Theorem 13.1. *p. 967* ***Fubini's Theorem for Double Integrals on Rectangular Regions***

Let $f(x, y)$ be continuous on the rectangular regions

$$D = \{\, (x, y)\, :\, a \le x \le b,\, c \le y \le d \,\}.$$

Then, the double integral of f over D may be evaluated by either of the two iterated integrals:

$$\iint\limits_D f(x,y)dA = \int\limits_c^d \int\limits_a^b f(x,y)\; dx\, dy = \int\limits_a^b \int\limits_c^d f(x,y)\; dy\, dx$$

In other words, the double integral may be evaluated as iterated integrals and the order of integration does not matter. In practice, one order of integration is often easier to evaluate than the other order.

Definition. *p. 970* ***Average Value of a Function over a Plane Region***

The **average value** of an integrable function f over a region D is

$$\bar{f} = \frac{1}{\text{area of } D} \iint\limits_D f(x,y)\, dA$$

Lesson 1: Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 13.1: Double Integrals over Rectangular Regions, p. 963 - 973

Definition. *p. 965* ***Partition of rectangular*** $D \subseteq \mathbb{R}^2$

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a two-variable, nonnegative function defined on a *rectangular* region $D \subseteq \mathbb{R}^2$ in the $xy-$plane, where

$$D = \{\, (x, y) \,:\, a \leq x \leq b,\, c \leq y \leq d \,\}.$$

A **partition** of the region D is formed by dividing D into n rectangular subregions using parallel lines to the $x-$ and $y-$axes (not necessarily uniformly spaced). We then number each rectangle from 1 to n in any systematic way so that we count all rectangles exactly once. The side lengths of the kth rectangle are denoted Δx_k and Δy_k, so that the area of the kth rectangle is

$$\Delta A_k = \Delta x_k \cdot \Delta y_k$$

Finally, we let the point (x_k^*, y_k^*) be any point in the kth rectangle, for $1 \leq k \leq n$.

To approximate the volume of the solid bounded by the surface $z = f(x, y)$ and the region D, we construct boxes on each of the n rectangles from our partition. Each box has a height of $f(x_k^*, y_k^*)$ and a base with area ΔA_k. Thus, the volume of the kth box is given by

$$V_k = f(x_k^*, y_k^*) \cdot \Delta A_k = f(x_k^*, y_k^*) \cdot \Delta x_k \cdot \Delta y_k.$$

The sum of the volumes of all n boxes gives an approximation to the volume of the entire solid

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} f(x_k^*, y_k^*) \cdot \Delta A_k = \sum_{k=1}^{n} f(x_k^*, y_k^*) \cdot \Delta x_k \cdot \Delta y_k.$$

Definition. *p. 965* ***Double Integrals***

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function defined on rectangular region $D \subseteq \mathbb{R}^2$ in the $xy-$plane. We say f is **integrable** on D if the limit

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

exists for all partitions of D and for all choices of points (x_k^*, y_k^*) within those partitions. The limit is the **double integral of** f **over** D, which we write as

$$\iint_D f(x, y)\; dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$