 \in$$



Cylindrical shell with finite height



$$C_s = \{(r,\theta,z) : 0 < a \le r \le b,\ h_1 \le z \le h_2,\ 0 \le \theta < 2\pi\}$$

The "positive" yz-plane through point (0,0,0) with $\vec{n} = \langle 1,0,0 \rangle$

$$P_{yz} = \{(r,\theta,z) : \theta = 0,\ r \in \mathbb{R} \text{ w/ } r \ge 0,\ z \in \mathbb{R}\}$$

The "positive" xz plane through point (0,0,0) with $\vec{n} = \langle 0,1,0 \rangle$

$$P_{xz} = \{(r,\theta,z) : \theta = 90° = \tfrac{\pi}{2}\}$$

The xy-Plane through point (0,0,0) with $\vec{n} = \langle 0,0,1 \rangle$

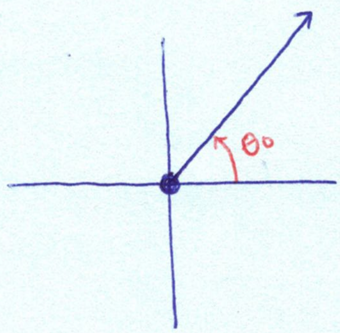
$$P_{xz} = \{(r,\theta,z) : z = 0\}$$

Any Horizontal plane with $\vec{n} = \langle 0,0,1 \rangle$ through point $(0,0,h_1)$

$$P = \{(r, \theta, z) : z = h_1\}$$

Any "Vertical" half plane:

$$P = \{(r, \theta, z) : \theta = \theta_0\}$$



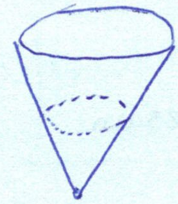
Since $0 \leq r \in \mathbb{R}$

we only get planes

extending in "Half"

the region.

Half cone w/ vertex at $(0,0,0)$



$$C_1 = \{ (r, \theta, z) \ : \ z = z(r) = a \cdot r \quad \text{for } a \in \mathbb{R} \}$$

height as a linear function of radius.

Example 13.5.1 p. 1008

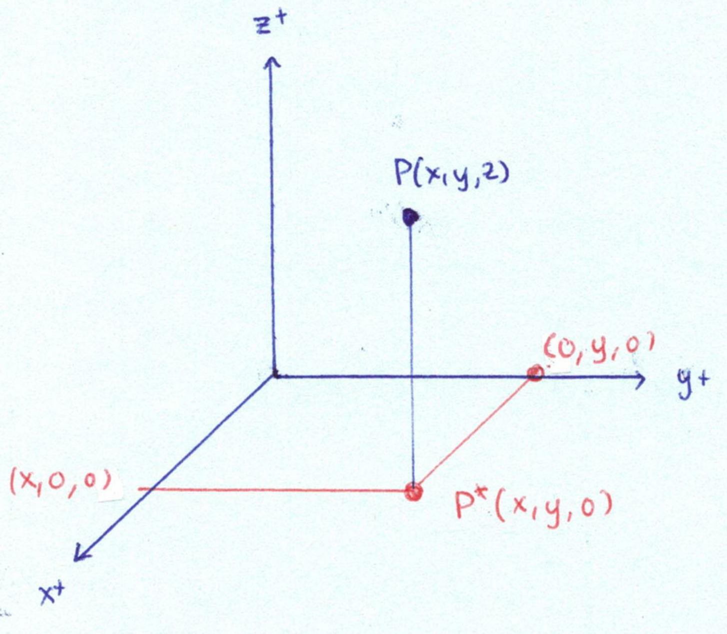
Identify and sketch the following sets encoded in cylindrical coordinats

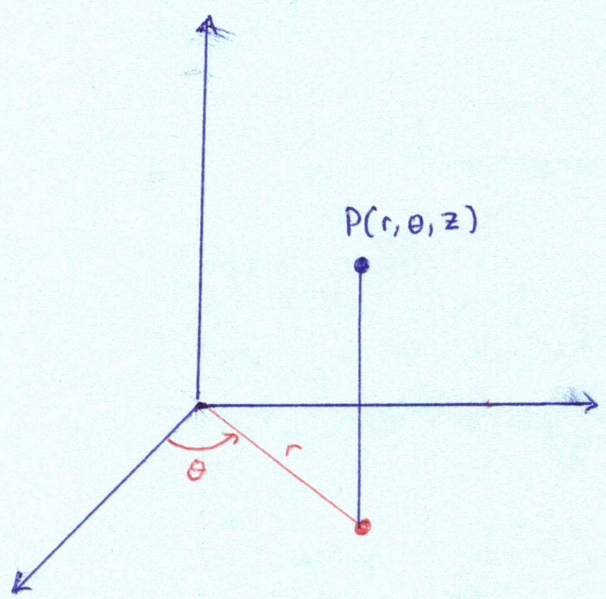
A. $Q = \{ (r, \theta, z) \ : \ 1 \leq r \leq 3 \ \text{ and } \ z \geq 0 \}$

B. $S = \{ (r, \theta, z) \ : \ z = 1 - r \ , \ 0 \leq r \leq 1 \}$

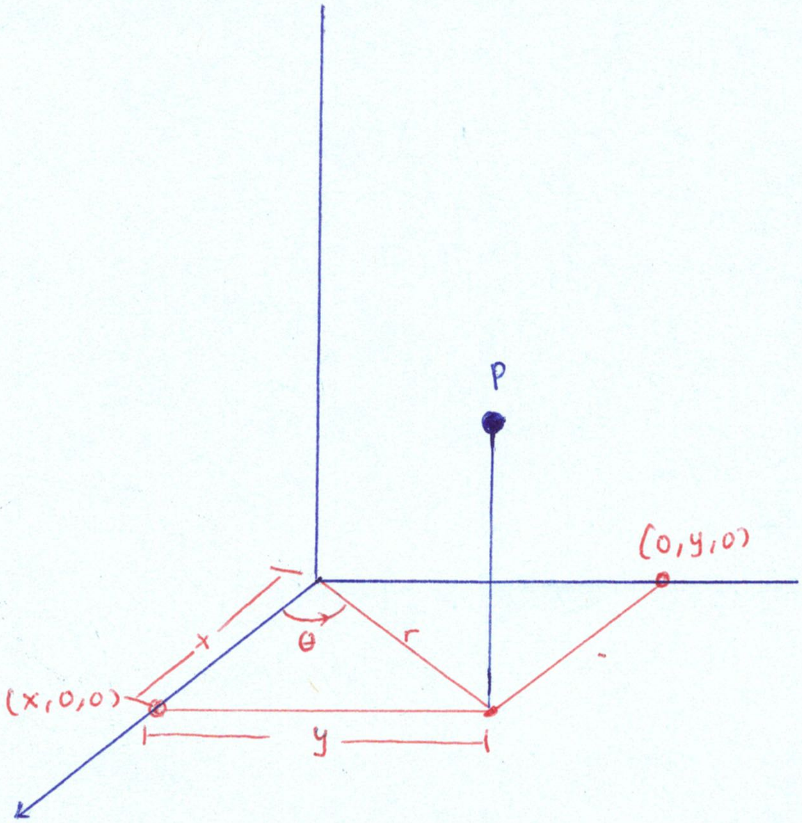
Transformations between Cartesian and Cylindrical Coordinates

Let's consider a point $P(x,y,z) \in \mathbb{R}^3$ encoded in cartesian coordinates and the same point $P(r,\theta,z)$ encoded in cylindrical coordinates:





The fundamental ideas behind tranformations:



$$\cos(\theta) = \frac{x}{r}$$

$$\sin(\theta) = \frac{y}{r}$$

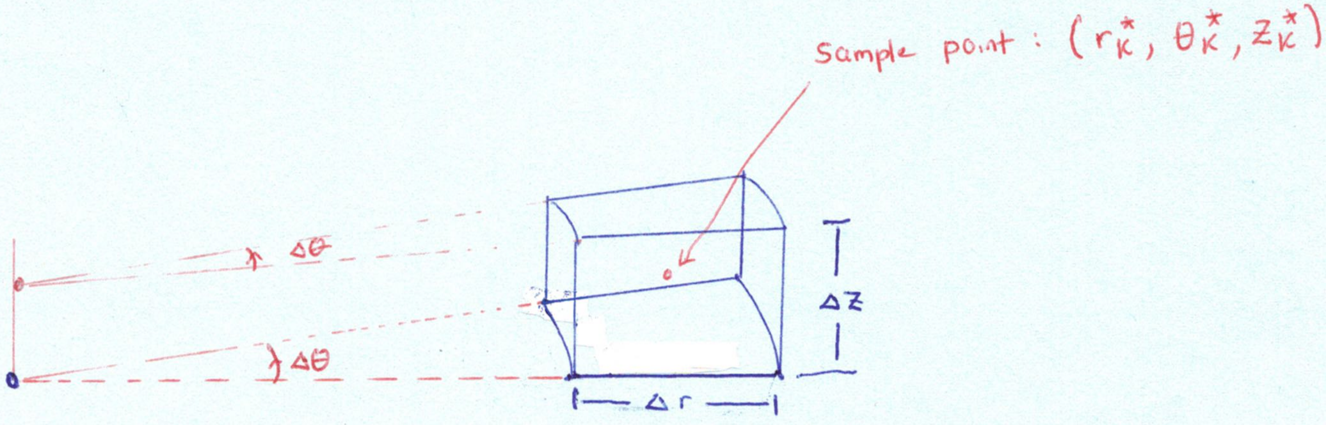
$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x} = \frac{\text{rise}}{\text{run}}$$

Triple Integrals in Cylindrical Coordinates.

Can you guess how to define $\int_D f \, dw$ using a partition of $D \subseteq \mathbb{R}^3$ encoded in cylindrical coordinates?

Let $D \subseteq \mathbb{R}^3$ be a region encoded in cylindrical coordinates. Now, partition D into cylindrical wedges



The "volume" of the Kth wedge is

$$\Delta V_K = \text{Area of base sector} \times \text{height}$$

$$= r_K^* \cdot \Delta r \cdot \Delta\theta \times \Delta z$$

$$= r_K^* \cdot \Delta r \cdot \Delta\theta \; \Delta z$$

where $r_K^* \text{ "="} \dfrac{r_s + r_L}{2}$

$$\Rightarrow \lim_{\Delta \to 0} \Delta V_K = r \cdot dr \; d\theta \; dz$$

Then, we imagine

$$\int_D f \, d\omega = \iiint_D f(r, \theta, z) \, d\omega$$ where $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$= \iiint_D f(r, \theta, z) \, dV$$

a "natural" choice for size measurements in domain region $D \subseteq \mathbb{R}^3$ correspond to volume

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) \cdot \Delta V_k$$

where $\Delta = \max\{\Delta r, \Delta\theta, \Delta z$

Example 13.5.3 p. 1011 - 1012

Find the mass of a solid D bounded above paraboloid $z = 4 - x^2 - y^2$ and below by plane $z=0$, where the density of any point in D is

$$f(x,y,z) = 5 - z$$

Solution: Let us first recall our density relation

$$\text{density} \overset{\text{``}}{=}\text{''} \frac{\text{mass}}{\text{unit volume}} \Rightarrow \text{mass} = \text{``density''} \times \text{unit volume}$$

density function

$$\Rightarrow \text{Mass} = \int_D f \, d\omega$$

size measured in volume

$$= \iiint_D f(x,y,z) \, dV \quad \text{where} \quad f(x,y,z) = 5 - z$$

and $D = \{(x,y,z) : -2 \le x \le 2, \ -\sqrt{4-x^2} \le y \le +\sqrt{4-x^2}, \ 0 \le z \le 4-x^2$

square roots are a headache!

Let's transform into cylindrical coordinates

Upper bound: $z = 4 - x^2 - y^2 = 4 - (x^2 + y^2) = 4 - r^2$

lower bound: $z = 0$

Integrand: $f(r, \theta, z) = 5 - z$

$$\Rightarrow \quad D = \{ (r, \theta, z) : \quad 0 \le r \le 2, \quad 0 \le \theta < 2\pi, \quad 0 \le z \le 4 - r^2 \}$$

$$\Rightarrow \quad \int_D f \, dw = \iiint_D f(r, \theta, z) \, dw$$

$$= \underbrace{\int_0^{2\pi} \underbrace{\int_0^{2} \underbrace{\int_0^{4-r^2} 5 - z \quad dz}_{\text{inner integral}} \cdot r \, dr}_{\text{middle integral}} \cdot d\theta}_{\text{outer integral}}$$

Side note: Inner Integral

$$\int_0^{4-r^2} 5 - z \, dz = 5z - \frac{z^2}{2} \Bigg|_0^{4-r^2}$$

$$= \left[5 \cdot (4 - r^2) - \frac{(4-r^2)^2}{2} \right] - \underbrace{\left[5 \cdot 0 - \frac{0^2}{2} \right]}_{0}$$

$$= 20 - 5r^2 - \frac{1}{2} \cdot (16 - 8r^2 + r^4)$$

$$= 20 - 5r^2 - 8 + 4r^2 - \frac{r^4}{2}$$

$$= 12 - r^2 - \frac{r^4}{2}$$

$$= \frac{1}{2} (24 - 2r^2 - r^4)$$

$$\Rightarrow M = \int_0^{2\pi} \int_0^2 \frac{1}{2}\left(24 - 2r^2 - r^4\right) \cdot r \cdot dr \quad d\theta$$

$$= \frac{1}{2} \boxed{\int_0^{2\pi} \boxed{\int_0^2 24r - 2r^3 - r^5 \; dr}_{\text{"middle"}} \; d\theta}_{\text{"outer"}}$$

see next page for middle calculations

$$= \frac{1}{2} \int_0^{2\pi} \frac{88}{3} \, d\theta$$

$$= \int_0^{2\pi} \frac{44}{3} \, d\theta$$

$$= \frac{44}{3} \theta \Big|_0^{2\pi}$$

$$= \frac{44}{3}(2\pi - 0) = \boxed{\frac{88}{3}\pi} . \checkmark$$

Side note: "Middle" Integral

$$\int_0^2 24r - 2r^3 - r^5 \, dr = 12r^2 - \frac{r^4}{2} - \frac{r^6}{6} \Big|_0^2$$

$$= \left(12 \cdot 2^2 - \frac{2^4}{2} - \frac{2^6}{6}\right) - \left(12 \cdot 0^2 - \frac{0^4}{2} - \frac{0^6}{6}\right)$$

$$= 12 \cdot 4 - 8 - \frac{32}{3}$$

$$= 48 - 8 - \frac{32}{3}$$

$$= 40 - \frac{32}{3}$$

$$= \frac{120 - 32}{3}$$

$$= \frac{88}{3}$$

Example 13.5.4 p. 1012 - 1013

Find the volume of solid D between the cone

$$Z_1 = \sqrt{x^2 + y^2}$$

and the paraboloid

$$Z_2 = 12 - x^2 - y^2$$

Solution: Let's begin by analyzing our region

$$D = \{(x,y,z) : \sqrt{x^2+y^2} \leq z \leq 12 - x^2 - y^2\}$$

This region is given in cartesian coordinates. Let's transform into rectangular coordinates:

$$Z_1 = \sqrt{x^2+y^2} = \sqrt{r^2} = +r \quad \text{since} \quad r \geq 0$$

$$Z_2 = 12 - x^2 - y^2 = 12 - (x^2+y^2) = 12 - r^2$$

Now let's find the intersection between the upper bound

$z = 12 - r^2$ and lower bound $z = r$

$\Rightarrow$ WTF r s.t. $12 - r^2 \geq r$

$\Rightarrow$ $r^2 + r - 12 \geq 0$

$\Rightarrow$ $(r+4) \cdot (r-3) \geq 0$

Now, we know $(r+4) \cdot (r-3) = 0 \Rightarrow r = -4$ or $r = 3$

In cylindrical coordinates we want $r \geq 0 \Rightarrow r = 3$

is our desired upper bound for r

$\Rightarrow$ $0 \leq r \leq 3$ & $0 \leq \theta < 2\pi$

represents region of r and θ

$$\Rightarrow \int_D f \, d\omega = \iiint_D f(r, \theta, z) \, dV$$

$$= \iiint_D 1 \, dV$$

$$= \underbrace{\int_0^{2\pi} \underbrace{\int_0^{3} \underbrace{\int_r^{12-r^2} 1 \, dz}_{\text{inner}} \; r \, dr}_{\text{middle}} \; d\theta}_{\text{outer}}$$

$$= \int_0^{2\pi} \int_0^{3} 12r - r^2 - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{99}{4} \, d\theta$$

$$= \frac{99}{4} \theta \Big|_0^{2\pi} = \frac{99}{4}(2\pi - 0) = \frac{99\pi}{2}$$

Side Note: Inner integral

$$\int_{r}^{12-r^2} 1 \, dz = z \Big|_{r}^{12-r^2}$$

$$= 12 - r^2 - r$$

$$= 12 - r - r^2$$

Side note: Middle integral

$$\int_{0}^{3} 12r - r^2 - r^3 \, dr = 6r^2 - \frac{r^3}{3} - \frac{r^4}{4} \Big|_{0}^{3}$$

$$= \left[6 \cdot 3^2 - \frac{3^3}{3} - \frac{3^4}{4} \right] - \left[6 \cdot 0^2 - \frac{0^3}{3} - \frac{0^4}{4} \right]$$

$$= 54 - 9 - \frac{81}{4} - 0$$

$$= 45 - \frac{81}{4} = \frac{99}{4}$$

Spherical Coordinate Systems

So far, we've seen two methods to encode points $P \in \mathbb{R}^3$ including

$$\underbrace{P(x, y, z)}_{\text{Cartesian coordinates}} = \underbrace{P(r, \theta, z)}_{\text{Spherical coordinates}}$$

We now introduce a third coordinate system in $\mathbb{R}^3$ called spherical coordinates. In spherical coordinates, every point in $\mathbb{R}^3$ is represented as an ordered-triplet

$$P(\rho, \phi, \theta)$$

Let's analyze, in detail, the notation for a point in spherical coordinats

$$P(\rho, \varphi, \theta)$$

1st coordinate: □ the symbol ρ (pronounced "rho" for the greek letter "rho")

represents the "distance" from the point P to the origin $O(0,0,0)$.

□ Think about ρ as a "radius" of a sphere in $\mathbb{R}^3$

□ BE CAREFUL: don't confuse ρ in spherical coordinates with r is cylindrical coordinates. In particular, r is a "radius" in $\mathbb{R}^2$ and encodes "distance" to z-axis.

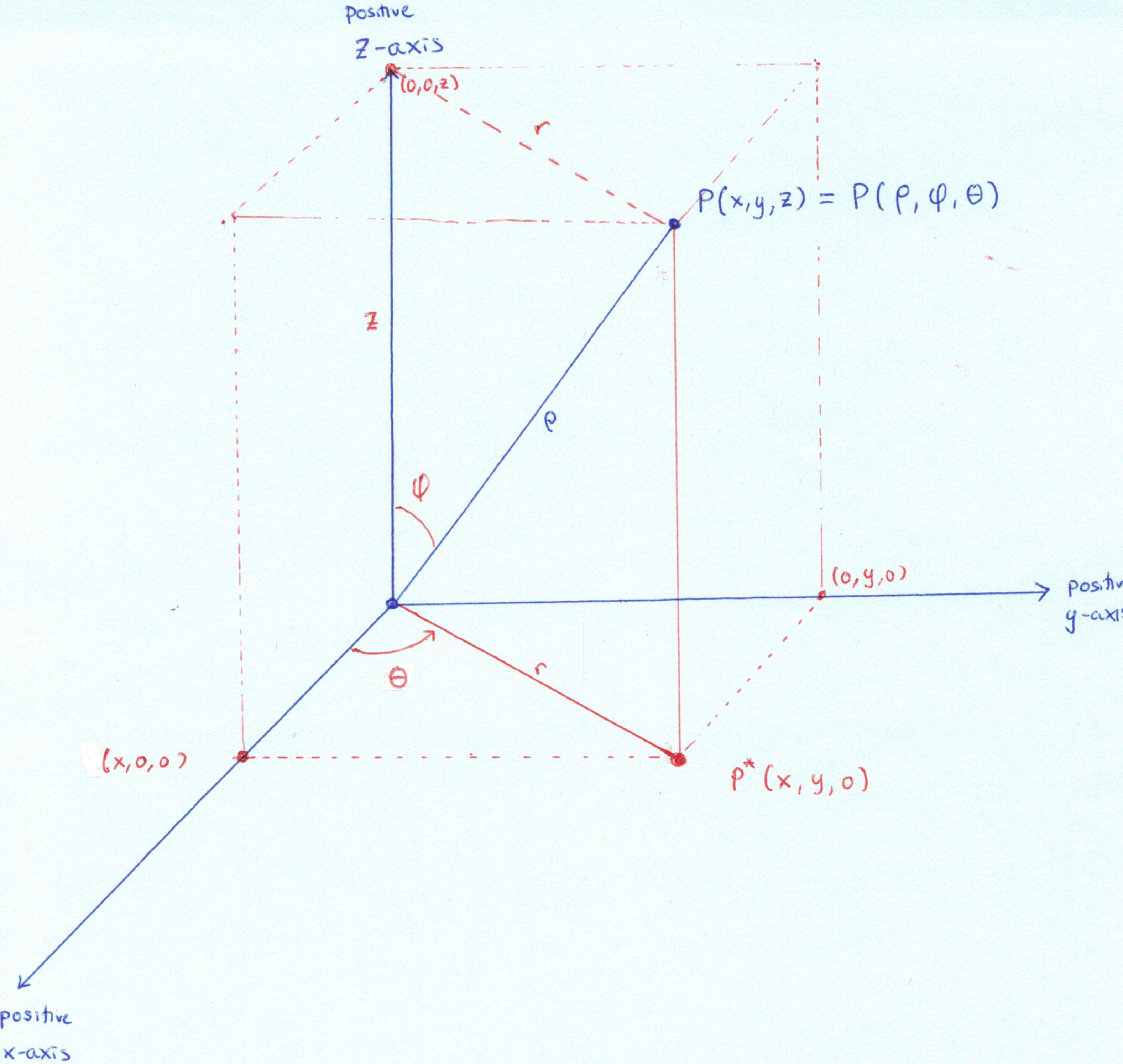
2nd coordinate: □ the scalar φ (prounounced "phi" for the greek letter "phi")

represents the "angle" between the positive z-axis and the line segment OP connecting origin to point P

□ The φ coordinate is sometimes called the colatitude because it is $\pi/2$ minus the "latitude" of points in the northern hemisphere.

3rd coordinate: □ the scalar θ is the same angle as in cylindrical coordinates

□ θ measures rotation about the z-axis relative to the positive x-axis.



$$\mathbb{R}^3 = \{(\rho, \varphi, \theta) : 0 \le \rho,\ 0 \le \varphi < 2\pi,\ 0 \le \theta < 2\pi\}$$

Let's make some useful observations: (How to translate spherical $\rightarrow$ rectangular)

- $$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

- $$\frac{r}{\rho} = \sin(\varphi) \Rightarrow \boxed{r = \rho \sin(\varphi)}$$

- $$\frac{x}{r} = \cos(\theta) \Rightarrow x = r \cdot \cos(\theta)$$

  $$\Rightarrow \boxed{x = \underbrace{\rho \cdot \sin(\varphi)}_{r} \cdot \cos(\theta)}$$

- $$\frac{y}{r} = \sin(\theta) \Rightarrow y = r \cdot \sin(\theta)$$

  $$\Rightarrow \boxed{y = \underbrace{\rho \cdot \sin(\varphi)}_{r} \cdot \sin(\theta)}$$

- $$\frac{z}{\rho} = \cos(\varphi) \Rightarrow \boxed{z = \rho \cos(\varphi)}$$

Transformations : $P(x,y,z) \longrightarrow P(\rho, \varphi, \theta)$

□ $\rho = \rho(x,y,z) = +\sqrt{x^2 + y^2 + z^2}$

□ $\dfrac{y}{x} = \tan(\theta) \Rightarrow \theta = \arctan\left(\dfrac{y}{x}\right)$

□ $\dfrac{z}{\rho} = \cos(\varphi) \Rightarrow \varphi = \arccos\left(\dfrac{z}{\rho}\right)$

$$\Rightarrow \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

OR

□ $\dfrac{r}{z} = \tan(\varphi) \Rightarrow \varphi = \arctan\left(\dfrac{r}{z}\right)$

$$\Rightarrow \varphi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

Let's create some regions $D \subseteq \mathbb{R}^3$ using spherical coordinates

- $D = \mathbb{R}^3 = \{ (\rho, \varphi, \theta) : 0 \leq \rho , \; 0 \leq \varphi < 2\pi , \; 0 \leq \theta < 2\pi \}$

- Sphere with radius $\rho = a$ and center $(0,0,0)$

$$D = \{ (\rho, \varphi, \theta) : \rho = a \}$$

-

Integration in Spherical Coordinates

Let $D \subseteq \mathbb{R}^3$ be a region encoded in spherical coordinates.

(See Mathematica Visual fig 13.58)

Let's partition our region into "spherical boxes" formed by the changes $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$.

Label all spherical boxes that lie within the region $D \subseteq \mathbb{R}^3$ in some systematic way from $k = 1$ to $k = n$.

Also choose sample point $(\rho_k^*, \varphi_k^*, \theta_k^*)$ from the kth subregion.

To construct the Riemann sum approximation to the integral, we set:

$$\int_D f \, d\omega = \iiint_D f(\rho, \varphi, \theta) \, dV \approx \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*) \, \Delta V_k$$

Let's focus on ΔV_k = "size" of kth subregion.

A "natural" choice here is the volume of kth subregion.

However, such a volume is non trivial to calculate exactly.
(extremely hard)

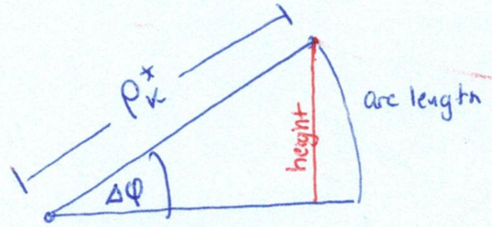
Instead, we will approximate the volume of our kth spherical box subregion by the volume of a related "box" in rectangular coordinates.

$$\underbrace{\Delta V_k}_{\substack{\text{size of } k\text{th} \\ \text{spherical box} \\ \text{subregion}}} \approx \overbrace{\text{width} \cdot \text{length} \cdot \text{height}}^{\text{volume of related rectangular box}}$$

$$\approx \boxed{\Delta\rho} \cdot \underbrace{\boxed{\rho_k^* \cdot \sin(\varphi_k^*) \cdot \Delta\theta}}_{\text{see next page}} \cdot \boxed{\rho_k^* \cdot \Delta\varphi}$$

approximation of height

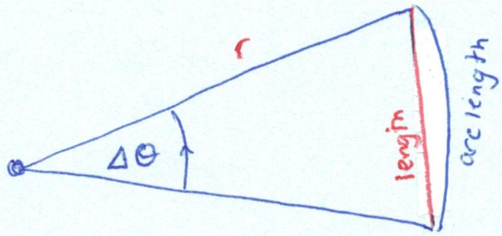
Recall: to find the "height" of an arc subtended by angle Δ with radius ρ_k^*, we can approximate by arclength



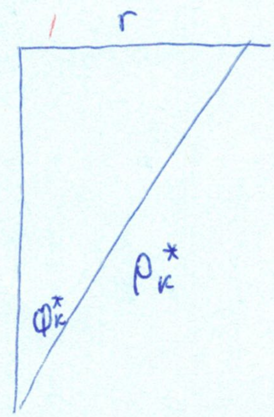
$$\text{arclength} = \rho_k^* \cdot \Delta\varphi$$

$$= \left[\rho_k^*\right]^2 \cdot \sin(\varphi_k^*) \cdot \Delta\rho \cdot \Delta\varphi \cdot \Delta\theta$$

To approximate the "length" of rectangular box, consider the following diagram



with



$$\frac{r}{\rho_k^*} = \sin(\varphi_k^*)$$

$$\Rightarrow r = \rho_k^* \sin(\varphi_k^*$$

$$\text{length} \approx \text{arclength} = r \cdot \Delta\theta$$

$$= \rho_k^* \sin(\varphi_k^*) \cdot \Delta\theta$$

If we set $\Delta = \text{maximum}\{\Delta\rho, \Delta\varphi, \Delta\theta\}$, then

we can define the integral

$$\int_D f\, d\omega = \iiint_D f(\rho, \varphi, \theta)\, d\omega$$

$$= \iiint_D f(\rho, \varphi, \theta)\, dV \leftarrow$$

dV is a function that assign sizes to spherical box subregions

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\, \Delta V_k$$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*) \cdot (\rho_k^*)^2 \cdot \sin(\varphi_k^*)\, \Delta\rho \cdot \Delta\varphi \cdot \Delta\theta$$

$$= \iiint_D f(\rho, \varphi, \theta) \cdot \underbrace{\rho^2 \cdot \sin(\phi)}_{\text{in the limit}} \cdot d\rho\, d\varphi\, d\theta$$

Example 13.5.6 p. 1017 - 1018

Evaluate the integral

$$\int_D f \, dw$$

where $f(x,y,z) = (x^2 + y^2 + z^2)^{-3/2}$ and D is

the region between the spheres of radius $\rho_1 = 1$

and radius $\rho_2 = 2$ contained in the first octant.

Solution: Let's start by analyzing our domain region

$$D = \{ (\rho, \varphi, \theta) : 1 \leq \rho \leq 2, \quad 0 \leq \varphi \leq \pi/2 , \quad 0 \leq \theta \leq \frac{\pi}{2} \}$$

Also, lets translate $f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ into spherical

coordinates :

$$f(\rho, \varphi, \theta) = (\rho^2)^{-3/2} = \rho^{-3} = \frac{1}{\rho^3}$$

Then, to take our integral, we have

$$\int_D f \, dw = \iiint_D f(\rho, \varphi, \theta) \, dw$$

$$= \iiint_D \frac{1}{\rho^3} \, dV$$

$$= \underbrace{\int_0^{\pi/2} \underbrace{\int_0^{\pi/2} \underbrace{\int_1^2 \frac{1}{\rho^3} \cdot \rho^2 \cdot \sin(\varphi) \, d\rho}_{\text{inner integral}} \, d\varphi}_{\text{middle integral}} \, d\theta}_{\text{outer integral}}$$

$$= \int_0^{\pi/2} \underbrace{\int_0^{\pi/2} \ln(2) \cdot \sin(\varphi) \, d\varphi}_{\text{middle integral}} \, d\theta$$

$$= \overbrace{\int_0^{\pi/2} \ln(2) \, d\theta}^{\text{outer integral}} = \ln(2) \cdot \theta \Big|_0^{\pi/2}$$

$$= \ln(2)\left(\frac{\pi}{2} - 0\right) = \frac{\pi \cdot \ln(2)}{2}.$$

Side note: Inner Integral

$$\int_1^2 \frac{1}{\rho^3} \rho^2 \sin(\varphi) \, d\rho = \int_1^2 \frac{\sin(\varphi)}{\rho} \, d\rho$$

$$= \sin(\varphi) \cdot \ln(\rho) \Big|_1^2$$

$$= \sin(\varphi) \cdot \left(\ln(2) - \ln(1) \right)$$

$$= \sin(\varphi) \cdot \ln(2)$$

Side note: Middle Integral

$$\int_0^{\pi/2} \ln(2) \cdot \sin(\varphi)\, d\varphi = \ln(2) \cdot \int_0^{\pi/2} \sin(\varphi)\, d\varphi$$

$$= \ln(2) \quad -\cos(\varphi) \Big|_0^{\pi/2}$$

$$= \ln(2) \cdot -1 \cdot \left(\cos(\pi/2) - \cos(0)\right)$$

$$= \ln(2) \cdot -1\,(0 - 1)$$

$$= +\ln(2)$$