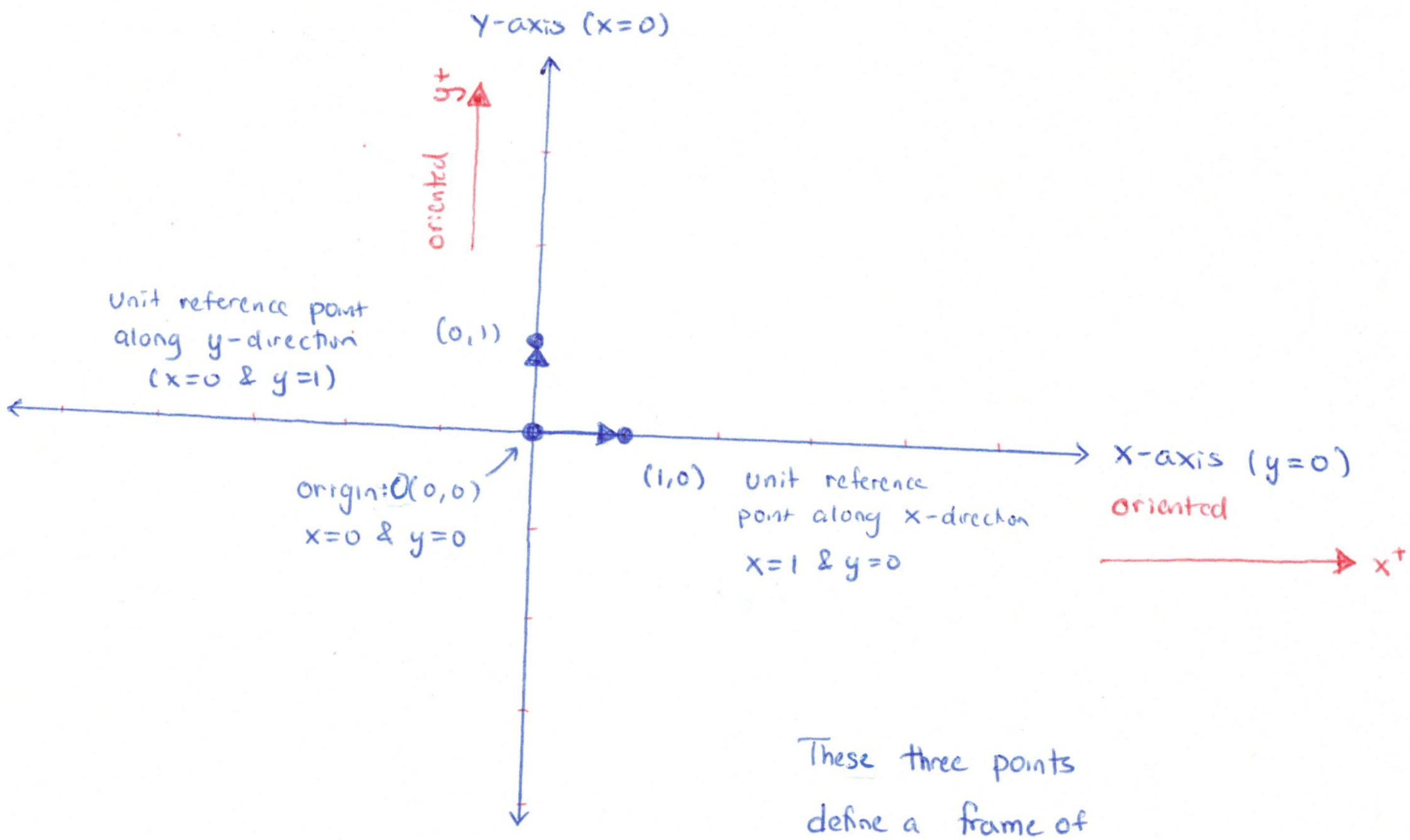
Lesson 3: Polar Coordinates

Recall that when we consider the space

$$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$$

we impose a very specific geometric structure

as we attempt to graph any ordered pair $P(x,y) \in \mathbb{R}^2$



These three points define a frame of reference for $\mathbb{R}^2$

(A model in physical space maps each point to physical objects)

The ordered pair (x, y) has a very special graphical encoding:

$$\left(\overset{\text{1st coordinate}}{\downarrow \atop x} \;,\; \overset{\text{2nd coordinate}}{\downarrow \atop y} \right)$$

The 1st coordinate:

- represents a signed (or oriented or directed) distance from the origin point $(0,0)$ to the point $(x, 0)$ by moving linearly x units along the $\langle 1,0 \rangle$ direction

- the sign (or orientation) is encoded in the scalar x compared with the imposed frame of reference

The 2nd coordinate: ☐ Represents a signed (oriented) distance from the origin point $(0,0)$ to the point $(0,y)$ by moving lineary y units along the $\langle 0,1\rangle$ direction

☐ the sign (or orientation) is encoded in the scalar y

With this in mind, we write

$$(x,y) = (x,0) + (0,y)$$

$$= \underset{\text{signed distance}}{x} \cdot \underbrace{(1,0)}_{\text{imposed geometry on 1st coordinate}} + \overset{\text{signed distance}}{y} \cdot \underbrace{(0,1)}_{\text{imposed geometry on second coordinate}}$$

This coordinate system is optimally designed to encode information using rectangular geometry. In other words, when modeling phenomena involving linear change in x any y, cartesean coordinates are a "natural" choice.

However, there are a large class of phenomena for which models that rely on cartesian coordinates are not optimal. In particular, when modeling a phenomena with periodic behavior or when modeling geometries that rely on circular, elliptical, or angular descriptions, the cartesian coordinate system can result in analytic descriptions that are quite difficult to manipulate algebraically.

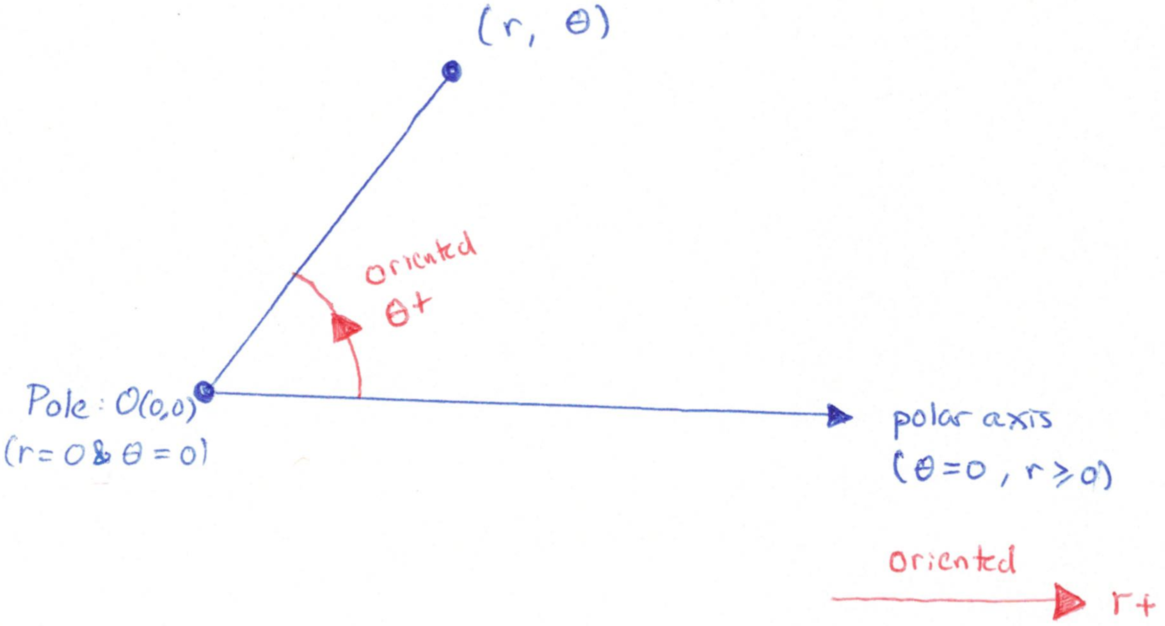
In these cases, we might be wise to use an alternative coordinate system that encodes angular/radial information. We will call this system Polar coordinates.

The Polar Coordinate System

- ☐ Used to store "points" in the "plane" $\mathbb{R}^2$

- ☐ imposes very special geometry to graph an point

as an ordered pair $P(r, \theta)$ where



The ordered pair (r, θ) has special geometric meaning

radial coordinate

angular coordinate

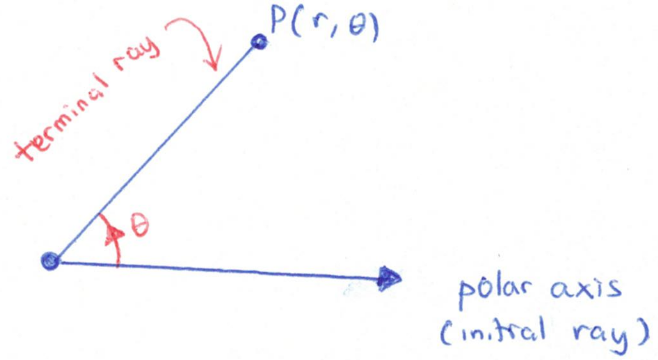
$$(r , \theta)$$

The radial coordinate:
(also called the radius)

- represents a signed (or oriented or directed) distance from the pole of the system to the end point of a ray

The angular coordinate: (also called the polar angle)

- ☐ desribes an angle whose initial ray is along the polar axis and whose terminal ray is the line segment that connects the pole $(0,0)$ to the point (r, θ)



- ☐ positive angles are oriented counterclockwise from the polar axis.

Example 10.2.1 p. 720 – 721

Graph the following points in polar coordinates

$$Q\left(1, \frac{5\pi}{4}\right), \quad R\left(-1, \frac{7\pi}{4}\right), \quad \text{and} \quad S\left(2, -\frac{3\pi}{2}\right)$$

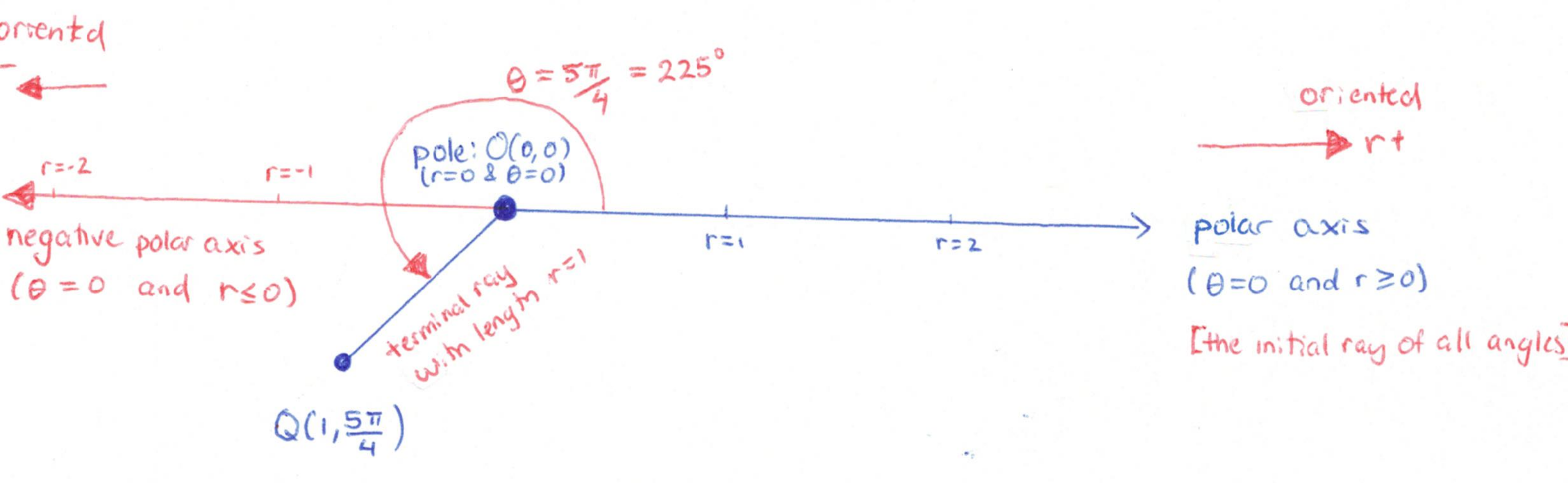
Solution: We begin this problem by focusing our attention on the point $Q\left(1, \frac{5\pi}{4}\right)$. To this end, consider

radial coordinate: $r = 1$

$$Q\left(1, \frac{5\pi}{4}\right)$$

angular coordinate: $\theta = \frac{5\pi}{4}$

$\boxed{\text{Graph the point } Q}$



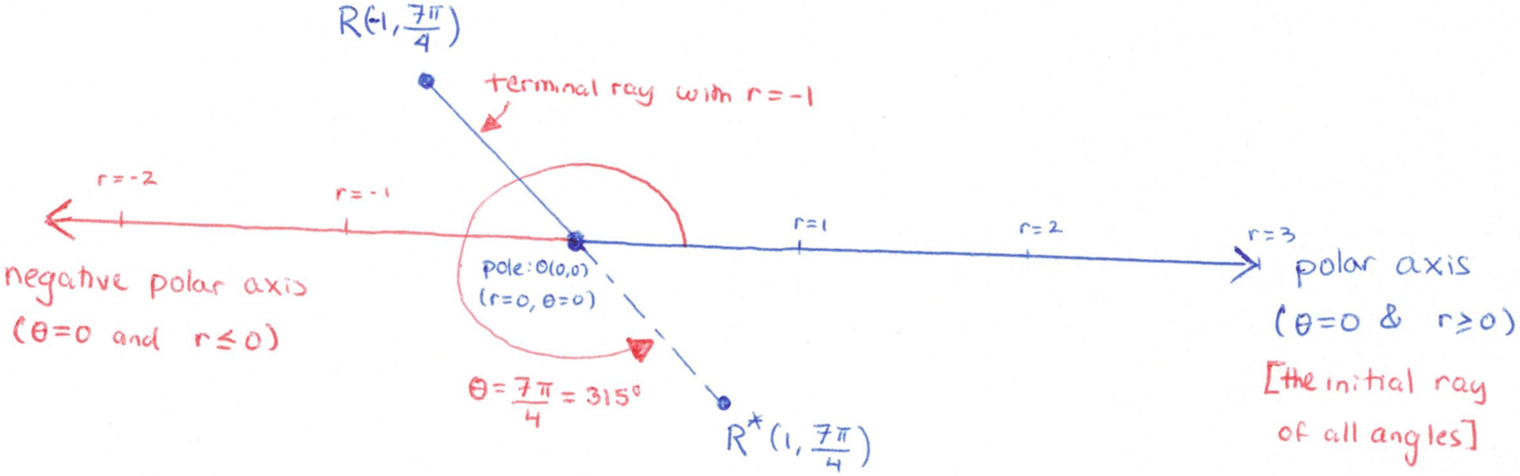
Graph the point R

Let's move on to the point $R(-1, \frac{7\pi}{4})$. To this end, consider

radial coordinate : $r=-1$

$$R(-1, \frac{7\pi}{4})$$

angular coordinate : $\theta = \frac{7\pi}{4}$

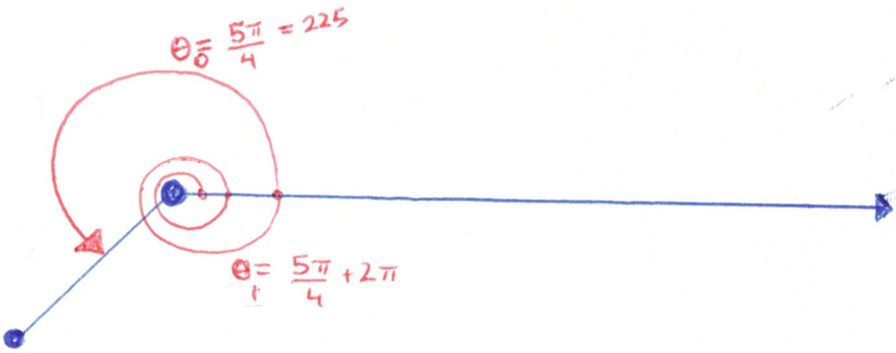


Non-unique nature of points in Polar coordinates

Notice that when graphing points in polar coordinates, we have many "equivalent" representations for each individual point.

Example 10.2.1 p. 720 – 721 continued ...

For the point $Q(1, \frac{5\pi}{4})$ we can find an infinite number of encodings:



$Q(-1, \hat{\theta}_m) = Q(1, \theta_n)$

where $\theta_n = \frac{5\pi}{4} + n \cdot 2\pi$ where $n = 0, 1, 2, 3, \ldots$

$\hat{\theta}_m = \frac{\pi}{4} + m \cdot 2\pi$ where $m = 0, 1, 2, 3, \ldots$

The non-uniqueness of points in polar coordinates results from the following two observations:

Observation 1: □ Angles in polar coordinates a unique up to multples of $2\pi = 360°$.

□ In other words, we have the following equivalencies

$$P(r, \theta) \text{ "=" } P(r, \theta + 2\pi)$$
$$\text{ "=" } P(r, \theta + 4\pi)$$
$$\text{ "=" } P(r, \theta + 6\pi)$$
$$\text{ "=" } P(r, \theta + 8\pi)$$
$$\vdots$$

$$\Rightarrow \quad P(r, \theta) \underset{\text{"equivalent to"}}{\equiv} P(r, \theta + 2n \cdot \pi) \quad \text{for } n \in \mathbb{Z}$$

□ We can sum this observation up succintly as follows: "adding any number of full turns to the angular coordinate θ does not change the cooresponding 'direction' of the terminal ray used to situate the point $P(r, \theta)$ in the plane." $\boxed{\text{L3, p. 11}}$

Observation 2: ☐ Since the radial coordinate r may be negative, we can reinterpret this point as traveling along a positive radius measured in the "opposite direction"

☐ In other words, we have the following equivalencies for $r \in \mathbb{R}$ with $r \geq 0$:

$$P(-r, \theta) \text{ "}=\text{" } P(r, \theta + \pi)$$
$$\text{"}=\text{" } P(r, \theta + 3\pi)$$
$$\text{"}=\text{" } P(r, \theta + 5\pi)$$
$$\text{"}=\text{" } P(r, \theta + 7\pi)$$
$$\vdots$$

$$\Rightarrow \quad P(-r, \theta) \equiv P(r, \theta + (2n+1)\pi) \quad \text{for } n \in \mathbb{Z}$$

Agreeing on a "Unique" Polar representation

Although each point in $\mathbb{R}^2$ can be encoded by an equivalence class of "points" encoded in polar coordinates, we can agree to choose a unique encoding.

We say the point $P(r, \theta)$ in polar coordinates in "unique" if and only if

condition (i.): the radial coordinate is nonnegative with $r \geq 0$

condition (ii): the angular coordinate is in the interval $[0, 2\pi)$

with $\theta \in [0, 2\pi)$ (radians) or $\theta \in [0°, 360°)$ (degrees)

Example 10.2.1 p. 720 – 721

Find the "unique" representations of each point below (assuming the points are written in polar coordinates

$$Q\left(1, \frac{5\pi}{4}\right), \quad R\left(-1, \frac{7\pi}{4}\right), \text{ and } S\left(2, -\frac{3\pi}{2}\right)$$

Solution: Recall, each point represents an entire equivalence class of polar representations. In order to be unique, we need $r \geq 0$ and $\theta \in [0, 2\pi)$. Then

- $Q(1, 5\pi/4)$ is already in desired form
- $R\left(-1, \frac{7\pi}{4}\right) = \boxed{R\left(1, \frac{3\pi}{4}\right)}$

unique representation

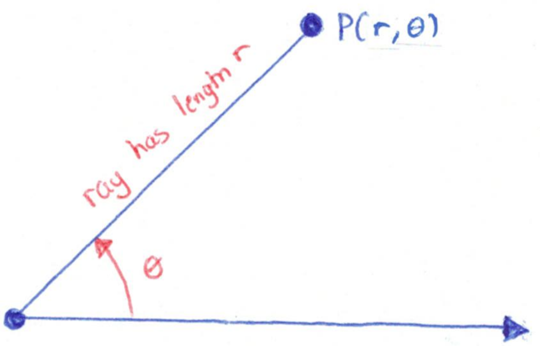
- $S\left(2, \frac{-3\pi}{2}\right) = \boxed{S\left(2, \frac{\pi}{2}\right)}$

Converting Between Cartesian and Polar Coordinates

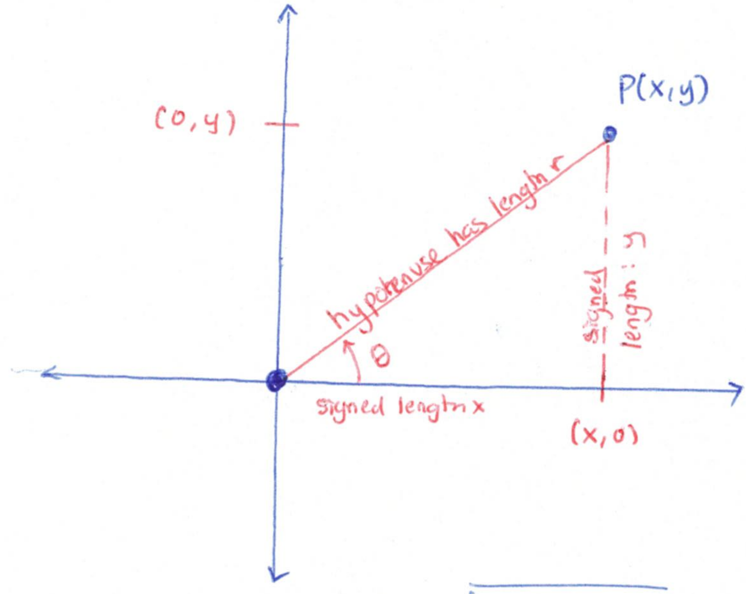
Because our education system (cerca 2018) tends to work predominately in the cartesian coordinate system for the majority of math classes prior to Math 1D and because polar coordinates are specialized tools, we may find it helpful to be able to convert between cartesian and polar coordinates.

The conversion equations we will use arise immediately from our work in trigonometry:

Polar Coordinates



Cartesian Coordinates



We recall: To convert from polar to cartesian, we have

- $\cos(\theta) = \dfrac{x}{r} \implies x = r \cdot \cos(\theta)$

$\implies x(r,\theta) = r \cdot \cos(\theta)$

where $x : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$
is a parameterization of x

- $\sin(\theta) = \dfrac{y}{r} \implies y = r \cdot \sin(\theta)$

$\implies y(r,\theta) = r \cdot \sin(\theta)$ where

$y : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a
parameterization of y

To convert from cartesian to the unique polar representation, we see

- $r^2 = x^2 + y^2 \Rightarrow r = \pm\sqrt{x^2 + y^2}$

$\Rightarrow r(x,y) = \pm\sqrt{x^2 + y^2}$

where $r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

is a parametorization of r

- $\tan(\theta) = \dfrac{x}{y} \Rightarrow \theta = \arctan\left(\frac{x}{y}\right)$ where

$-\pi/2 \leq \theta \leq \pi/2$ is used to determine the principal value of angle θ

Example 10.3.2 p. 721 – 722

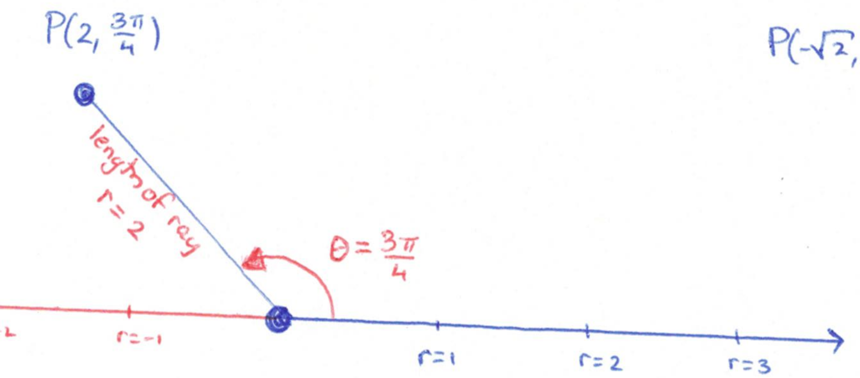
A. Express the point $P(2, \frac{3\pi}{4})$ in cartesian coordinates.

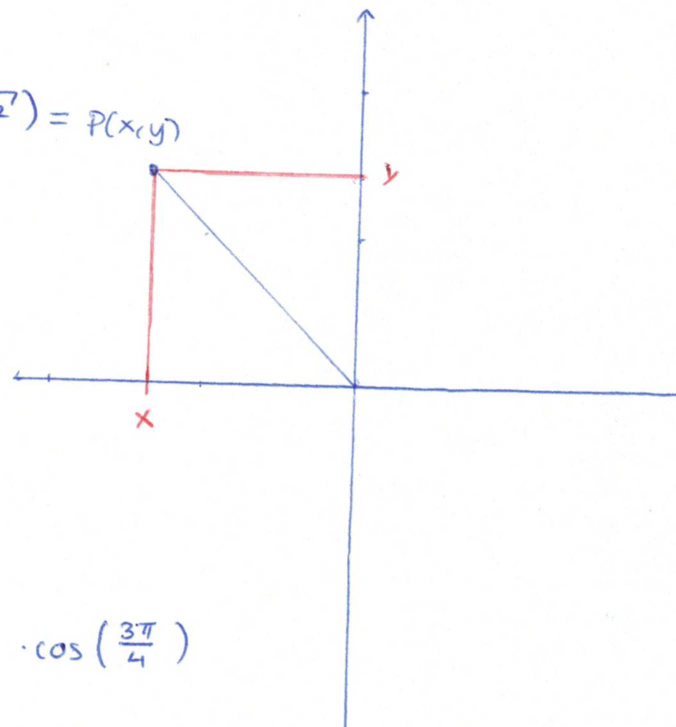
B. Express the point $Q(1, -1)$ in polar coordinates.

Solution to A Let's start by converting the point $P(2, \frac{3\pi}{4})$ into cartesian coordinates

Polar coordinates

Cartesian coordinates





□ $x = x(r, \theta) = x(2, \frac{3\pi}{4}) = 2 \cdot \cos(\frac{3\pi}{4})$

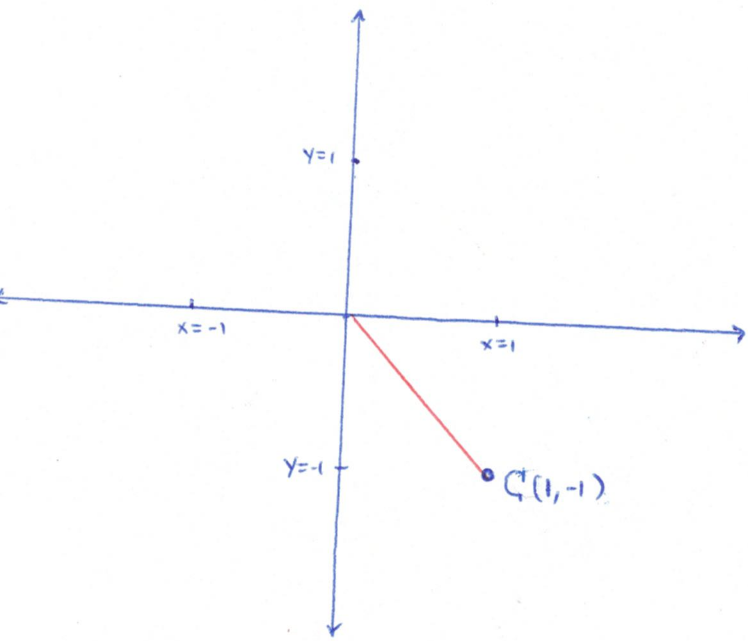
$= 2 \cdot \frac{-1}{\sqrt{2}} = -\sqrt{2}$

□ $y = y(r, \theta) = y(2, \frac{3\pi}{4}) = 2 \sin(\frac{3\pi}{4})$

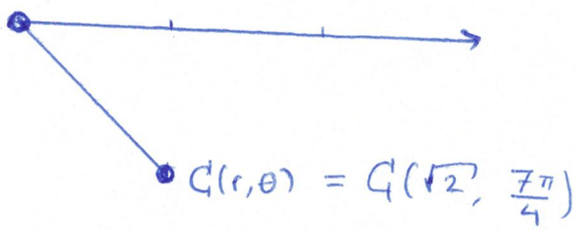
$= 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

$\boxed{\text{Solution to B}}$ Now we will convert point $G(1,-1)$ into polare coordinates:

Cartesian coordinates



Polar Coordinates



$$r = r(x,y) = +\sqrt{x^2+y^2} = \sqrt{(1)^2+(-1)^2} = +\sqrt{2}$$

$$\theta = \theta(x,y) = \arctan\left(\frac{x}{y}\right) = \arctan(-1) = -\frac{\pi}{4} \equiv 7\pi/4$$