Lesson 1: Double Definite Integrals over rectangular regions

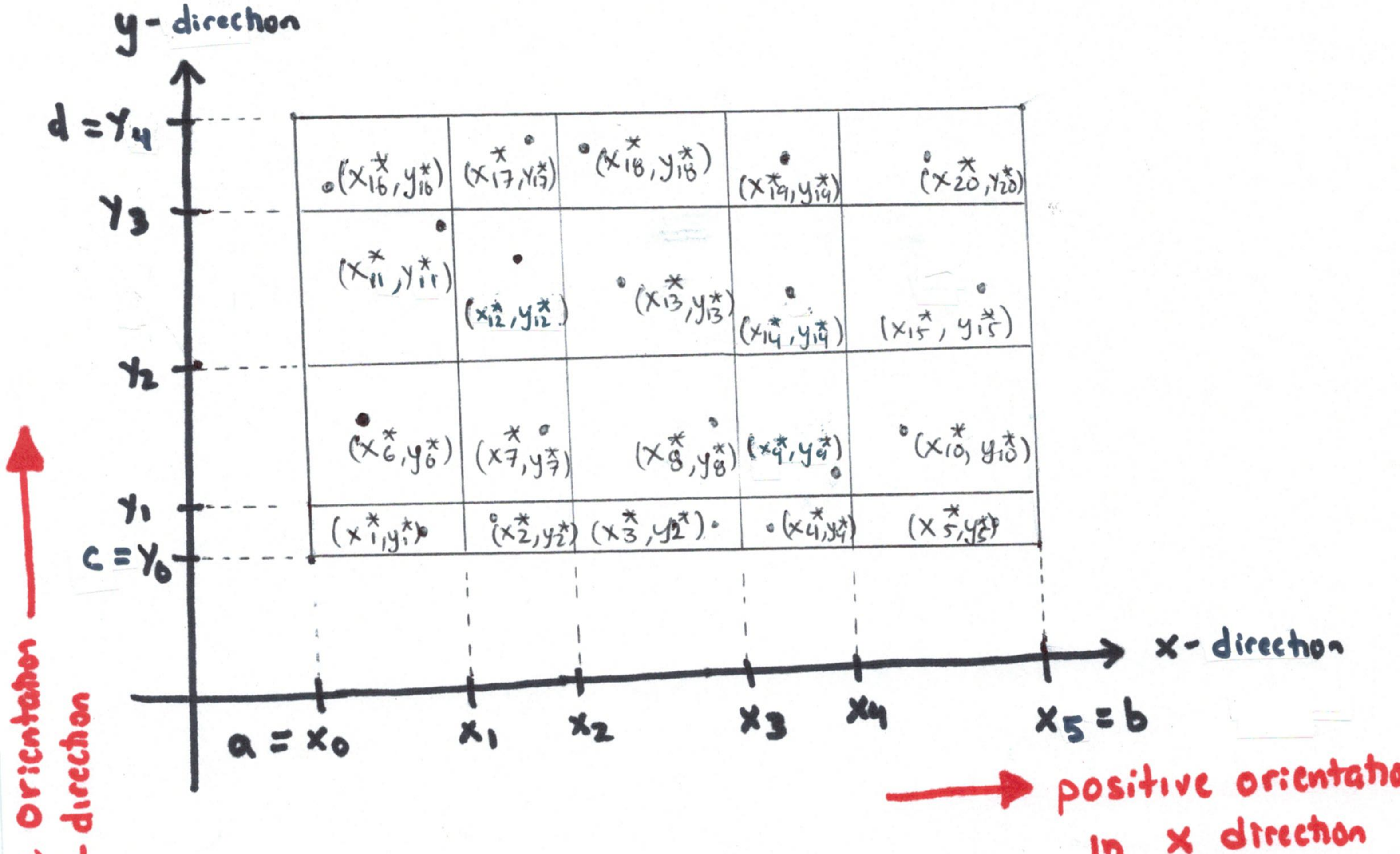
Let $z = f(x,y)$ be a nonnegative, explicit function representation of a surface in $\mathbb{R}^3$ with

$$f : \underbrace{D}_{\substack{\text{Domain region} \\ \text{(two-variable input)}}} \subseteq \mathbb{R}^2 \longrightarrow \underbrace{\mathbb{R}}_{\substack{\text{codomain} \\ \text{(real-valued output)}}}$$

where D is assumed to be a rectangular region given by

$$D = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$
$$= \{(x,y) : x \in [a,b],\ y \in [c,d]\}$$

We form a <u>general</u> partition of D by dividing D into n rectangular subregions using lines parallel to the x- and y-axis. In the case of a general partition, these lines are not necessarily uniformly spaced.



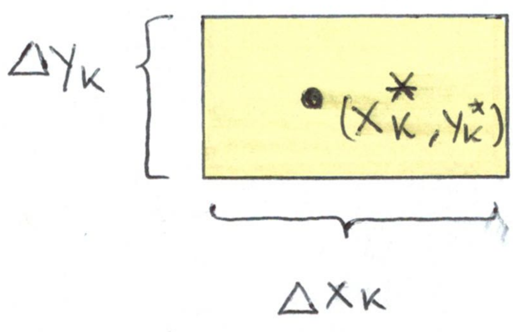
In the diagram above, $n = 20$

The rectangles forming each subregion may be numbered via any systemic method. The orientation of doman region is useful to keep in mind for this mapping.

We denote the "size" of the Kth rectangle as

$$\Delta A_K$$

A natural choice to measure size is to use the area of this rectangle



If we set :

Δx_k = "size" of subinterval in x

along kth rectangular subregion

Δy_k = "size" of subinterval in y

along kth rectangular subregion

then, the "area" of the kth rectangular subregion will be given by

$$\Delta A_k = \Delta x_k \, \Delta y_k$$

Finally, we let (x_k^*, y_k^*) be any point in the kth rectangular subregion for $k \in \{1, 2, \ldots, n\}$.

To estimate the "volume" of the solid bounded by the surface $z = f(x, y)$ and the region D, we construct n boxes using each of the n-rectangular subregions of D, with

$$f(x_k^*, y_k^*) = \text{"height" of } k\text{th box}$$

$$\Delta A_k = \text{"area" of base of } k\text{th box}$$

for $1 \le k \le n$.

Then, the volume of the Kth box is

$$V_K = f(x_K^*, y_K^*)\,\Delta A_K$$

$$= f(x_K^*, y_K^*)\,\Delta x_K\,\Delta y_K$$

Moreover, we can approximate the exact volume of the solid by summing the volume of the n boxes

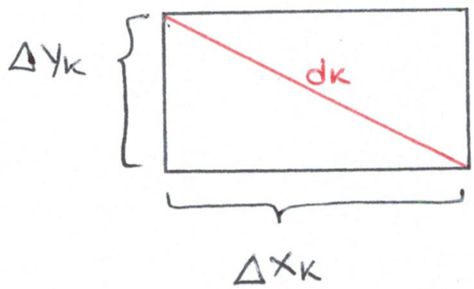
$$V \approx \sum_{K=1}^{n} V_K = \sum_{K=1}^{n} f(x_K^*, y_K^*)\,\Delta A_K$$

$$= \sum_{K=1}^{n} f(x_K^*, y_K^*)\,\Delta x_K\,\Delta y_K$$

To make this approximation exact, we set

$$\Delta = \text{max length of diagonal of rectangles.}$$

$$= \max_{k} \{ d_k \}$$



Let $d_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$

$$\Rightarrow \Delta = \max \{ d_1, d_2, d_3, \cdots, d_n \}$$

Notice that as $\Delta \to 0 \Rightarrow d_k \to 0$ for all k

$$\Rightarrow \Delta A_k \to 0$$

$$\Rightarrow n \to \infty$$

Moreover, we say the double integral of

f over D is given by

$$\iint_D f(x,y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta x_k\, \Delta y_k$$

Note on notation:

I might choose to be a little more

cavelier with my notaton

$$\int_D f\, dA = \iint_D f(x,y)\, dA = \iint_D f\, dA$$

the double integral notation suggests $D \subseteq \mathbb{R}^2$ and thus that $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

Example 13.1.1 and 13.1.2 p. 965 – 967

Evaluate $\iint_D f(x,y)\, dA$ where

$f(x,y) = 6 - 2x - y$ and domain

$D = \{(x,y) : 0 \le x \le 1 \ \& \ 0 \le y \le 2\}$

Solution 1: Consider the integral

$$V = \iint_D f(x,y)\, dA = \iint_D 6 - 2x - y \ dA$$

differential form to measure "size" of subregions in domain region

Using the general "slicing" method in which we cut solid using planes in the $\langle 1,0,0 \rangle$ direction with equation $x = \alpha$ for $a \leq \alpha \leq b$ we can solve the volume problem in two steps:

Step 1: Use "single-variable" integration to find area under each yz-trace as a function of x

Step 2: Integrate over all values of x in D to find total volume.

Then $V = \iint_D 6-2x-y \, dA$

$$= \int_0^1 A(x)\, dy \qquad \text{where } A(x) = \int_0^2 6-2x-y\, dy$$

Side note:

If we treat x as a "constant" and find area under yz trace for $0 \le x \le 1$, then

$$A(x) = \int_0^2 6-2x-y\, dy$$

$$= 6y - 2xy - \frac{y^2}{2} \Big|_0^2$$

$$= \left(6\cdot 2 - 2x\cdot 2 - \frac{2^2}{2}\right)$$

$$- \left(6\cdot 0 - 2x\cdot 0 - 0^2\right)$$

$$= 12 - 2 - 4x - 0$$

$$= 10 - 4x$$

$$\Rightarrow \quad V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 10 - 4x \, dx$$

$$= 10x - 2x^2 \Big|_0^1$$

$$= (10\cdot 1 - 2\cdot 1^2) - (10\cdot 0 - 2\cdot 0^2)$$

$$= 10 - 2$$

$$= \boxed{8} \text{ units}^3$$

L1, p1

Solution 2: In the previous solution 1,

we used yz traces to find volume.

We could have also sliced up the solid using

xz-traces for all y-values between $0 \leq y \leq 2$:

$$V = \iint_D f(x,y)\, dA$$

$$= \int_0^2 A(y)\, dy \qquad \text{where} \qquad A(y) = \int_0^1 f(x,y)\, dx$$

Side note:

$$A(y) = \int_0^1 6-2x-y\, dx$$

$$= 6x - x^2 - xy \Big|_0^1$$

$$= 5 - y$$

$$\Rightarrow \quad V = \int_0^2 A(y)\, dy$$

$$= \int_0^2 5 - y \, dy$$

$$= 5y - \frac{y^2}{2} \Big|_0^2$$

$$= \left(5 \cdot 2 - \frac{2^2}{2}\right) - \left(5 \cdot 0 - \frac{0^2}{2}\right)$$

$$= 10 - 2$$

$$= \boxed{8} \checkmark$$

Notice that the two solution mechanisms produce the same answer ... Hmm?

In other words, we have

$$V = \int_D f \, dA = \iint_D f(x,y) \, dA$$

$$= \int_a^b A(x) \, dx$$

$$= \int_{x=a}^{x=b} A(x) \, dx \qquad \text{where} \qquad A(x) = \int_c^d f(x,y) \, dy$$

$$\Rightarrow A(x) = \int_{y=c}^{y=d} f(x,y) \, dy$$

assume each $x \in [a,b]$ is held constant and find area under yz trace

$$= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y) \, dy \right) dx$$

evaluate inner integral with respect to y first, holding x fixed which results in a function of x.

$$= \int_a^b \int_c^d f(x,y) \, dy \, dx$$

To finish, integrate outer integral $\int_a^b A(x) \, dx$

L1, p15

We also saw that

$$V = \int_D f \, dA = \iint_D f(x,y)\, dA$$

$$= \underbrace{\int_c^d A(y)\, dy}_{\text{outer integral}} \quad \text{where } A(y) = \underbrace{\int_a^b f(x,y)\, dx}_{\text{inner integral}}$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

order of integration in iterated integral:

1st: hold y constant and move over all x to get function of y

2nd: integrate each function of y over bounds in y

Notice that our differential form dA to measure "size" of subregion in domain is $dA = dx\,dy$ or $dA = dy\,dx$ depending on which iterated slicing technique we use.

Theorem 13.1. *p. 967* ***Fubini's Theorem for Double Integrals on Rectangular Regions***

Let $f(x, y)$ be continuous on the rectangular regions

$$D = \{\,(x, y)\;:\; a \le x \le b,\; c \le y \le d\,\}.$$

Then, the double integral of f over D may be evaluated by either of the two iterated integrals:

$$\iint\limits_{D} f(x, y)dA = \int\limits_{c}^{d}\int\limits_{a}^{b} f(x, y)\; dx\, dy = \int\limits_{a}^{b}\int\limits_{c}^{d} f(x, y)\; dy\, dx$$

In other words, the double integral may be evaluated as iterated integrals and the order of integration does not matter. In practice, one order of integration is often easier to evaluate than the other order.

Proof:

Example 13.1.3 p. 968 - 969

Find the volume of solid bounded

by the surface

$$f(x,y) = 4 + 9x^2y^2$$

over the region $D = \{(x,y) : -1 \le x \le 1 \text{ and } 0 \le y \le 2\}$

Solution: First we note that $f(x,y) \ge 0$ ✓

Next, we consider

$$V = \int_D f \, dA = \iint_D f(x,y) \, dA$$

$$= \int_a^b A(x) \, dx \qquad \text{where } A(x) = \int_c^d f(x,y) \, dy$$

$$= \int_c^d A(y) \, dy \qquad \text{where } A(y) = \int_a^b f(x,y) \, dx$$

L1, P1

$$\Rightarrow \iint_D f(x,y)\,dA = \int_a^b A(x)\,dx$$

Side note:

$$A(x) = \int_c^d f(x,y)\,dy$$

$$= \int_0^2 4 + 9x^2y^2\,dy$$

$$= 4y + 3x^2y^3 \Big|_0^2$$

$$= 4\cdot 2 + 3\cdot x^2 \cdot 2^3 - 0$$

$$= 8 + 24x^2$$

$$= \int_{-1}^{1} 8 + 24x^2\,dx$$

$$= 8x + 8x^3 \Big|_{-1}^{1}$$

$$= (8\cdot 1 + 8\cdot 1^3) - (-8 - 8)$$

$$= 16 + 16 = \boxed{32} \checkmark$$

Notice also that

$$\iint_D f(x,y)\,dA = \int_c^d A(y)\,dy$$

Side note:

$$A(y) = \int_a^b f(x,y)\,dx$$

$$= \int_{-1}^{1} 4 + 9x^2y^2\,dx$$

$$= 4x + 3x^3y^2 \Big|_{-1}^{1}$$

$$= (4 + 3y^2) - (-4 - 3y^2)$$

$$= 8 + 6y^2$$

$$= \int_0^2 8 + 6y^2\,dy$$

$$= 8y + 2y^3 \Big|_0^2$$

$$= (8\cdot 2 + 2\cdot 2^3) - 0$$

$$= 16 + 16 = \boxed{32}\ \checkmark$$

Example 13.1.4 p. 969

Evaluate $\iint_D f(x,y)\, dA$ where $f(x,y) = y\, e^{x \cdot y}$

and $D = \{(x,y) : 0 \le x \le 1$ and $0 \le y \le \ln(2)$

Solution: Based on Fubini's theorem, we can find this volume using either integrated integral. However, one method is significantly less difficult than the other:

Method 1: $\iint_D f(x,y)\, dA = \int_a^b A(x)\, dx$ where $A(x) = \int_c^d f(x,y)\, dy$

OR Method 2: $\iint_D f(x,y)\, dA = \int_c^d A(y)\, dy$ where $A(y) = \int_a^b f(x,y)\, dx$

Method 1: $$\iint_D f(x,y)\,dA = \int_0^1 A(x)\,dx$$

Side note:

$$A(x) = \int_0^{\ln(2)} y \cdot e^{xy}\,dy$$

let $u = y$

$dv = e^{xy}\,dy$

$\Rightarrow v = \dfrac{e^{xy}}{x}$

$du = 1$

$$= \frac{y}{x} e^{xy}\Big|_0^{\ln(2)} - \int_0^{\ln(2)} \frac{e^{xy}}{x}\,dy$$

$$= \frac{\ln(2)}{x} \cdot e^{\ln(2)\cdot x} - \frac{e^{xy}}{x^2}\Big|_0^{\ln(2)}$$

$$= \frac{\ln(2)\, e^{\ln(2)\cdot x}}{x} - \frac{e^{\ln(2)x}}{x^2} + \frac{1}{x^2}$$

AHH! This is extremely ugly!

No thank you.

Method 2: $$\iint_D f(x,y)\, dA = \int_c^d A(y)\, dy$$

Side note:

$$A(y) = \int_a^b f(x,y)\, dx$$

$$= \int_0^1 y e^{xy}\, dx$$

$$= e^{xy} \Big|_0^1$$

$$= e^y - 1$$

$$= \int_0^{\ln(2)} e^y - 1 \, dy$$

$$= e^y - y \Big|_0^{\ln(2)}$$

$$= \left[e^{\ln(2)} - \ln(2)\right] - \left[e^0 - 0\right]$$

$$= 2 - \ln(2) - 1$$

$$= \boxed{1 - \ln(2)} \checkmark$$