Lesson 4: Double Integrals in Polar Coordinates

In lessons 1 and 2, we focused on developing theory to integrate $z = f(x,y)$ where $f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and $D \subseteq \mathbb{R}^2$ was a region encoded using rectangular coordinates. In this case, we had

$$\int_D f \, d\omega = \iint_D f(x,y)\, dA$$

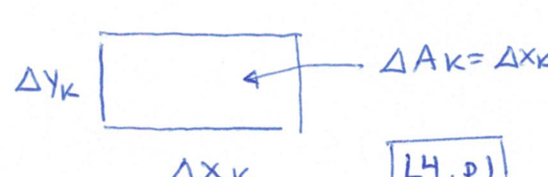
$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta x_k \, \Delta y_k$$

Fubini's Th

In this case, we created the differential form $dA = dx\,dy \overset{\downarrow}{=} dy\,d$

using the geometry of the subregion:



However, In some instances, we may try to integrate a function $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ in which the domain region of integration $D \subseteq \mathbb{R}^2$ is more convienently encoded using polar coordinates.

A great example of such a region is a polar rectangle

$$D = \{(r, \theta) : 0 \leq a \leq r \leq b \text{ and } 0 \leq \alpha \leq \theta \leq \beta < 2\pi\}$$

Polar rectangles are analogous to cartesian rectangles except that all coordinates in polar rectangles are stored as pairs (r, θ).

One concrete example that motivates this study is on the next page.

Find the volume of the solid bounded by the paraboloid

$$z = f(x,y) = 9 - x^2 - y^2$$

and the xy-plane given by equation $z=0$

To find the region of integration, let's consider the xy-trace

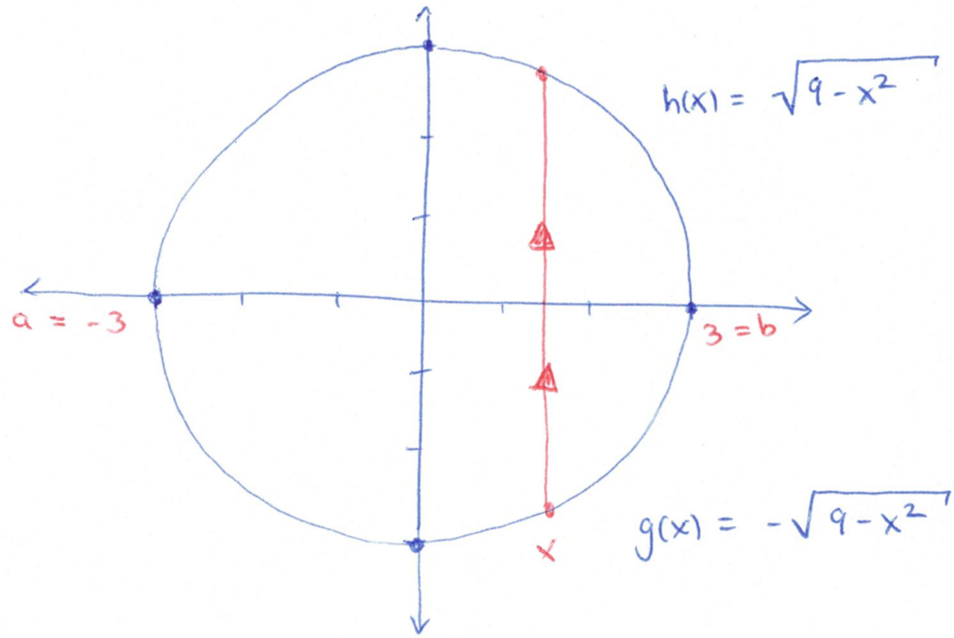
$$z = 0 = 9 - x^2 - y^2$$

$$\Rightarrow \quad x^2 + y^2 = 9$$

$$\Rightarrow \quad D = \{(x,y) : x^2 + y^2 \le 9\}$$

If we are going to convert this disk into a y-simple region, we need to find functions $y_1 = g(x)$ and $h(x) = y_2$ that represent the upper- and lower-bounds of region.

To this end, consider the diagram:



$$\Rightarrow \quad \int f \, d\omega = \iint_D f(x,y) \, dA = \int_a^b A(x) \, dx \quad \text{where} \quad A(x) = \int_{g(x)}^{h(x)} f(x,y) \, dy$$

$$= \int_a^b \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \, dy \, dx$$

This integral is hideous
(try it for yourself)

However, this circular region $D \subseteq \mathbb{R}^2$ can be expressed much more simply using polar coordinates

$$D = \{ (x,y) : x^2 + y^2 \leq 9 \}$$

$$= \{ (x,y) : y^2 \leq 9 - x^2 \}$$

$$= \{ (x,y) : \sqrt{y^2} \leq \sqrt{9-x^2} \}$$

$$= \{ (x,y) : |y| \leq \sqrt{9-x^2} \}$$

$$= \{ (x,y) : -\sqrt{9-x^2} \leq |y| \leq \sqrt{9-x^2} \}$$

$$= \{ (r,\theta) : 0 \leq r \leq 3 \text{ and } 0 \leq \theta < 2\pi \}$$

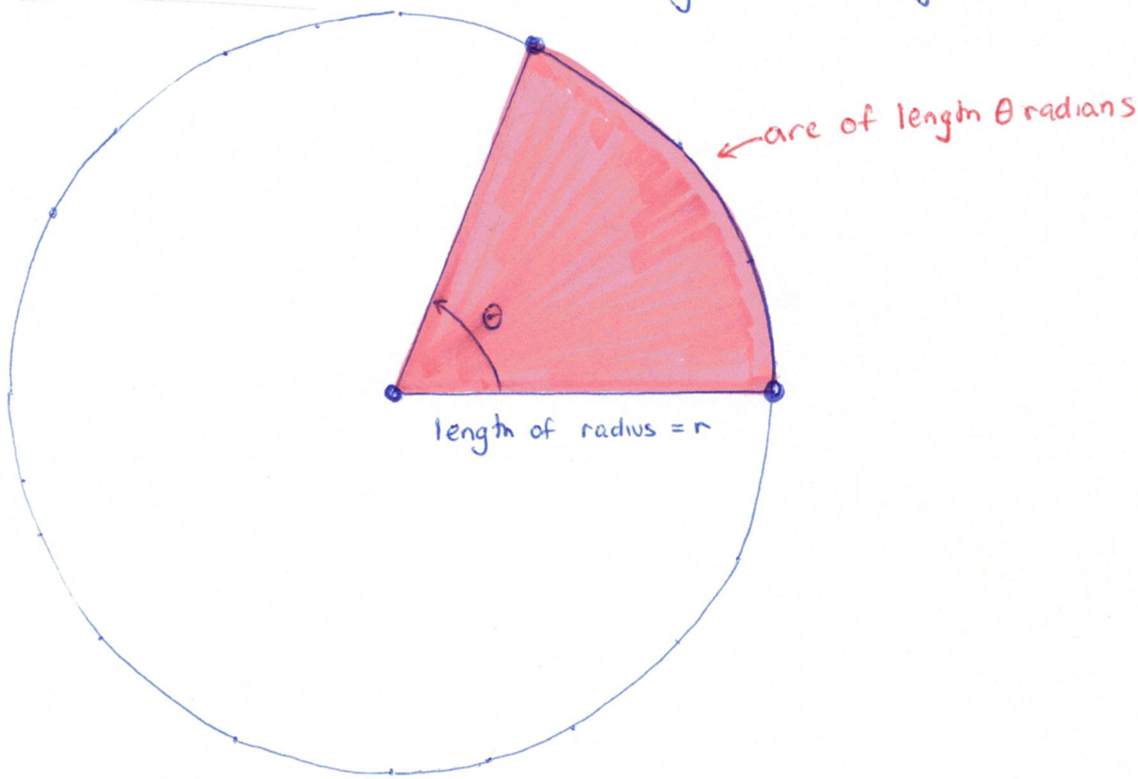
We will call this a polar rectangle. Let's see if we can define a double integral on such a region.

$\boxed{\text{L4,p6}}$

Recall: The area of a circle with radius r is "given" by equation

$$\text{area} = \pi \cdot r^2$$

A circular sector is the portion of a circle with radius r "subtended" by an angle θ



$$\frac{\text{Area of circular sector}}{\text{Area of circle}} = \frac{A}{\pi \cdot r^2} = \frac{\text{angle } \theta \text{ in radians}}{\text{total angle that defines circle}}$$

$$\Longrightarrow \quad \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

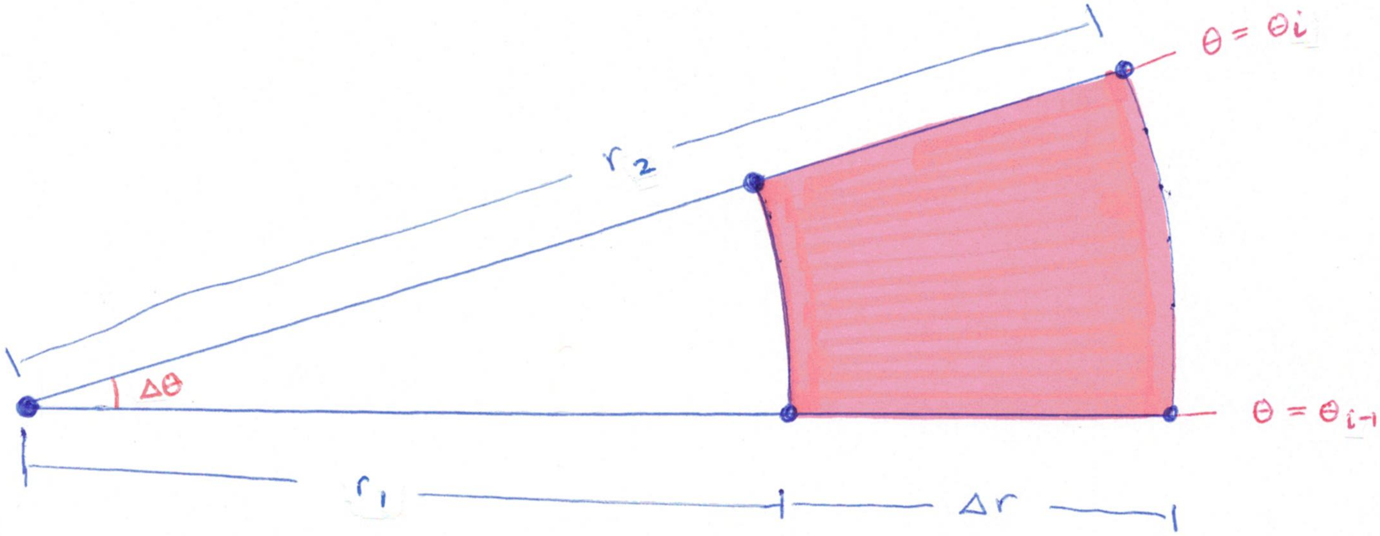
$$\Longrightarrow \quad A = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$= \frac{1}{2} \cdot \frac{\pi}{\pi} \cdot \frac{r^2}{1} \cdot \theta$$

$$= \frac{1}{2} \cdot r^2 \cdot \theta$$

$$= \frac{\theta \cdot r^2}{2}$$

The area of polar-rectangular subregion:



In this case the differential form dA measures the area of the polar-rectangular subregion of our partition. We approximate this area as

$$\Delta A = \text{total area of given sector} - \text{area of yellow sector}$$

$$= \frac{1}{2}\Delta\theta \cdot r_2^2 - \frac{1}{2}\Delta\theta \cdot r_1^2$$

$$= \frac{1}{2}\Delta\theta\,(r_2^2 - r_1^2)$$

$$\Rightarrow \Delta A = \frac{1}{2} \Delta\theta \, (r_2^2 - r_1^2)$$

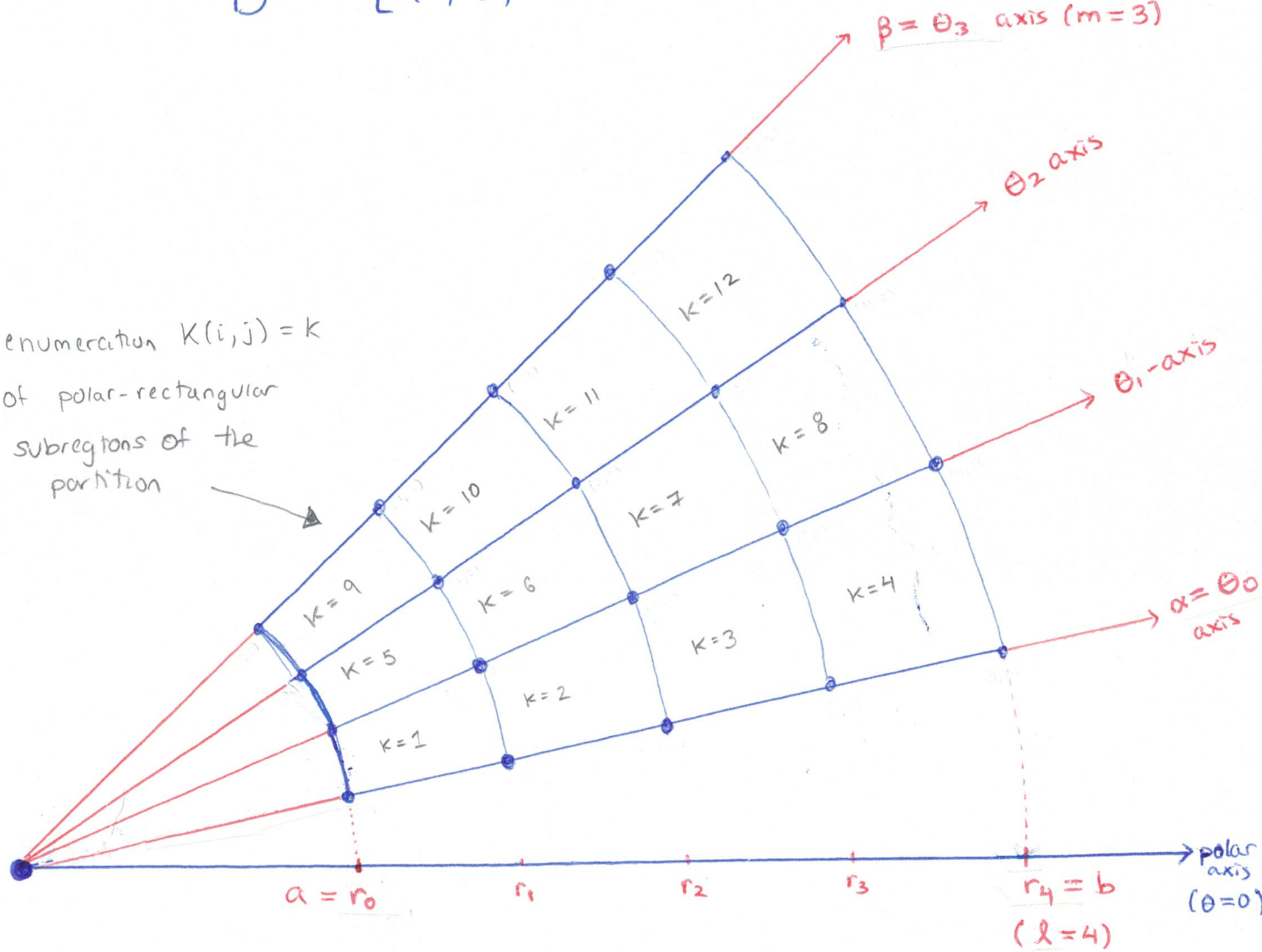
$$\Rightarrow \Delta A = \frac{1}{2} \Delta\theta \, \underbrace{(r_2 - r_1)}_{\Delta r} \cdot (r_2 + r_1)$$

$$\Rightarrow \Delta A = \frac{r_2 + r_1}{2} \cdot \Delta\theta \cdot \Delta r$$

$$\Rightarrow \Delta A = r_K^* \cdot \Delta r \cdot \Delta\theta \qquad \text{where} \quad r_K^* = \frac{r_1 + r_2}{2}$$

Let $D \subseteq \mathbb{R}^2$ be a polar rectangle with

$$D = \{(r, \theta) : 0 \le a \le r \le b \;\&\; 0 \le \alpha \le \theta \le \beta < 2\pi\}$$



We with start with a uniform discretization of both intervals $r \in [a, b]$ and $\theta \in [\alpha, \beta]$.

Let's start with radius: Cut radial interval into ℓ pieces w/ $\ell \in \mathbb{N}$

- Set lower endpoint closer to the pole: $a = r_0$
- Set upper endpoint further from te pole: $b = r_\ell$
- Set the length of each subinterval: $\Delta r_k = \frac{b-a}{\ell} = \underbrace{\Delta r}_{\text{uniform}}$
- Enumerate the endpoints on each subregions w/r to radius partition

$$a = r_0 < r_1 < r_2 < r_3 < \cdots < r_{\ell-1} < r_\ell = b$$

where

$$r_1 = r_0 + \Delta r_1 = r_0 + \Delta r$$

$$r_2 = r_1 + \Delta r_2 = r_1 + \Delta r = r_0 + \Delta r + \Delta r$$

$$= r_0 + 2 \cdot \Delta r$$

$$r_3 = r_2 + \Delta r_3 = r_2 + \Delta r = r_0 + 3 \cdot \Delta r$$

$$\vdots$$

$$r_j = r_{j-1} + \Delta r_j = r_{j-1} + \Delta r = r_0 + j \cdot \Delta r$$

$$\vdots$$

$$r_\ell = r_0 + \Delta r_\ell = r_{\ell-1} + \Delta r = r_0 + \ell \cdot \Delta r = b$$

Let's continue with Angle cut angular interval into m pieces w/ $m \in \mathbb{N}$

- ☐ Set lower endpoint with smaller angle: $\alpha = \theta_0$
- ☐ Set upper end point with larger angle: $\beta = \theta_m$
- ☐ Set length of each subinterval : $\Delta\theta_K = \frac{\beta - \alpha}{m} = \Delta\theta$
- ☐ Enumerate endpoint of each subregion w/r to angular partition

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_m = b$$

where

$$\theta_1 = \theta_0 + \Delta\theta_1 = \theta_0 + \Delta\theta$$

$$\theta_2 = \theta_1 + \Delta\theta_2 = \theta_1 + \Delta\theta = \theta_0 + 2 \cdot \Delta\theta$$

$$\theta_3 = \theta_2 + \Delta\theta_3 = \theta_2 + \Delta\theta = \theta_0 + 3\Delta\theta$$

$$\vdots$$

$$\theta_i = \theta_{i-1} + \Delta\theta_i = \theta_{i-1} + \Delta\theta = \theta_{i-1} + i \cdot \Delta\theta$$

$$\vdots$$

$$\theta_m = \theta_{m-1} + \Delta\theta_m = \theta_{m-1} + \Delta\theta = \theta_{m-1} + m \cdot \Delta\theta = \beta$$

Now that we've partitioned both the radial interval $r \in [a,b]$ into l pieces and the angular interval $\theta \in [\alpha, \beta]$ into m pieces, we will enumerate the $n = l \cdot m$ polar-rectangular subregions

Let's suppose that we enumerate our radial subintervals

$$[a,b] = \underbrace{[r_0, r_1]}_{\text{1st subinterval}} \cup \overbrace{[r_1, r_2]}^{\text{2nd subinterval}} \cup \cdots \cup \underbrace{[r_{j-1}, r_j]}_{\text{jth subinterval}} \cup \cdots \cup \overbrace{[r_{\ell-1}, r_\ell]}^{\ell\text{th subinterval}}$$

$\Rightarrow$ the jth radial subiterval is $[r_{j-1}, r_j]$

for $1 \le j \le \ell$ (or $j = 1, 2, 3, \ldots, \ell$)

Now we can also enumerate our angular subintervals

$$[\alpha, \beta] = \underbrace{[\theta_0, \theta_1]}_{\text{1st subinterval}} \cup \overbrace{[\theta_1, \theta_2]}^{\text{2nd subinterval}} \cup \cdots \cup \underbrace{[\theta_{i-1}, \theta_i]}_{\text{ith subinterval}} \cup \cdots \cup \overbrace{[\theta_{m-1}, \theta_m]}^{\text{mth subinterval}}$$

$\Rightarrow$ the ith subinterval is $[\theta_{i-1}, \theta_i]$

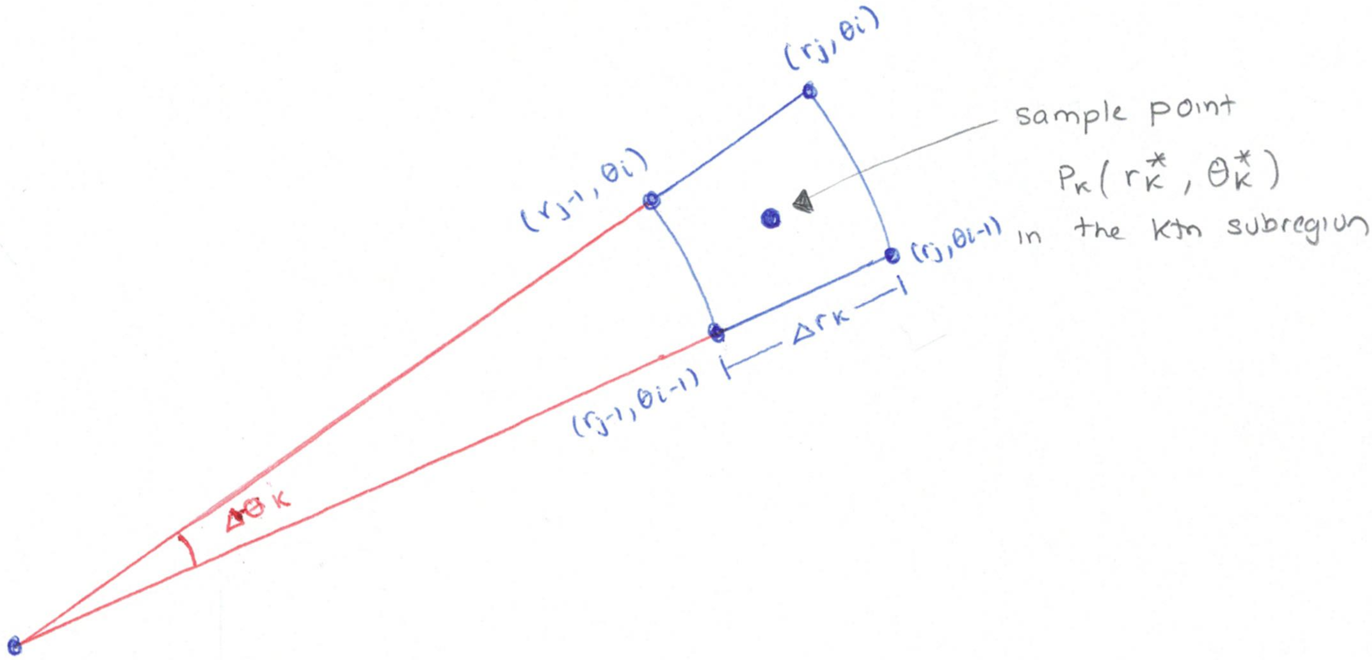
for $1 \le i \le m$ (or $i = 1, 2, \ldots, m$)

In our diagram we have agreed upon the following enumeraton rule to enurmate subregions : $\boxed{K = (i-1)\cdot \ell + j = K(i,j)}$

We can deduce this rule from the following table

index j of radial region for $1 \le j \le \ell$	index i of angular subregion for $1 \le i \le m$	index K of subregion for $1 \le K \le n$
$j=1 \Leftrightarrow [r_0, r_1]$	$i=1 \Leftrightarrow [\theta_0, \theta_1]$	$K=1 = 0\cdot 4 + 1$
$j=2 \Leftrightarrow [r_1, r_2]$	$i=1 \Leftrightarrow [\theta_0, \theta_1]$	$K=2 = 0\cdot 4 + 2$
$j=3 \Leftrightarrow [r_2, r_3]$	$i=1 \Leftrightarrow [\theta_0, \theta_1]$	$K=3 = 0\cdot 4 + 3$
$j=4 \Leftrightarrow [r_3, r_4]$	$i=1 \Leftrightarrow [\theta_0, \theta_1]$	$K=4 = 0\cdot 4 + 4$
$j=1 \Leftrightarrow [r_0, r_1]$	$i=2 \Leftrightarrow [\theta_1, \theta_2]$	$K=5 = 1\cdot 4 + 1$
$j=2 \Leftrightarrow [r_1, r_2]$	$i=2 \Leftrightarrow [\theta_1, \theta_2]$	$K=6 = 1\cdot 4 + 2$

Lets consider the kth polar-rectangular subregion for $1 \le k \le n$ where $n = l \cdot m$. We can visualize this as follows:



We will use the "midpoint" rule to choose the sample point (r_k^*, θ_k^*) inside the subregion:

$$\text{let} \quad r_k^* = r_{j-1} + \frac{r_j - r_{j-1}}{2}$$

$$= \frac{r_{j-1} + r_j}{2}$$

$$= r_{j-1} + \frac{\Delta r}{2}$$

$$\Rightarrow r_j = r_{j-1} + \Delta r = \underbrace{r_{j-1} + \frac{\Delta r}{2}}_{r_k^*} + \frac{\Delta r}{2} = r_k^* + \frac{\Delta r}{2}$$

Similarly, we have

$$r_{j-1} + \frac{\Delta r}{2} = r_k^* \Rightarrow r_{j-1} = r_k^* - \frac{\Delta r}{2}$$

Then, the area of the polar-rectangular subregion is

$$\Delta A_k = \frac{1}{2} \cdot r_j^2 \cdot \Delta\theta - \frac{1}{2} r_{j-1}^2 \, \Delta\theta$$

$$= \frac{1}{2} \Delta\theta \cdot \left[\left(r_k^* + \frac{\Delta r}{2}\right)^2 - \left(r_k^* - \frac{\Delta r}{2}\right)^2 \right]$$

$$= \frac{1}{2} \Delta\theta \left[\cancel{(r_k^*)^2} + 2 \frac{\Delta r}{2} \cdot r_k^* + \cancel{\frac{\Delta r^2}{4}} - \cancel{(r_k^*)^2} + 2 \cdot \frac{\Delta r}{2} \cdot r_k^* - \cancel{\frac{\Delta r^2}{4}} \right]$$

$$= \frac{1}{2} \Delta\theta \cdot 2 \Delta r \cdot r_k^*$$

$$= \boxed{r_k^* \cdot \Delta r \cdot \Delta\theta}$$

here is the source of the product by r-term from a geometric interpretation.

Then, to approximate the volume of the solid region we sum the volume of each "box" whose base is the kth polar rectangle and whose height is $f(r_k^*, \theta_k^*)$.

Since the volume of the kth box is

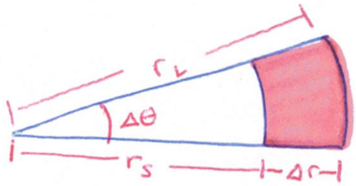
$$f(r_k^*, \theta_k^*) \cdot \Delta A_k$$

for $k = 1, 2, \cdots, n$ and ΔA_k is the area of each polar-rectangular subregion.

Thus the volume of the solid region beneath the surface $z = f(r, \theta)$ over domain region D is approximately

$$V \approx \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \, \Delta A_k$$

This is another type of Riemann sum in which the "size" measurements refer to sectors of a circle in the form



Area of pink region

$$\Delta A = \frac{r_s + r_L}{2} \cdot \Delta r \cdot \Delta\theta$$

Let $\Delta = \max\{\Delta r_1, \Delta\theta_1, \Delta r_2, \Delta\theta_2, \dots, \Delta r_n, \Delta\theta_n\}$.

Then, we can define the double integral

$$\int_D f \, d\omega = \iint_D f(r,\theta)\, dA$$

$$= \lim_{\Delta \to 0} \sum_{K=1}^{n} f(r_K^*, \theta_K^*)\, \Delta A_K$$

$$= \lim_{\Delta \to 0} \sum_{K=1}^{n} f(r_K^*, \theta_K^*)\, r_K^* \, \Delta r_K \, \Delta\theta_K$$

$$= \int_{\alpha}^{\beta} A(\theta)\, d\theta \quad \text{where} \quad A(\theta) = \int_a^b f(r,\theta) \cdot r \, dr$$

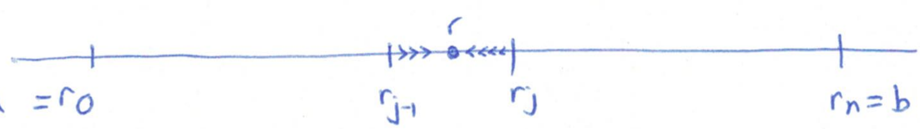
L4, p19

It is important to note that in the limit

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\, r_k^* \, \Delta r_k \, \Delta \theta_k$$

the sample $r_k^* = \dfrac{r_{j-1} + r_j}{2}$ for $k = 1, 2, \ldots, n$ converges

to an individual r value :



$$\lim_{\Delta \to 0} r_k^* = \lim_{\Delta \to 0} \frac{r_{j-1} + r_j}{2}$$

$$= \lim_{n \to \infty} \frac{r_{j-1} + r_j}{2}$$

$$= \frac{r + r}{2}$$

$$= \frac{2r}{2} = r$$

Example 13.3.1 p. 986

Find the volume of solid bounded by the paraboloid

$$z = f(x,y) = 9 - x^2 - y^2$$

and the xy-plane.

Solution: Let's begin by analyzing the domain region over which we are integration. To find $D \subseteq \mathbb{R}^2$, we look for all ordered pairs $(x,y) \in \mathbb{R}^2$ such that

$$0 \leq f(x,y) \implies 0 \leq 9 - x^2 - y^2$$

$$\implies x^2 + y^2 \leq 9$$

$$\implies x^2 + y^2 \leq 3^2$$

$$\implies D = \{(x,y) : x^2 + y^2 \leq 3^2\} \subseteq \mathbb{R}^2$$

this domain region is encoded using cartesian coordinates

$$\implies D = \{(r,\theta) : \theta \in [0, 2\pi) \text{ and } r \in [0, 3]\}$$

this domain region is encoded using polar coordinates.

$\boxed{L4, p21}$

Then, when we consider the function

$$z = f(x,y) \quad \text{where} \quad f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

the fact that the input variables are written as "x" and "y" is designed to suggest that surface is written over domain in Cartesian coordinates

initially, this region is assumed to be written in cartesian coordinates

$$\Rightarrow z = f(x,y) = 9 - x^2 - y^2$$

the values of variables x and y represent linear, signed, rectangular distances along x and y axis

$$= 9 - (x^2 + y^2)$$

$$\Rightarrow z = f(x,y) = f(x(r,\theta), y(r,\theta))$$

$$= 9 - (x(r,\theta))^2 - (y(r,\theta))^2$$

$$= 9 - ((x(r,\theta))^2 + (y(r,\theta))^2)$$

$$\Rightarrow z = f(r,\theta) = 9 - r^2$$

this is now a polar representation of output as a function of r and θ

In this case, we see $f(r, \theta) = 9 - r^2$ where

suggests that domain is written in polar coordinates

$f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $D = \{(r, \theta) : 0 \leq r \leq 3 \text{ and } 0 \leq \theta < 2\pi$

$$\Rightarrow \int_D f \, d\omega = \iint_D f(x, y) \, dA$$

$$= \iint_D f(r, \theta) \, dA$$

$$= \int_0^{2\pi} A(\theta) \, d\theta \quad \text{where} \quad A(\theta) = \int_0^3 (9 - r^2) \cdot r \cdot dr$$

Side note:

$$A(\theta) = \int_{r=0}^{r=3} (9 - r^2) \cdot r \, dr$$

$$= \int_{r=0}^{r=3} 9r - r^3 \, dr$$

$$= \frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^3$$

$$= \left(\frac{9 \cdot 3^2}{2} - \frac{3^4}{4}\right) - \left(9 \cdot 0^2 - \frac{0^4}{4}\right)$$

$$= \frac{81}{2} - \frac{81}{4} = \frac{81}{4}$$

$\boxed{\text{L4, p23}}$

$$\Rightarrow \iint_D f(r,\theta)\,dA = \int_0^{2\pi} \frac{81}{4}\,d\theta$$

$$= \frac{81}{4}\,\theta \Big|_0^{2\pi}$$

$$= \frac{81}{4} \cdot (2\pi - 0)$$

$$= \boxed{\frac{81\pi}{2}}$$

Example 13.3.2 p. 986 - 987

Find the volume of the region bounded between the paraboloids $z_1 = x^2 + y^2$ and $z_2 = 8 - x^2 - y^2$.

Solution: Let's begin this problem by trying to find the region D of integration. to this end, let's find the contour curve C that sits at the intersection between both surfaces. The two paraboloids intersect when

$$x^2 + y^2 = 8 - x^2 - y^2$$

$$\Rightarrow \quad 2x^2 + 2y^2 = 8$$

$$\Rightarrow \quad x^2 + y^2 = 4$$

region is better described via polar coordinates

$$\Rightarrow \quad D = \{ (x,y) : \quad x^2 + y^2 \leq 4 \}$$

region encoded in cartesian coordinates

$\boxed{\text{L4, p25}}$

$$\Rightarrow \quad D = \{(r,\theta) : 0 \le r \le 2 \quad \text{and} \quad 0 \le \theta < 2\pi\}$$

region is now encoded via polar coordinates

Then, for the solid described in this problem, we see that the upper bounding surface of the solid is

$$z = 8 - x^2 - y^2$$
$$= 8 - (x^2 + y^2)$$
$$= 8 - r^2$$

On the lower bound, we have

$$z = x^2 + y^2 = r^2$$

Then, the desired volume of our solid is

$$V = \int_D f \, d\omega = \iint_D f(r,\theta) \, dA$$

region best parameterized by polar coordinates

$$\Rightarrow \quad V = \int_0^{2\pi} A(\theta)\, d\theta \qquad \text{where} \qquad A(\theta) = \int_0^{2} f(r,\theta)\cdot r\, dr$$

Side note:

$$A(\theta) = \int_0^{2} (8 - r^2 - r^2)\cdot r\, dr$$

$$= \int_{r=0}^{r=2} 8r - 2r^3\, dr$$

$$= 4r^2 - \frac{r^4}{2} \Big|_0^2$$

$$= \left(4\cdot 2^2 - \frac{2^4}{2}\right) - \left(4\cdot 0^2 - \frac{0^4}{2}\right)$$

$$= 16 - 8$$

$$= 8$$

$$\Rightarrow \quad V = \int_0^{2\pi} 8\, d\theta = 8\theta \Big|_0^{2\pi} = 8\cdot 2\pi = \boxed{16\pi} \checkmark$$

Note: an annulus is the region between two concentric circles

Example 13.3.3 p. 987

Find the volume beneath the surface

$$z = f(x,y) = 10 + x \cdot y$$

and above the "annular" region

$$D = \{(r,\theta) : 2 \leq r \leq 4 \ \& \ 0 \leq \theta < 2\pi\}$$

Solution: Since we are given an accurate description of our domain region of integration $D \subseteq \mathbb{R}^2$ in polar coordinates yet our function $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is represented in cartesian coordinates, we begin by converting our function into polar coordinates

$$f(x,y) = 10 + xy \qquad \begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}$$

$$\Rightarrow f(r,\theta) = 10 + r \cdot \cos(\theta) \cdot r\sin(\theta)$$

$$\Rightarrow f(r,\theta) = 10 + r^2 \cos(\theta)\sin(\theta)$$

$$\Rightarrow \quad f(r,\theta) = 10 + \frac{1}{2} r^2 \sin(2\theta)$$

Recall: $\sin(2\theta) = \sin(\theta + \theta)$

$= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)$

$= 2 \sin(\theta)\cos(\theta)$

$\Rightarrow \quad \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$

$$\Rightarrow \quad V = \int_D f \, d\omega = \iint_D f(r,\theta)\, dA$$

$$= \int_0^{2\pi} A(\theta)\, d\theta \qquad \text{where} \quad A(\theta) = \underbrace{\int_2^4 \left(10 + \tfrac{1}{2}r^2 \sin(2\theta)\right) r\, dr}_{\text{see next page}}$$

$$= \int_0^{2\pi} 30 \sin(2\theta) + 60 \, d\theta$$

$$= -15 \cos(2\theta) + 60\theta \Big|_0^{2\pi}$$

$$= 60 \cdot 2\pi \underbrace{- 15\cos(4\pi) + 15\cos(0)}_{\text{combine to zero}}$$

$$= \boxed{120\pi} \checkmark$$

$\boxed{\text{L4, p29}}$

Side note for example 13.3.3 p. 987

$$A(\theta) = \int_2^4 \left(10 + \frac{r^2}{2} \cdot \sin(2\theta)\right) r \, dr$$

$$= \int_2^4 10r + \frac{r^3}{2} \sin(2\theta) \, dr$$

$$= 5r^2 + \frac{r^4}{8} \sin(2\theta) \Big|_2^4$$

$$= \left(5 \cdot 4^2 + \frac{4^4}{8} \cdot \sin(2\theta)\right) - \left(5 \cdot 2^2 + \frac{2^4}{8} \sin(2\theta)\right)$$

$$= 5 \cdot (4^2 - 2^2) + \frac{\sin(2\theta)}{8} (4^4 - 2^4)$$

$$= 5 \cdot (16 - 4) + \sin(2\theta) \left(\frac{2^8 - 2^4}{2^3}\right)$$

$$= 5 \cdot 12 + \sin(2\theta) (2^5 - 2)$$

$$= 60 + 30 \sin(2\theta).$$

$\boxed{\text{L4, p30}}$

Example 13.3.5. p. 990

Compute the area of the region where $x \geq 0$

outside the circle

$$r = r_1(\theta) = \sqrt{2}$$

and inside the lemniscate

$$r^2 = (r_2(\theta))^2 = 4 \cos(2\theta)$$

Solution: Recall by example 13.2.6 p. 980 that we can use double intgrals to find the area of a region in $\mathbb{R}^2$ by setting $f(r,\theta) = 1$. Now that we have our integrand, let's find our domain of integration

$$D = \{(r,\theta) : \quad r_1(\theta) \leq r \leq r_2(\theta)\}$$

Notice, we can find the point(s) where $r_1(\theta)$ intersects $r_2(\theta)$ by solving the algebraic equation

$$2 = 4\cos(2\theta)$$

$$\Rightarrow \cos(2\theta) = \frac{1}{2}$$

$$\Rightarrow 2\theta = \arccos\left(\frac{1}{2}\right)$$

$$\Rightarrow 2\theta = \pm\frac{\pi}{3}$$

$$\Rightarrow \theta = \pm\frac{\pi}{6}$$

$$\Rightarrow D = \left\{(r,\theta) : \sqrt{2} \le r \le 2\sqrt{\cos(2\theta)} \ \ \& \ \ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}$$

Then, we can set up our double integral to measure area.

$$A = \int_D f \, dw = \iint_D f(r,\theta) \, dA$$

$$= \int_{-\pi/6}^{\pi/6} A(\theta) \, d\theta \qquad \text{where} \quad \underbrace{A(\theta) = \int_{\sqrt{2}}^{2\sqrt{\cos(2\theta)}} 1 \cdot r \, dr}_{\text{See next page}}$$

$$= \int_{-\pi/6}^{\pi/6} 2\cos(2\theta) - 1 \, d\theta$$

$$= \sin(2\theta) - \theta \, \Big|_{-\pi/6}^{\pi/6}$$

$$= \left(\sin(\pi/3) - \pi/6\right) - \left(\sin(-\pi/3) + \pi/6\right)$$

$$= 2\sin(\pi/3) - \pi/3$$

$$= 2 \cdot \frac{\sqrt{3}}{2} - \pi/3$$

$$= \sqrt{3} - \pi/3.$$

$\boxed{\text{L4, p33}}$

Side Note:

$$A(\theta) = \int_{\sqrt{2}}^{2\sqrt{\cos(2\theta)}} r \, dr$$

$$= \frac{r^2}{2} \Bigg|_{\sqrt{2}}^{2\sqrt{\cos(2\theta)}}$$

$$= \frac{1}{2}\left(\left(2\sqrt{\cos(2\theta)}\right)^2 - \left(\sqrt{2}\right)^2 \right)$$

$$= \frac{1}{2}\left(4 \cdot \cos(2\theta) - 2 \right)$$

$$= 2\cos(2\theta) - 1.$$