Lesson 5: Triple Integrals in Cartesian Coordinates

When considering integrals of type

integrand

$$\int_D f \; dw$$

differential form

region of integration

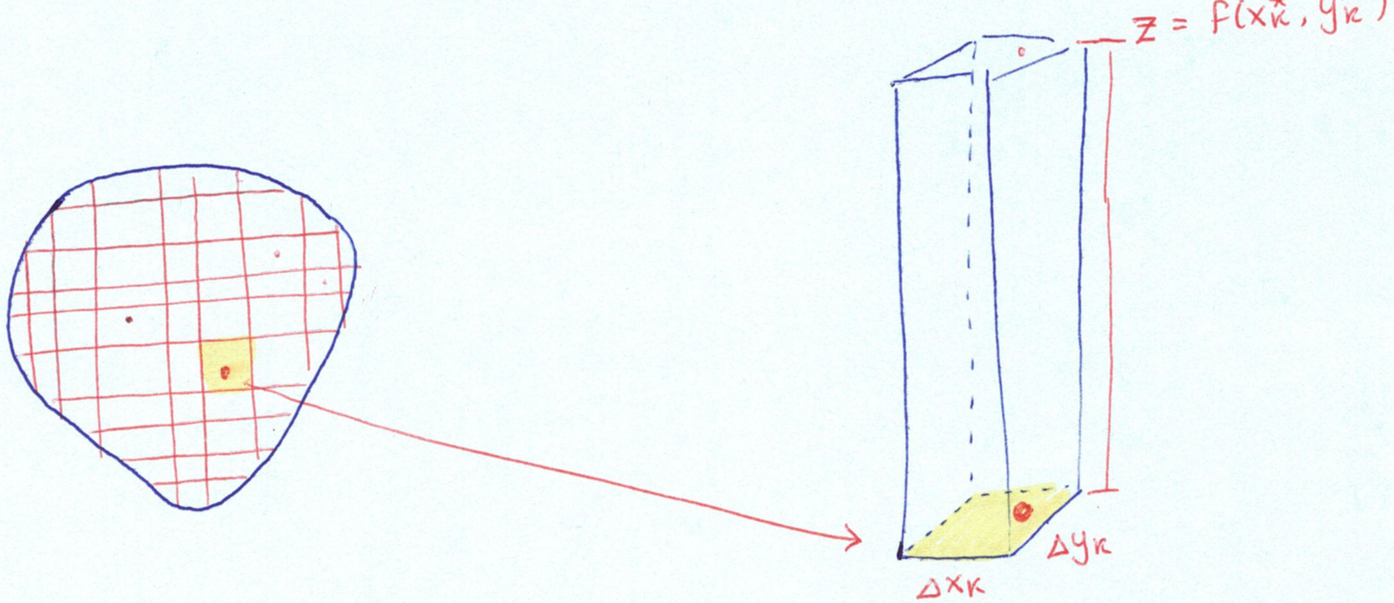
we have focused on two paradigms so far:

Single-variable integration: $\displaystyle\int_D f \, dw = \int_D f(x) \, dx$

$$= \int_a^b f(x) \, dx$$

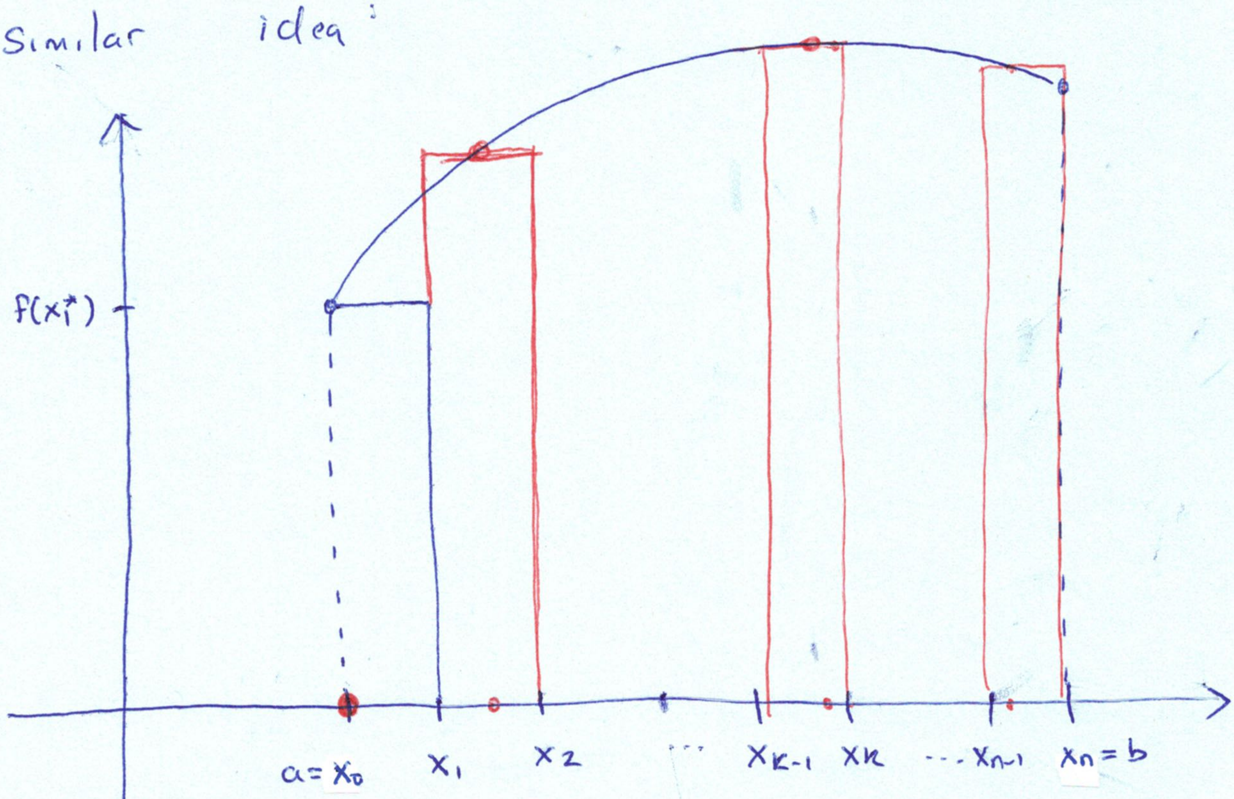
Two-variable integration: $\displaystyle\int_D f \, dw = \iint_D f(x,y) \, dA$

Similarly, when considering the volume bounded between surface $z = f(x,y)$ over domain region $D \subseteq \mathbb{R}^2$, we partitioned D into n subregions, sampled each subregion at a point (x_k^*, y_k^*) and measured height:



We interpreted each sample $f(x_k^*, y_k^*)$ as the "height" of a box above kth rectangular subregion.

In both cases, we "constructed" the integral by relying heavily on a geometric interpretation of Riemann sums as (height) × (size of subinterval). Really both of the derivations we used came from a similar idea:



$$\text{Area} = \int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x_k$$

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

However, the "area-centric" interpretation is not neccssary and in fact gets in the way when trying to "visualize"

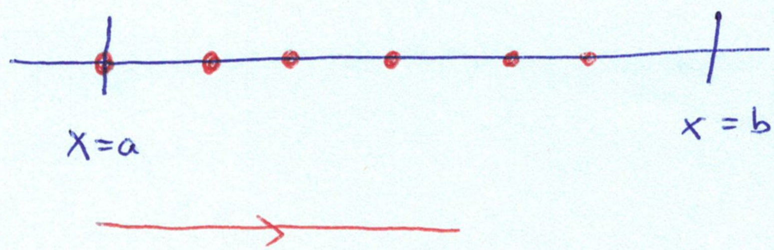
$$\int_D f\, d\omega = \iiint_D f(x,y,z)\, dV$$

Instead, let's approach the problem of integration from a modeling perspective and purposefully ignore geometry

Re-interpret single integral w/out geometry

$$\int_D f \, dw = \int_D f(x) \, dx$$

$$= \int_a^b f(x) \, dx$$



Reinterpretation: think of each $x \in D=[a,b$
as having an assigned weight: $f(x)$
eg: temperature, density, charge, etc.

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \cdot \underbrace{\Delta x_k}_{\text{"size" of Kth subregion}}$$

sampled "weight" function from Kth subregion

Note: previously we thought of the "weight" as a height. This is not necessary!

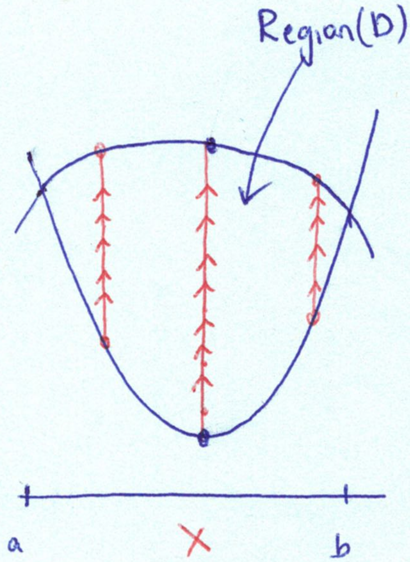
Re-interpret Double Iterated integrals w/out geometry

$$\int_D f\, dw = \iint_D f(x,y)\, dA$$

Not necessarily Area

$$= \int_R F(x)\, dx \quad \text{where} \quad F(x) = \int_{y=g(x)}^{y=h(x)} f(x,y)\, dy$$

and $R = \text{Proj}_{\mathbb{R}}(D)$



$R = [a, b]$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} \underbrace{f(x_k^*, y_k^*)} \cdot \underbrace{\Delta A_k}_{\text{size of the kth subregion}}$$

Sampled weight function from kth subregion

In this reinterpretation, the integral

$$\int_D f \, d\omega$$

represents an "infinite" inner product (dot product) of sampled function "weights" with the corresponding "size" measurements associated with each "point" in the domain region of integration.

From this perspective, the partition of the region D is designed to guarantee that we sample all points in D as $\Delta \to 0$ in the limit. This will also be true for the triple integral

$$\int_D f \, d\omega = \iiint_D f(x,y,z) \, dV$$

Lesson 5: Triple Integrals of Scalar fields

Let $w = f(x,y,z)$ be an explicit function

with

$$f : \underbrace{D}_{\substack{\text{domain region of integration} \\ \text{(three-variable input)}}} \subseteq \mathbb{R}^3 \longrightarrow \underbrace{\mathbb{R}}_{\substack{\text{codomain} \\ \text{(real-valued output)}}}$$

For our derivations, we will assume D is a "closed and bounded" region in $\mathbb{R}^3$.

To construct the triple integral

$$\int_D f \, dw = \iiint_D f(x,y,z) \, dV$$

we will move away from interpreting $f(x_k^*, y_k^*, z_k^*)$ as a "height" and instead think of this as a "weight".

more general

We form a general partition of $D \subseteq \mathbb{R}^3$ by "slicing" this region with a collection of planes.

In particular, we use sets of planes in three "directions"

Direction 1: $\vec{n} = \alpha \cdot \langle 0, 1, 0 \rangle$: planes that are in the same direction as the xz-plane

Direction 2: $\vec{n} = \alpha \cdot \langle 1, 0, 0 \rangle$: planes that are "in the same direction" as the yz-plane

Direction 3: $\vec{n} = \alpha \cdot \langle 0, 0, 1 \rangle$: planes that are "in the same direction" as the xz plane

This partition subdivides D into small "boxes".

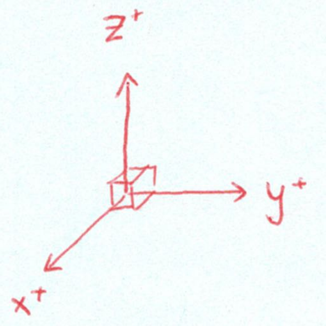
We can then choose all the boxes from the partition that are wholly contained in the region $D \subseteq \mathbb{R}^3$ and enumerate these boxes from $K=1$ to $K=n$ where $n \in \mathbb{N}$.

To assign these index variables, we need only to create a systemic process to assign each box a unique index value $K = 1, 2, \dots, n$.

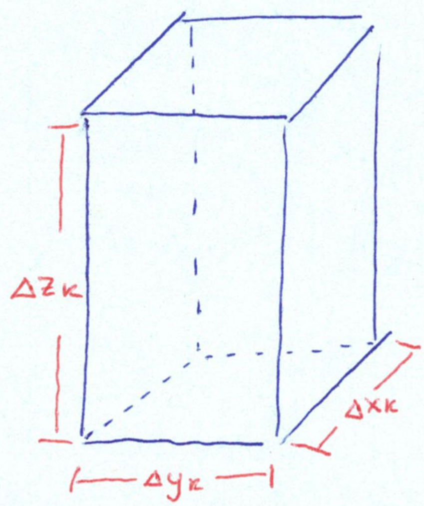
Then, the kth box has side "lengths"

given by Δx_k, Δy_k, Δz_k

Frame of Reference:



Recall the right-hand rule from Math IC



Then, we can impose a "natural" size measurement for the volume ΔV_k of this box with

$$\Delta V_k = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$$

To sample the output $w = f(x, y, z)$ on the kth box, we choose a sample point (x_k^*, y_k^*, z_k^*) from the kth box for $k = 1, 2, \ldots, n$.

Then, our general Riemann sum to approximate the "triple" integral is given by

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \cdot \Delta V_k$$

If we set $d_k = \sqrt[2]{\Delta x_k^2 + \Delta y_k^2 + \Delta z_k}$

and let $\Delta = \max\{d_1, d_2, \cdots, d_n\}$

The value of Δ represents the maximum length of the diagonals of all the boxes

then $\int_D f \, dw = \iiint_D f(x,y,z)\, dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta V_k$

(as $\Delta \to 0$, we must have that $n \to 0$)

Example 13.4.1 p. 997

A solid box $D \subseteq \mathbb{R}^3$ is bounded by the planes

$$x = 0 \qquad x = 3$$

$$y = 0 \qquad y = 2$$

$$z = 0 \qquad z = 1$$

The "density" of the box decreases linearly in the positive z direction and is modeled as

$$f(x, y, z) = 2 - z$$

Find the mass of the box.

Note 1: density $= \dfrac{\text{mass}}{\text{unit volume}}$

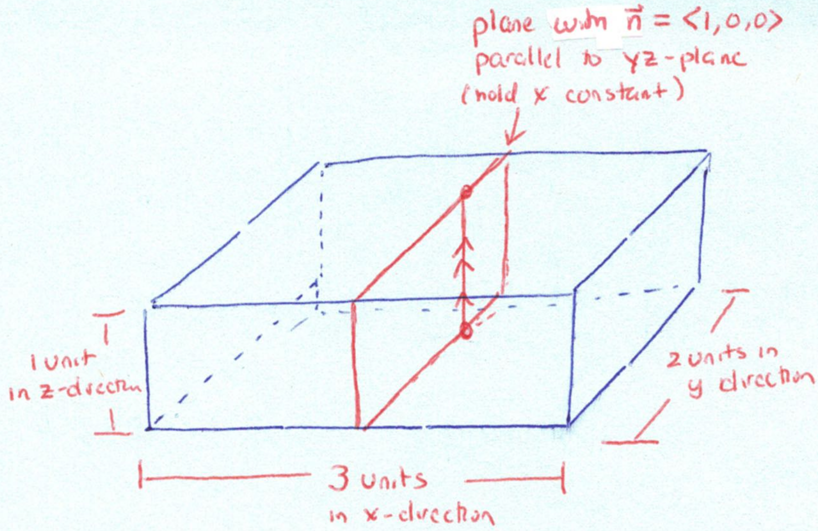
Note 2: mass = density · Volume (think of mass as the sum of each "piece" of dense material)

Solution:

To find the mass of the box, we will "add up" the total density of each "molecule" at positions (x, y, z) in the box.

Example 13.4.1 ...

In other words



$$M = \int_D f \, d\omega$$

$$= \iiint_D f(x,y,z) \, dV$$

$$= \int_0^3 \int_0^2 \int_0^1 2 - z \; dz \; dy \; dx$$

Side note 1: Integral w/r to z

$$\int_0^1 2 - z \, dz = 2z - \frac{z^2}{2} \Big|_0^1$$

$$= \left(2 - \frac{1}{2}\right) - 0$$

$$= \frac{3}{2}$$

$$= \int_0^3 \int_0^2 \frac{3}{2} \; dy \; dx$$

Example 13.4.1 ...

Let's try a different order

$$M = \int_0^2 \int_0^1 \int_0^3 2-z \; dx \; dz \; dy$$

$$= \int_0^2 \int_0^1 6-3z \; dz \; dy$$

$$= \int_0^2 \frac{9}{2} \; dy$$

$$= \frac{9}{2}y \Big|_0^2$$

$$\boxed{= 9.}$$ ✓

Side note 1:

$$\int_0^3 2-z\,dx = 2x - xz \Big|_0^3$$

$$= (2-z)(3-0)$$

$$= 6-3z$$

Side note 2:

$$\int_0^1 6-3z\,dz = 6z - \frac{3z^2}{2} \Big|_0^1$$

$$= \left(6 - \frac{3}{2}\right) - 0$$

$$= \frac{9}{2}$$

Example 13.4.1 ...

Side note 2: Integral w/r to y

$$\int_0^2 \frac{3}{2}\,dy = \frac{3}{2}y \Big|_0^2$$

$$= 3 - 0$$

$$= 3$$

$$\Rightarrow M = \int_0^3 3\,dx$$

$$= 3x \Big|_0^3$$

$$= 9 - 0 = 9$$

Interpretation: The density of the box varied linearly from top to botton with $f(x,y,z) = 2 - z$

and $0 \le z \le 1$.

Example 13.4.1 ...

Table 13.3: Triple integral and order of integration

middle integral

$$\int f \, dw = \int_0^3 \int_0^2 \int_0^1 2-z \, dz \, dy \, dx$$

inner integral

outer integral

Integral location	variable	Interval
Inner	z	$0 \le z \le 1$
middle	y	$0 \le y \le 2$
outer	x	$0 \le x \le 3$

In this problem, we have it as nice as we can get it ... Region is a pure box: we can switch order of integration quite easily

Finding limits of integration:

Suppose that $D \subseteq \mathbb{R}^3$ is a region bounded below by surface $z_1 = G(x,y)$ and above by surface $z_2 = H(x,y)$.

Then, these surfaces determine the limits of integration in the z-direction with

$$G(x,y) \leq z \leq H(x,y)$$

Example 13.4.2 p. 997 - 998

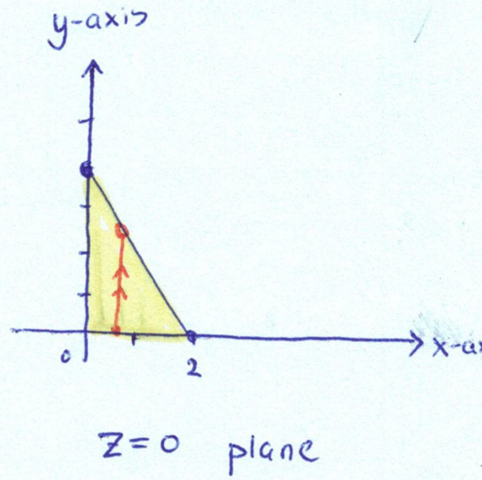
Find the volume of the prism $D \subseteq \mathbb{R}^3$ in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ bounded by planes

plane 1: $y = 4 - 2x$

plane 2: $z = 6$

Solution: we begin by analyzing region $D \subseteq \mathbb{R}^3$

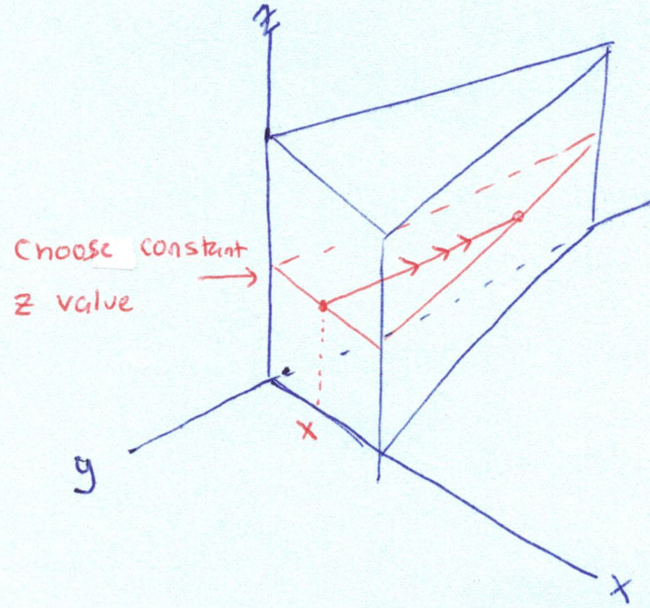
$$D = \{(x,y,z) : 0 \leq x, \quad 0 \leq y \leq 4-2x, \quad 0 \leq z \leq 6\}$$



$z = 0$ plane

$$\Rightarrow D = \{(x,y,z) : 0 \leq x \leq 2, \quad 0 \leq y \leq 4-2x, \quad 0 \leq z \leq 6\}$$

$$\Rightarrow \quad V = \int_D f \, d\omega = \iiint_D f(x,y,z) \, dV$$



$$= \boxed{\int_0^6 \boxed{\int_0^2 \boxed{\int_0^{4-2x} 1 \, dy} \, dx} \, dz}$$

Example 13.4.3 p. 998 – 999

Set up triple integral used to

Find the volume of the region $D \subseteq \mathbb{R}^3$

bounded by the paraboloids

$$y_1 = x^2 + z^2 \qquad \text{and} \qquad y_2 = 16 - 3x^2 - z^2$$

Solution: To find our desired volume, we will set

the weight function

$$f(x,y,z) = 1$$

and calculate $\displaystyle\int_D f \, dw = \iiint_D 1 \, dV = V$

We begin our calculation by analyzing the

domain region $D \subseteq \mathbb{R}^3$ in detail.

Example 13.4.3 ...

We notice $D = \{ (x,y,z) : x^2 + z^2 \leq y \leq 16 - 3x^2 - z^2 \}$

The restrictions on y have implicit bounds on x and z.

Let's figure out exactly what these bounds are! We begin

by finding the curve at the intersection of our two paraboloids

$$x^2 + z^2 = 16 - 3x^2 - z^2$$

$$\Rightarrow 4x^2 + 2z^2 = 16$$

Note: Get bounds on z w/r to x

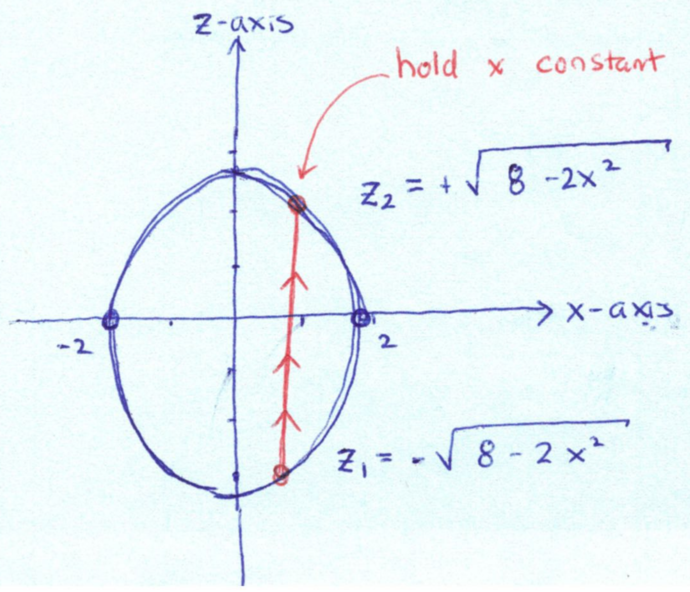
$\Rightarrow z^2 = 8 - 2x^2$

$\Rightarrow z = \pm\sqrt{8 - 2x^2}$

$$\Rightarrow \frac{x^2}{4} + \frac{z^2}{8} = 1$$

□ ellipse in xz variables with x-semiaxis length of 2 and z-semiaxis length of $2\sqrt{2}$

$$\Rightarrow \frac{x^2}{2^2} + \frac{z^2}{(2\sqrt{2})^2} = 1$$



$\Rightarrow \; D = \{(x,y,z) : -2 \le x \le 2, \; -\sqrt{8-2x^2} \le z \le \sqrt{8-2x^2}, \; x^2+z^2 \le y \le 16-3x^2-z^2\}$

$\Rightarrow \; V = \int_D f \, d\omega = \iiint_D 1 \, dV$

middle integral

$$= \int_{-2}^{2} \int_{-\sqrt{8-2x^2}}^{+\sqrt{8-2x^2}} \int_{x^2+z^2}^{16-3x^2-z^2} 1 \; dy \; dz \; dx$$

inner integral

outer integral

Side note 1: Inner integral

$$\int_{x^2+z^2}^{16-3x^2-z^2} 1 \; dy = y \Big|_{x^2+z^2}^{16-3x^2-z^2}$$

$$= 16 - 3x^2 - z^2 - x^2 - z^2$$

$$= 16 - 4x^2 - 2z^2$$

$$= \int_{-2}^{2} \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} 16 - 4x^2 - 2z^2 \, dz \, dx$$

Side note 2: Middle Integral

$$\int_{-\sqrt{8-2x^2}}^{+\sqrt{8-2x^2}} 16 - 4x^2 - 2z^2 \, dz = (16-4x^2)z - \frac{2z^3}{3} \Bigg|_{-\sqrt{8-2}}^{\sqrt{8-2}}$$

$$= (16-4x^2) \cdot 2\sqrt{8-2x^2} - \frac{4}{3}\left(\sqrt[2]{8-2x^2}\right)^3$$

Hmm... beastly integral.

Example 13.4.3 ...

Integral location	Variable	Interval
inner integral	y	$x^2 + z^2 \le y \le 16 - 3x^2 - y^2$
middle integral	z	$-\sqrt{8-2x^2} \le z \le +\sqrt{8-2x^2}$
outer integral	x	$-2 \le x \le 2$

$$\Rightarrow \int_D f \, dw = \iiint_D f(x,y,z) \, dV$$

$$= \int_{x=a}^{x=b} \int_{z=g(x)}^{z=h(x)} \int_{y=G(x,z)}^{y=H(x,z)} f(x,y,z) \, dy \, dz \, dx$$