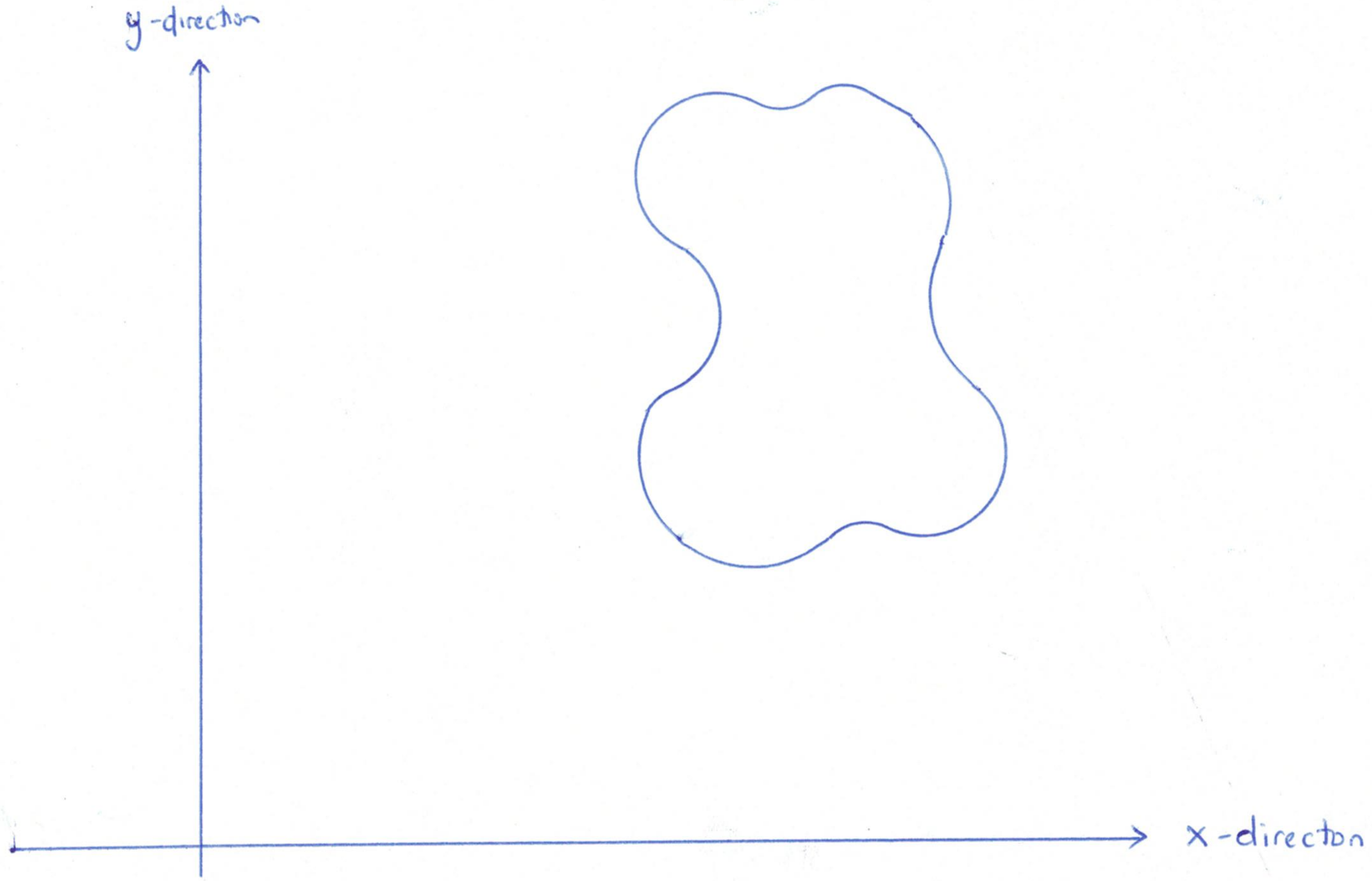
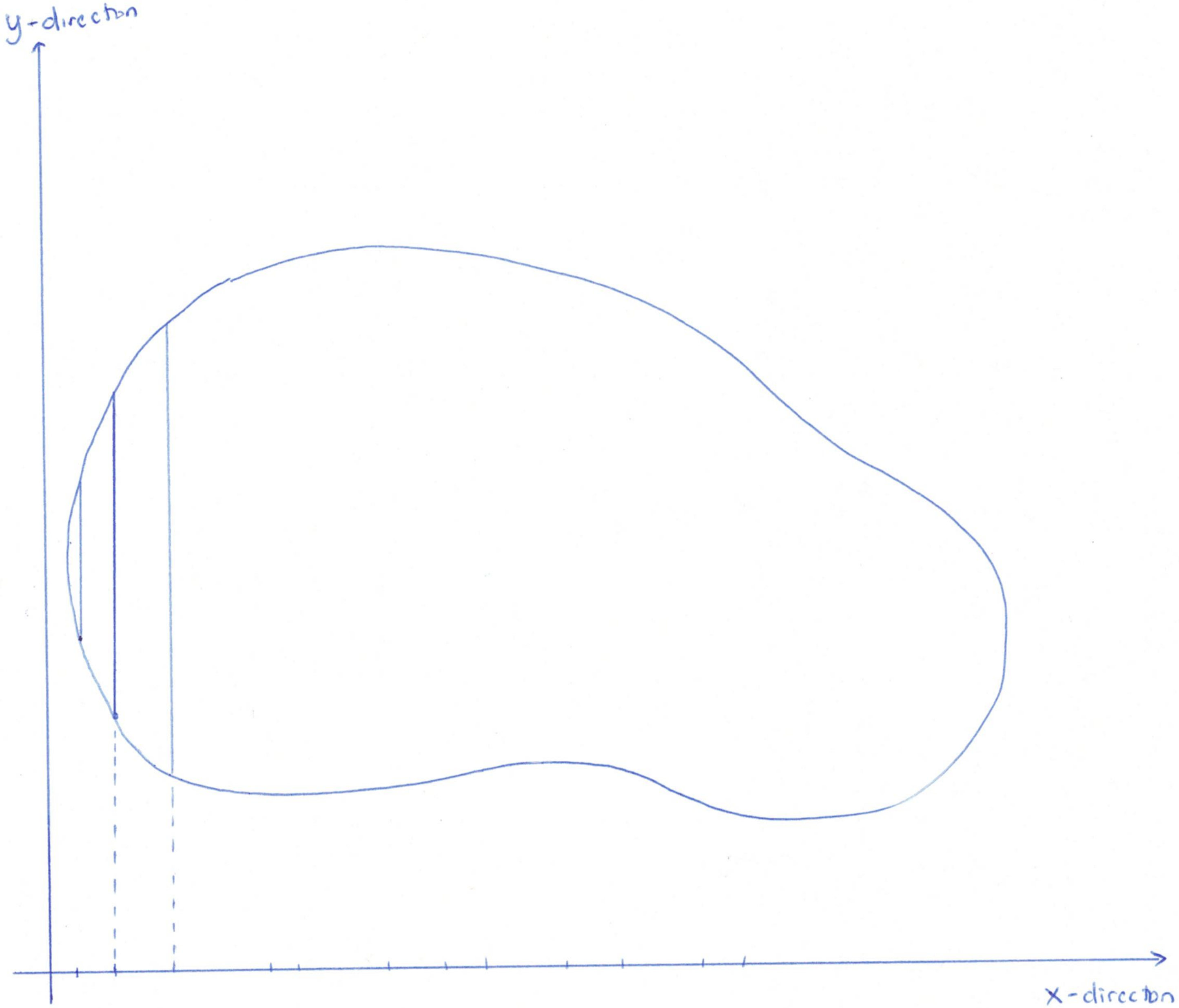
Math 1D: Lesson 2 Double Integrals over General Regions $D \subseteq \mathbb{R}^2$

Let $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function on a closed and bounded nonrectangular region D.



As with our study of rectangular regions, we will create a general partition of D consisting of rectangles.

However, in general this finite partition will not exactly cover D. In the case of a non rectangular region, we count only the n rectangles that lie completely within the region D.



y-direction
x-direction

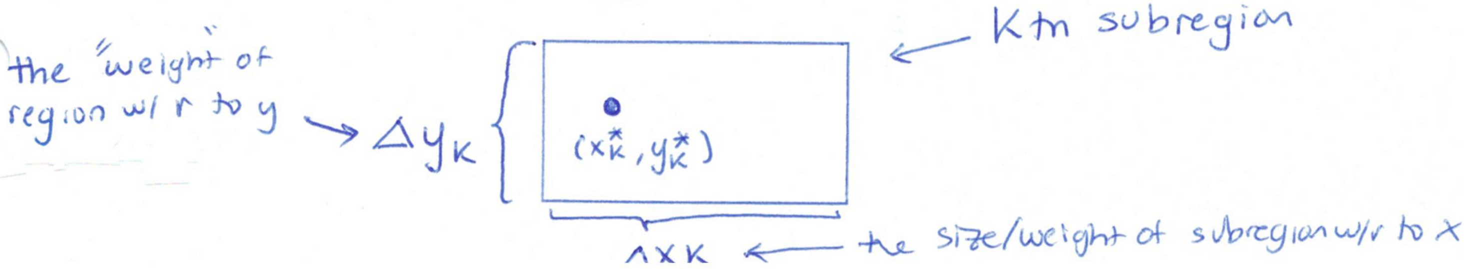
For surface $z = f(x,y)$ where $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, the net volume of the solid bounded between the surface $f(x,y)$ and region D in the xy-plane can be approximated by the Riemann sum

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

where (x_k^*, y_k^*) is in the kth rectangular subregion inside D and ΔA_k represents the measured size of the kth subregion in this partition. A "natural" choice is to set

$$\Delta A_k = \Delta x_k \, \Delta y_k$$

associated with the diagram



$$\Rightarrow \quad V \approx \sum_{K=1}^{n} f(x_K^*, y_K^*)\, \Delta A_K$$

$$= \sum_{K=1}^{n} f(x_K^*, y_K^*)\, \Delta x_K\, \Delta y_K$$

Now we set $d_K = \sqrt[2]{\Delta x_K^2 + \Delta y_K^2}$ for $K = 1, 2, \ldots, n$

and let $\Delta = \max\{d_1, d_2, \ldots, d_n\}$.

Type I: Nonrectangular Regions $D \subseteq \mathbb{R}^2$ that are "y-simple" (simple w/r to y)

simpler to integrate over y first, then over x next

Suppose we have two continuous, real-valued functions

$$g : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R} \quad \text{and}$$

$$h : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

that satisfy $g(x) \leq h(x)$ for all $x \in [a,b]$

Let D be the region defined by the set of all points (x,y) such that $x \in [a,b]$ and $g(x) \leq y \leq h(x)$

$$\Rightarrow D = \{(x,y) : x \in [a,b] \ \& \ g(x) \leq y \leq h(x)\}$$

$$\Rightarrow D = \{(x,y) : a \leq x \leq b \ \& \ g(x) \leq y \leq h(x)\}$$

Such a region is said to be y-simple.

(i.e. the region is described in a "simple" way with $y = y(x)$"

Example 13.2.1 p. 974)

Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function

$$f(x,y) = 2x^2 y$$

where $D = \{ (x,y) : \quad 3x^2 \leq y \leq 16 - x^2 \}$.

is the region bounded by parabolas $y_1 = 3x^2$ and $y_2 = 16 - x^2$. Find $\iint_D f(x,y)\, dA = \int_D f \, dA$

Solution: We begin this problem by graphing the region D. Let

$$g(x) = 3x^2 \quad \text{and} \quad h(x) = 16 - x^2$$

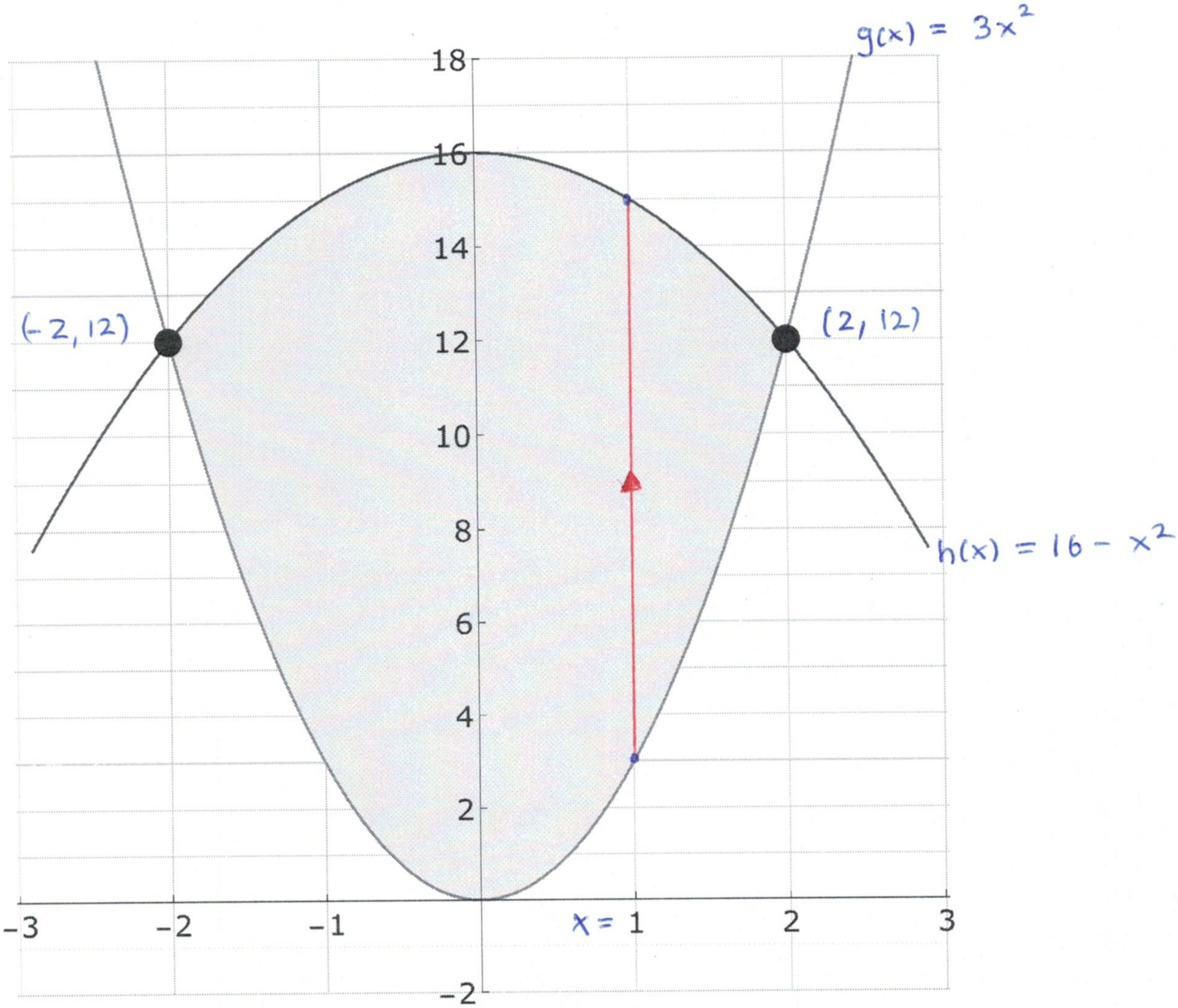
$$\Rightarrow \quad g(x) = h(x) \iff 16 - x^2 = 3x^2$$

$$\iff 4x^2 = 16$$

$$\iff x^2 = 4$$

$$\iff x = \pm 2$$

Then, we consider a geometric interpretation of the region D :



If we hold x constant with $x \in [-2, 2]$ and intersect the surface f(x,y) with a plane parallel to the yz-plane ($\vec{n} = \langle 1, 0, $

we can define the "area" function under the resulting trace

$$A(x) = \int_{3x^2}^{16-x^2} 2x^2 y \, dy$$

Then, by integrating over each of these cross sectional areas with $x \in [-2,2]$, we produce an iterated integral that encodes the volume of surface over region D:

$$\int_D f \, dA = \iint_D f(x,y) \, dA$$

$$= \iint_D 2x^2 y \, dA$$

$$= \int_{-2}^{2} A(x) \, dx \quad \text{where} \quad A(x) = \int_{3x^2}^{16-x^2} 2x^2 y \, dy$$

$$= \int_{-2}^{2} \int_{3x^2}^{16-x^2} 2x^2 y \, dy \, dx$$

Since this is an iterated integral, we start by considering

$$A(x) = \int_{3x^2}^{16-x^2} 2x^2 y \, dy$$

$$= x^2 y^2 \Big|_{3x^2}^{16-x^2}$$

$$= x^2 \cdot (16 - x^2)^2 - x^2 \cdot (3x^2)^2$$

$$= x^2 (16 - x^2) \cdot (16 - x^2) - 9x^6$$

$$= x^2 (256 - 32x^2 + x^4) - 9x^6$$

$$= -8x^6 - 32x^4 + 256x^2$$

Then, we can find the volume:

$$V = \int_D f \, dA = \int_{-2}^{2} A(x) \, dx$$

$$= \int_{-2}^{2} -8x^6 - 32x^4 + 256x^2 \, dx$$

$$= \underbrace{-\frac{8}{7}x^7 - \frac{32}{5}x^5 + \frac{256}{3}x^3}\Bigg|_{x=-2}^{x=2}$$

this is not very beautiful... much better to use Mma or calculator than to do by hand

$$= \frac{69{,}632}{105} \approx 663.2$$

Type II: Nonrectangular Regions $D \subseteq \mathbb{R}^2$ that are $\boxed{x\text{-simple}}$

simpler to integrate over x first and then over y next

Suppose we have two continuous, real-valued functions

$$g : [c, d] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

$$h : [c, d] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

that satisfy $g(y) \leq h(y)$ for all $y \in [c, d]$.

Let the nonrectangular region D be given as

$$D = \{(x, y) : y \in [c, d] \ \& \ g(y) \leq x \leq h(y)\}$$

$$\Rightarrow D = \{(x, y) : c \leq y \leq d \ \& \ g(y) \leq x \leq h(y)\}$$

Such a region is called <u>x-simple</u>

Example 13.2.2 p. 976

Find the volume of the solid below the surface

$$f(x,y) = 2 + \frac{1}{y}$$

where $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and region D is given by the three inequalities

Inequality 1: $y \geq 1 \implies y \geq 1$

Inequality 2: $y \leq 8 - x \implies x + y \leq 8$

Inequality 3: $y \leq x \implies x - y \geq 0$

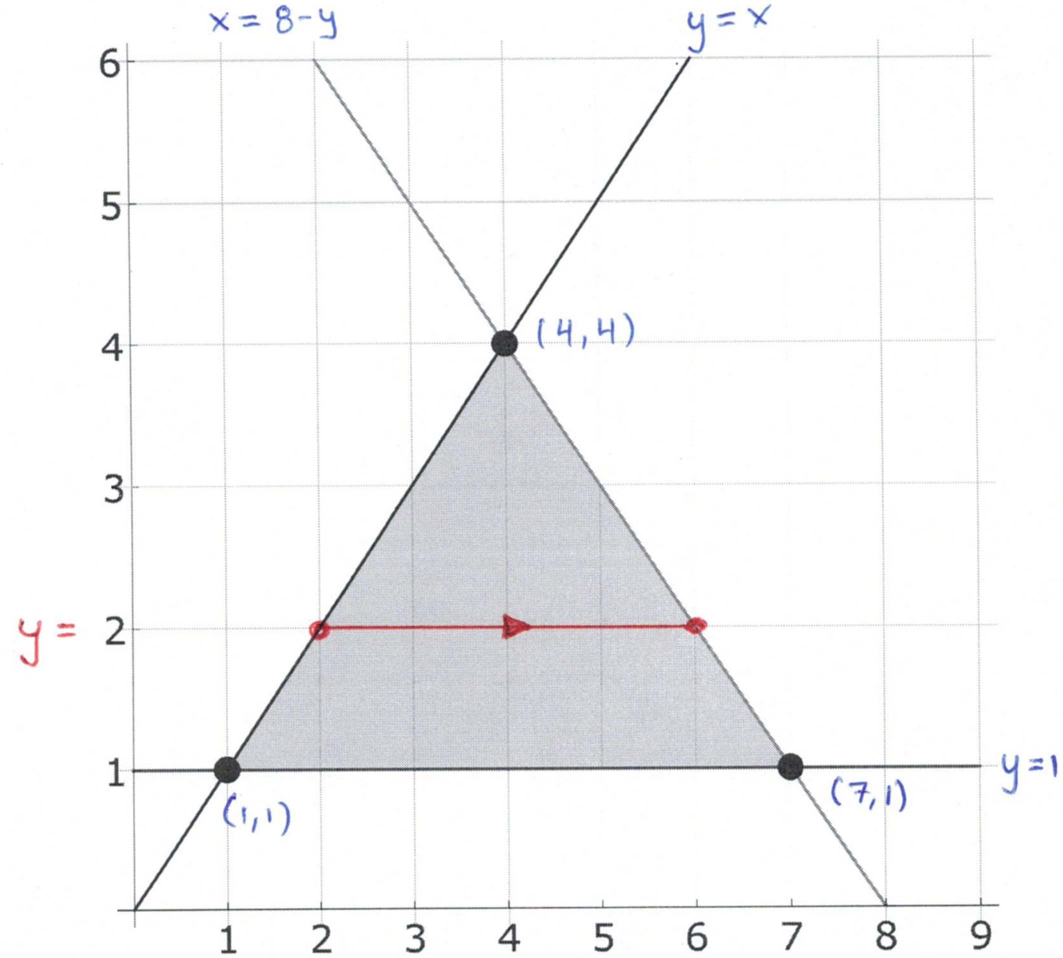
Solution: We begin this problem by analyzing and graphing the region D. To this end, let's find the point where line: $y_2 = 8 - x$ intersects line $y_3 = x$:

$$8 - x = x \iff 2x = 8$$

$$\iff x = 4$$

$\iff$ point of intersection at (4,4

Now let's graph our region D:



In this case, we see that slices in D work much more effectively if we hold y constant with $y \in [1,4]$ and intersect the surface $f(x,y)$ with a plane parallel to the xz-plane (with normal vector $\vec{n} = \langle 0,1,0 \rangle$).

The intersection between the surface and the plane forms a curve (the trace). For each y value, we can find area under this curve.

$$A(y) = \int_{y}^{8-y} 2 + \frac{1}{y} \, dx$$

$$= 2x + \frac{x}{y} \Big|_{y}^{8-y}$$

$$= \left(2(8-y) + \frac{8-y}{y}\right) - \left(2y + \frac{y}{y}\right)$$

$$= 16 - 2y + \frac{8}{y} - 1 - 2y - 1$$

$$= 14 - 4y + \frac{8}{y}$$

Then, the volume under surface on given region is

$$V = \int_D f \, dA = \iint_D f(x,y) \, dA$$

$$= \int_1^4 A(y) \, dy \qquad \text{where} \quad A(y) = \int_y^{8-y} 2 + \frac{1}{y} \, dx$$

$$= \int_1^4 14 - 4y + \frac{8}{y} \cdot dy$$

$$= 14y - 2y^2 + 8 \ln(y) \Big|_1^4$$

$$= \left(14 \cdot 4 - 2 \cdot 4^2 + 8 \ln(4)\right) - \left(14 - 2 \cdot 1^2 + 8 \ln(1)\right)$$

$$= 56 - 32 + 8 \cdot \ln(4) - 14 + 2$$

$$= \boxed{12 + 8 \ln(4)} \approx 23.09 .$$

Example 13.2.6 p. 980

Find the area of region $D \subseteq \mathbb{R}^2$ where the ordered pairs $(x, y) \in D$ satisfy the following inequalities

Inequality 1: $y \geq x^2$

Inequality 2: $y \leq 4x + 12$

Inequality 2: $y \leq 12 - x$

Solution: we begin by analyzing our region $D \subseteq \mathbb{R}^2$.

Inequalities 1&2: $x^2 = 4x + 12 \Rightarrow x^2 - 4x - 12 = 0$

$\Rightarrow (x-6)\cdot(x+2) = 0$

$\Rightarrow \boxed{x = -2}$ or $x = 6$

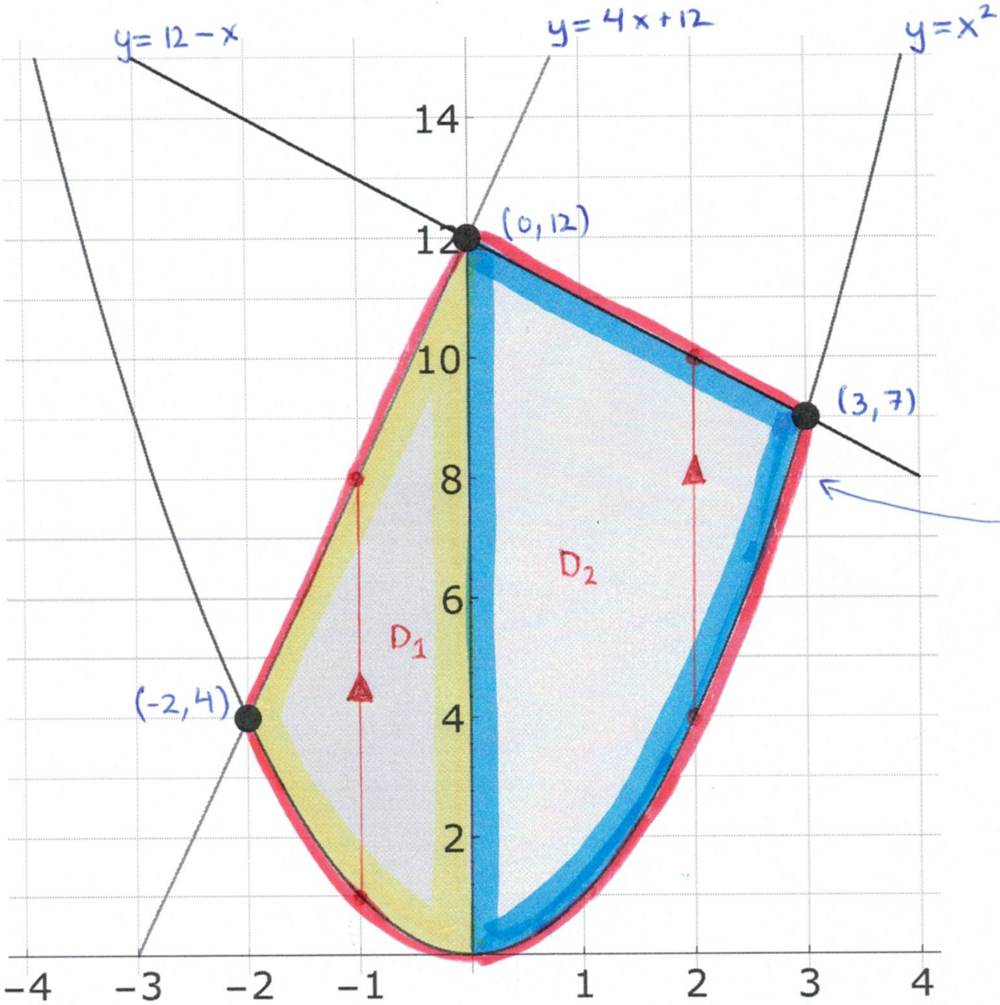
Inequalities 2&3: $4x + 12 = 12 - x \Rightarrow 5x = 0$

$\Rightarrow \boxed{x = 0}$

Inequalities 1&3: $x^2 = 12 - x \Rightarrow x^2 + x - 12 = 0$

$\Rightarrow (x+4)\cdot(x-3) = 0 \Rightarrow \boxed{x = 3}$

We can use this analysis to create a region plot:



the region inside red boundary is neither x-simple nor y-simple. But, we can partition it into two subregions that are x-simple:

$D = D_1 \cup D_2$

Then $\text{Area}(D) = \text{Area}(D_1) + \text{Area}(D_2)$

$$\Rightarrow \int_D 1\, dA = \int_{D_1} 1\, dA + \int_{D_2} 1\, dA$$

$$\Rightarrow \iint_D 1\, dA = \iint_{D_1} 1\, dA + \iint_{D_2} 1\, dA$$

Now, let's solve each subproblem separately:

$$V_1 = \iint_{D_1} 1 \, dA = \int_{-2}^{0} A(x)\, dx \qquad \text{where } A(x) = \int_{x^2}^{4x+12} 1 \, dy$$

Side note 1:

We can verify that dy dx is the "best" since

$D_1 = \{(x,y) : -2 \leq x \leq 0 \ \& \ x^2 \leq y \leq 4x+12\}$

this is y-simple
(integrate w/ dy dx)

Side note 2:

$$A(x) = \int_{x^2}^{4x+12} 1 \, dy = y \Big|_{x^2}^{4x+12}$$

$$= 4x + 12 - x^2$$

$$\Rightarrow V = \int_{-2}^{0} 4x + 12 - x^2 \, dx$$

$$= 2x^2 + 12x - \frac{x^3}{3} \Big|_{-2}^{0}$$

$$= 0 - \left(8 - 24 + \frac{8}{3}\right) = 16 - \frac{8}{3} = \frac{40}{3} \checkmark$$

Next we have

$$V_2 = \iint_{D_2} 1 \, dA = \int_0^3 A(x) \, dx \qquad \text{where} \int_{x^2}^{12-x} 1 \, dy = A(x)$$

Side note 1:

$$D_2 = \{(x,y) : 0 \le x \le 3 \ \& \ x^2 \le y \le 12-x\}$$

Side note 2:

$$A(x) = \int_{x^2}^{12-x} 1 \, dy$$

$$= y \Big|_{x^2}^{12-x}$$

$$= 12 - x - x^2$$

$$\Rightarrow V_2 = \int_0^3 12 - x - x^2 \, dx$$

$$= 12x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^3$$

$$= 36 - \frac{9}{2} - \frac{27}{3} = 27 - \frac{9}{2} = \frac{54-9}{2} = \frac{45}{2}$$

Then, we can find the total

$$\text{Area}(D) = \text{Area}(D_1) + \text{Area}(D_2)$$

$$\Rightarrow \iint_D 1\,dA = \iint_{D_1} 1\,dA + \iint_{D_2} 1\,dA$$

$$= \frac{40}{3} + \frac{45}{2}$$

$$\boxed{= \frac{215}{6}}$$