Lesson 14: Green's Theorem

- Recall that in single-variable integral calculus, we studied Part 2 of the fundamental theorem of calculus which stated

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}[F(x)]\,dx$$

$$= F(b) - F(a)$$

This theorem relateds the integral of $f(x) = \frac{d}{dx}[F(x)]$ on region $[a,b] \subseteq \mathbb{R}$ to the values of the function $F(x)$ on the "boundary" of our region $[a,b]$

- In Lesson 13, we studied the fundamental theorem of line integrals (Thm 14.4, p. 1081) which statd that line integrals of conservative vector fields are "path independent".

In other words, the fundamental theorem of line integrals states that the vector field $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ is conservative if and only if

$$\int_{C} \vec{F} \circ d\vec{r} = \int_{C} \vec{F} \circ \vec{T}\, ds$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \circ \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \underbrace{\|\vec{r}\,'(t)\|_2\, dt}_{ds}$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

$$= \int_{a}^{b} \vec{\nabla}\phi(\vec{r}(t))\, \vec{r}\,'(t)\, dt$$

$$= \int_{a}^{b} \frac{d}{dt}\left[\phi(\vec{r}(t))\right] dt$$

$$= \phi(\underbrace{\vec{r}(b)}) - \phi(\underbrace{\vec{r}(a)})$$

$$= \phi(B) - \phi(A)$$

- This fundamental theorem of line integrals relates the "line" integral of a gradient vector field

$$\vec{\nabla}\phi(x,y) = \langle \phi_x(x,y), \phi_y(x,y) \rangle = \vec{F}(x,y)$$

on a piecewise-smooth, oriented curve C to the values of potential function $\phi(x,y)$ on the boundary of the curve. It's worth noting that the boundary of the curve C consists of the two endpoints of the curve, call them $A = \vec{r}(a)$ and $B = \vec{r}(b)$ where $\vec{r}(t)$ is a parameterization of C with $a \leq t \leq b$.

- In each of the remaining lessons of this course, we will extend these fundamental theorems to apply to more general domain regions and integrals in both $\mathbb{R}^2$ and $\mathbb{R}^3$. In the following pages, we will give a brief overview of these extensions and generalizations.

The cummulative (integrated) effect of the derivatives) of a function throughout a region is determined by the "values" of the function on the boundary of that region

Table 14.4 p. 1144 Section 14.8:

The Fundamental Theorem of (Single-variable) Calculus

(Foothill's Math 1B course)

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}[F(x)]\,dx$$

$$= F(b) - F(a)$$

The Fundamental Theorem of line integrals

(Lesson 13 of this Math 1D course)

$$\int_C \vec{F} \circ d\vec{r} = \int_C \vec{F} \circ \vec{T}\,ds$$

$$= \int_C \vec{F}(\vec{r}(t)) \circ \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \underbrace{\|\vec{r}\,'(t)\|_2\,dt}$$

$$= \int_C \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\,dt$$

$$= \int_C \vec{\nabla}\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[\phi(\vec{r}(t))\right] dt$$

$$= \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

$$= \phi(B) - \phi(A)$$

- Relates the line integral of a vector field over a simple, closed curve in $\mathbb{R}^2$ to a double integral over the region enclosed by this curve

→ Green's Theorem in Circulation Form

(Lesson 14 of this Math 1D course)

$$\oint_C \underbrace{\vec{F} \cdot d\vec{r}}_{\text{circulation}} \text{ "=" } \oint_C \langle f(t), g(t) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$\text{"="} \oint_C f\,dx + g\,dy$$

$$= \oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \oint_C \vec{F} \cdot \vec{T} \, ds$$

$$= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$= \iint_D \underbrace{(g_x - f_y)}_{\text{curl or rotation in } \mathbb{R}^2} dA$$

$$\text{"="} \iint_D (\vec{\nabla} \times \vec{F}) \cdot \vec{k} \, dA$$

where C is a simple, closed, piecewise-smooth curve oriented counterclockwise that encloses the connected and simply-connected region $D \subseteq \mathbb{R}^2$ (i.e. C is the boundary of region D)

Jeff's Thought Bubble: let's kick street knowledge (or abuela language)

- ☐ The net or accumulated rotation of a vector field over all points in a 2D region $D \subseteq \mathbb{R}^2$ is equal to the net circulation (as a line integral) on the boundary C of region D.
- ☐ If $\vec{F}$ models a "fluid flow", then Green's Thm in circulation form says that the net or cumulative rotation of the flow over all points inside D equals the circulation along the boundary C of region D.

Relates the line integral of a vector field $\vec{F} = \langle f, g \rangle$ where $\vec{F}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ over a simple, closed, oriented curve in a plane to a double integral over the region enclosed by this curve

Green's Theorem in Flux form: $\displaystyle\oint_C \vec{F} \circ \vec{n}\, ds = \oint_C \vec{F}(\vec{r}(t)) \circ \vec{n}(t)\, ds$

(Lesson 14 of this Math 1D course)

$$= \oint_C \langle f(t), g(t) \rangle \cdot \langle y'(t), -x'(t) \rangle\, dt$$

$$= \oint_C f(t) \cdot y'(t) - g(t)\, x'(t)\, dt$$

$$= \oint_C f\, dy - g\, dx$$

$$= \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$$

$$= \iint_D (f_x + g_y)\, dA$$

where C is a simple, closed, piecewise-smooth curve oriented counterclockwise that encloses the connected and simply-connected region $D \subseteq \mathbb{R}^2$ (i.e. C is boundary of D)

Jeff's Thought Bubble: let's translate this into more intuitive language

- ☐ The net or accumulatated "expansion" or "contraction" of a vector field over all points inside a 2D region $D \subseteq \mathbb{R}^2$ is equal to the net flux (as a line integral) on the boundary C of region D.

- ☐ If $\vec{F}$ models "fluid flow" or the transport of material, then the flux form of Green's Thm says that the cumulative effect of the sources (or sinks) of the flow within D equals the net flow across the boundary C of region D.

Relates the line integral over a simple, closed, oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is that curve (extends Green's thm in circulation form

Stoke's Theorem (the 3D version of the circulation form of Green's Theorem)

(Lesson 17 of this math 1D course)

$$\oint_C \vec{F} \circ d\vec{r} = \oint_C \langle f, g, h\rangle \cdot \langle dx, dy, dz\rangle$$

$$= \oint_C f\,dx + g\,dy + h\,dz$$

$$= \oint_C \vec{F}(x(t), y(t), z(t)) \circ \vec{r}\,'(t)\,dt$$

$$= \oint_C \vec{F} \circ \vec{T}\,dt$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \circ \vec{n}\,dS$$

where S is an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed curve C around it's boundary whose orientation is consistent with the orientation of surface S and $\vec{n}$ is the unit normal vector to S determined by the orientation of S.

Jeff's Thought Bubble: let's spit street Knowledge (or abuela language)

- ☐ Let's interpret stoke's theorem using abuela language: under certain conditions, the net or accumulated rotation of a vector field over a surface S (as given by the normal component of the curl) equals the net circulation (as a line integral) on the boundary C of surface S.

Relates an integral over a closed, oriented surface $S \subseteq \mathbb{R}^3$ to a triple integral over the region $D \subseteq \mathbb{R}^3$ enclosed by that surface (extendeds Green's Thm in Flux form to $\mathbb{R}^3$)

→ Divergence Theorem (the 3D version of the flux form of Green's Thm) (Lesson 18 of this math 1D course)

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV$$

$$= \iiint_D \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_D (f_x + g_y + h_z)\, dV$$

$$= \iiint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dV$$

where $S \subseteq \mathbb{R}^3$ is a closed, oriented surface that encloses a connected and simply-connected region $D \subseteq \mathbb{R}^3$.

Jeff's Thought Bubble:

- Under certain conditions, the net or accumulated expansion (or contraction) of a vector field over every point in a region $D \subseteq \mathbb{R}^3$ (as given by the cumulative divergence) equals the net flux (as a surface integral) of the vector field across the boundary S of the region D.

☐ In Lesson 14, we focus our attention on developing two different forms of Green's Theorem including:

Part I: Green's Thm in Circulation Form: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (g_x - f_y)\, dA$

Part II: Green's Thm in Flux Form $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (f_x + g_y)\, dA$

Each of these theorems will relate the double integral of the "derivatives" of a vector-valued function over all points inside a region $D \subseteq \mathbb{R}^2$ to line integrals on the boundary C of the region D.

Circulation Form of Green's Theorem

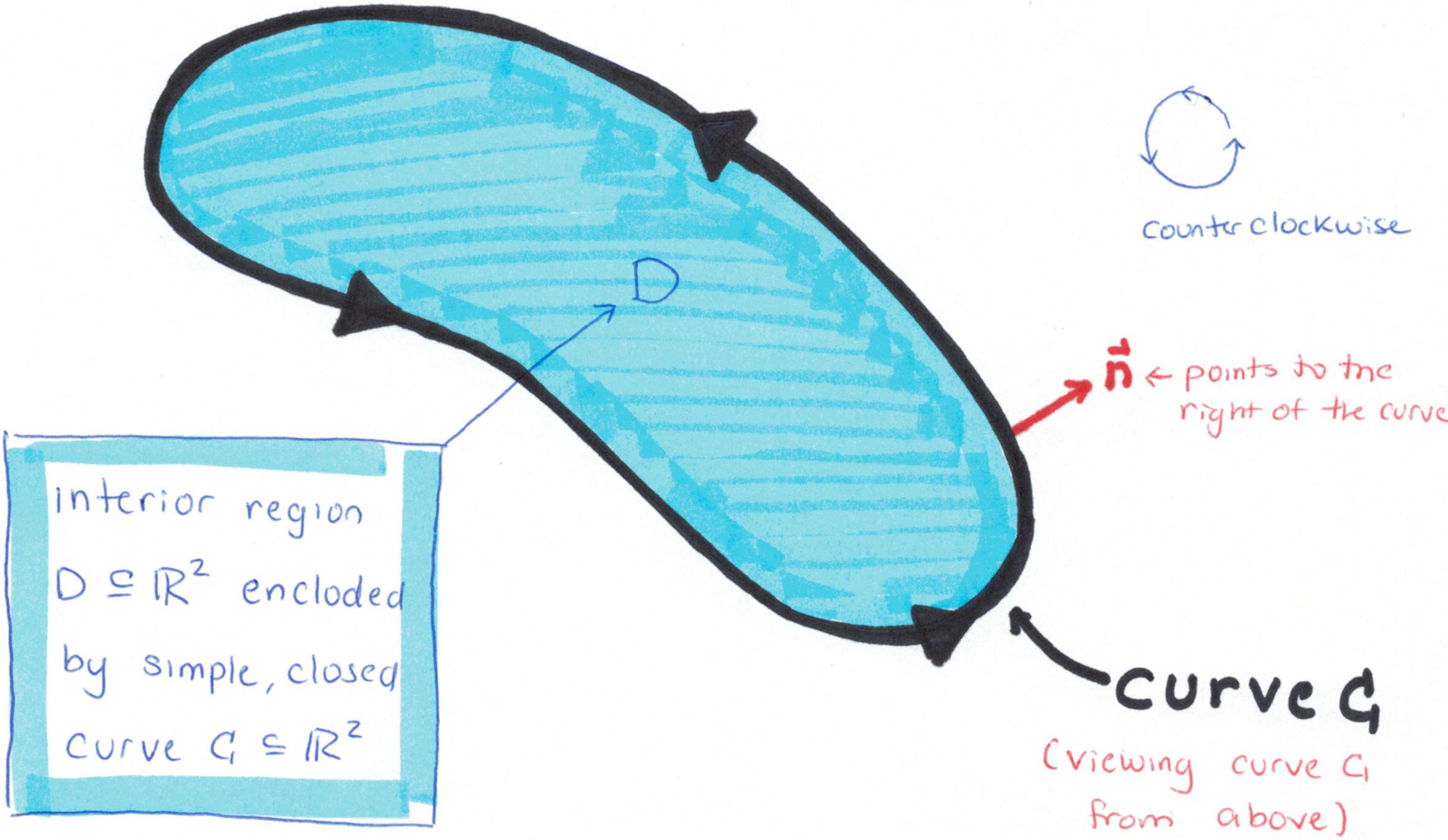
- Throughout our work in this lesson, we assume that any curve in $\mathbb{R}^2$ that we use is simple, closed, piecewise-smooth, and oriented, unless otherwise stated.

- There is a famous (and sophisticated) result in Topology known as the Jordan Curve Theorem* that implies that all closed, simple curves $C \subseteq \mathbb{R}^2$ have a well-defined interior region such that when we traverse the curve in the counterclockwise direction (as viewed from above), the interior of the region is on the left.

* please see wikipedia article: "Jordan Curve Theorem"
Also, welcome to upper-division / graduate level math where we quote and use results that require years of deep thought to understand and prove ... :)
Now, the fun part of your journey can beginning...

- Let's visualize this scenario on the next page:

☐ With our counterclockwise orientation



we have a unique outward-pointing unit normal vector $\vec{n} \in \mathbb{R}^2$ that points to the right of the curve (at all points that the curve is smooth.

☐ We've also assume that our curve $C \subseteq \mathbb{R}^2$ is contained in a larger region that is BOTH connected and simply connected.

- Suppose the vector field

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

is defined with $F: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $D \subseteq \mathbb{R}^2$ is the interior region whose boundary is the closed curve $C \subseteq \mathbb{R}^2$.

- The first form of Green's Theorem, known as Green's Theorem in Circulation form relates the circulation of $\vec{F}$ on C to a double integral over region D of a quantity that measures the "rotation" of $\vec{F}$ at each point $(x,y) \in D$.

Before we give the formal statement of Green's Theorem in Circulation form, let's develop our intuition on some of the important ideas underlying this theorem

- Recall that the circulation of $\vec{F}$ on C_1 is given by the "line" integral

$$\oint_{C_1} \vec{F} \cdot \vec{T}\, ds = \oint_{C_1} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \underbrace{\|\vec{r}\,'(t)\|_2\, dt}_{ds}$$

dot product

$$= \oint_{C_1} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \oint_{C_1} \vec{F}(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle\, dt$$

$$= \oint_{C_1} \langle f(x(t), y(t)), g(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle\, dt$$

scalar multiplication

$$= \oint_{C_1} f(x(t), y(t)) \cdot x'(t) + g(x(t), y(t)) \cdot y'(t)\, dt$$

$$\Rightarrow \oint_C \vec{F}\cdot\vec{T}\,ds = \oint f(x(t),y(t))\,x'(t) + g(x(t),y(t))\cdot y'(t)\,dt$$

$$\text{"="} \oint_a^b f(t)\,x'(t) + g(t)\,y'(t)\,dt$$

fuzzy math: expert notation that is mathematically spurious

$$\text{"="} \oint_a^b f(t)\underbrace{x'(t)\,dt}_{dx} + g(t)\underbrace{y'(t)\,dt}_{dy}$$

$$\text{"="} \oint_a^b f\,dx + g\,dy$$

$$\text{"="} \oint_C \langle f,g\rangle \cdot \langle dx,dy\rangle$$

dot product

$$\text{"="} \oint_C \vec{F}\; d\langle x,y\rangle$$

dot product

$$\text{"="} \oint \vec{F}\cdot d\vec{r}$$

Recall that $\oint_C \vec{F}\cdot\vec{T}\,ds$ measured the net or accumulated components of $\vec{F}$ in the direction of the unit tangent vectors $\vec{T}(t)$ at every point along C (using our ideas of projection from Math 1C)

Jeff's Thought Bubble: Interpreting Circulation via Modeling Perspectives

- One way to encode the idea of circulation $\oint_C \vec{F}\cdot\vec{T}\,ds$ in our mind is to assume that the vector field $\vec{F} = \langle f, g\rangle$ models the <u>velocity</u> of a fluid moving in $\mathbb{R}^2$. The circulation $\oint_C \vec{F}\cdot\vec{T}\,ds$ gives a measurement of how much the fluid is rotating in the same direction as the orientation of curve C. A great example to keep in mind is Example 14.2.7 b p. 1070 where $\vec{F} = \langle -y, x\rangle$ along the unit circle C centered at the origin in counterclockwise direction. When we measure "rotation", we are speaking about macrorotation (as opposed to micro rotation).

- With the fluid model in mind, we might imagine that a nonzero circulation on the closed curve C should indicate that the vector field must have some special feature(s) inside the curve on region D that produces the circulation on the curve C. In other words, we might expect some connection between the circulation and the behavior inside D.

To build our intuition about Greens theorem in circulation form which states

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (g_x - f_y)\, dA$$

we will investigate the following questions:

$\boxed{\text{Question 1:}}$ ☐ What is $\underbrace{\text{macroscopic rotation}}$ in a vector field?
↳ aka macrorotation

$\boxed{\text{Question 2:}}$ ☐ What is $\underbrace{\text{microscopic rotation}}$ in a vector field?
↳ aka microrotaten

$\boxed{\text{Question 3:}}$ ☐ Can macro rotation exists without microrotation?

$\boxed{\text{Question 4:}}$ ☐ Can microrotation exist without macrorotation?

$\boxed{\text{Question 5:}}$ ☐ How do we measure, using numbers, the macrorotation of a vector field?

$\boxed{\text{Question 6:}}$ ☐ How do we quantify* miccorotations of a vector field?

*Jeff's Thought Bubble: **Note** on notation - the verb quantify

☐ One of the most beautiful, challenging, and subtle assumptions of applied math is that we (as scientists) can observe complex phenomena and introduce numerical measurement systems to collect data associated with such phenomena. This process is what we mean when we use the verb "to quantify."

Let's begin our discussion with an attempt to answer $\boxed{\text{question 1:}}$

☐ What is "macroscopic rotation" in a vector field?

From this point on, we will focus our minds to assume that the vector fields we study model the velocity of a fluid. In other words, when we write

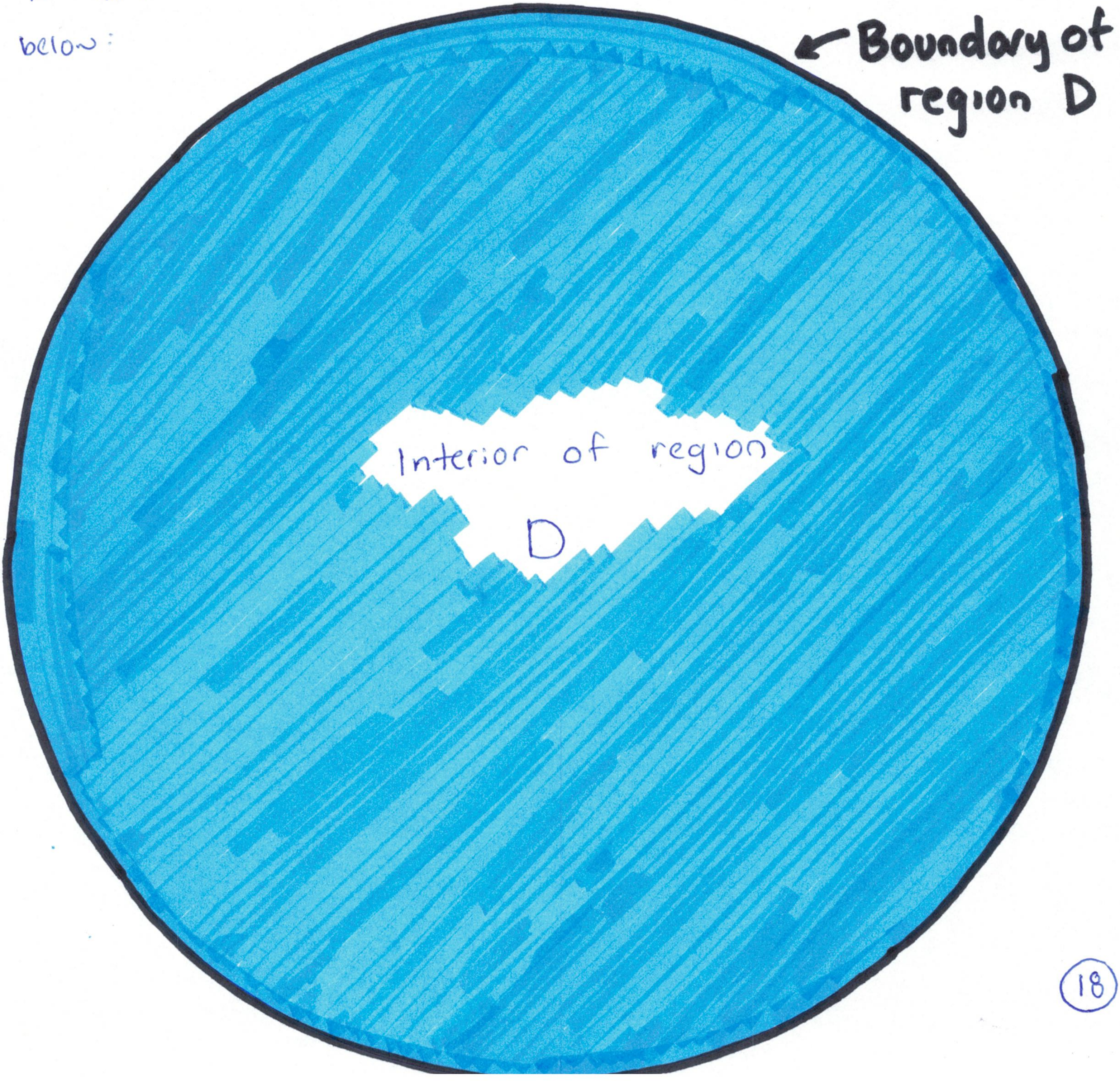
$$F(x,y) = \langle f(x,y), g(x,y) \rangle$$

$$= \langle f(x,y), 0 \rangle + \langle 0, g(x,y) \rangle$$

$$= f(x,y) \cdot \check{i} + g(x,y) \cdot \vec{j}$$

with $F: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we will impose a bunch of model-based meaning on each of the symbols.

To this end, suppose that the region $D \subseteq \mathbb{R}^2$ is a mathematical model of a two-dimensional region in space that is covered with a layer of fluid. For simplicity, let's assume our region D is a circle and our fluid is water. We "draw" a diagram to symbolize this situation below:



☐ One very important aspect of this modeling paradigm is that we will assume we are modeling "fluid statics"*; focusing on modeling velocity vectors

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

at a single point in time.

☐ To create such a static* model, we must imagine we have a very special super power. In particular, when looking down on our region of water from above, we must imagine watching the water flow, mix, swirl, and move over time (a dynamic* system). As we watch the water move over time, we take a snapshot or picture of the fluid at an instant in time.

* Jeff's Thought Bubble: what's up with the word "statics"

☐ This modeling paradigm is an introduction to the field of fluid mechanics. Fluid mechanics is a subfield of the larger field of mechanics. Mechanics is the physical science that studies, explains, and models the effects of forces on objects.

☐ Mathematical theory and techniques allow us to rigourously express and develop the principles of mechanics. Thus, mathematics will play a crucial role in solving practical problems via applied math modeling.

* Jeff's thought bubble continued ...

- The field of mechanics is partitioned into two subfields which are known as

Statics :
- the study of objects/bodies under action of forces without explicit reference to changes in time
- usually involves analysis of equilibrium positions of individual objects in a larger system that includes many different objects and forces interacting with each other at a single, static point in time.

Dynamics :
- the study of objects/bodies under action of forces that may change over time and includes models that explicitly account for changes in time
- usually involves analysis of differential equations that model changes in position of individual objects in a larger system that includes many different objects and variable forces interacting dynamically with each other over time.

Jeff's Thought Bubble continued...

- ☐ In our case, we will be focusing on creating a mathematical model to desribe the position and velocity of each drop of fluid (water) in our region at an instant in time. Thus, our modeling paradigm falls into the field of fluid Statics. We start with the static case rather than the dynamic phenomena to make our introduction to this modeling scenario more accessible and less complicated. The over all goal, of course, is to fully understand the static model so that we can generalize to the more interesting case of fluid dynamics.

- ☐ Remember, we are about to introduce the capstone results from 400+ years of development of the field of Calculus. As your coach, mentor, and guide, I implore you to create, develop, and refine a very strong modeling-based component of your concept image associated with the mathematica ideas we are studying. My suggestion (and approach) is to focus on developing strong intuition about these ideas by focusing on a model for the "position" and "velocity" of water molecules in a region at a static/constant instant in time.

(the moment we take our photo)

- At that instant, not only do we have the (super human) ability to look at the photo and identify each individual drop of water in the entire picture, but we also can assign each drop of water <u>three</u> important quantities that will help to describe the behavior of the drop at the instant the photo was taken. These include:

quantity 1

- for each drop of water we see, we can assign a position (x,y) to quantify the precise location of this drop in our region at the time the photo was taken.

Quantity 2&3

- for each drop, we can assign a velocity vector $\vec{F} = \langle f, g \rangle$ that has two components:

quantity 3

$$\vec{F} = \vec{F}(x,y) = \langle f, g \rangle$$

quantity 2

$$= \langle f(x,y), g(x,y) \rangle$$

$$= \boxed{\langle f(x,y), 0 \rangle} + \boxed{\langle 0, g(x,y) \rangle}$$

$$= f(x,y) \cdot \vec{i} + g(x,y) \cdot \vec{j}$$

Jeff's Thought Bubble: Is this model really only base on three quantities?

- On the last page, I claimed that our model of water focused on assigning each drop of water three quantites. <u>I LIED</u>! Or, better said, I $\boxed{\text{vastly oversimplified}}$ our modeling paradigm.

- In reality, the phenomena of the motion of water molecules in a region is way too complex to be described by three ideas! However, a quantity (measurement) usually encodes many ideas implicitly hidden underneith. To be sure that we really develop all the underlying ideas properly and make the connections between the quantities we model and the physical phenomena we are hoping to describe, let's recapitulate the phenomena we are trying to model:

MODELING SCENARIO

We begin by assuming that we watch a fluid (water) from above as the fluid moves, mixes, flows, swirls, splashes and bubbles. Then, at some point during our observation, we take a picture of the fluid (water) with a magic camera...

This MODELING SCENARIO is incredibly complex. Let's break it down into the individual, basic concepts that we need to identify to create our model:

Concept 1: Space

- The body of water that we are watching exists in <u>physical space</u>. If the water evaporated, we could still see and touch physical objects that created a boundary/bed/region in which the water existed.

- This physical space can be modeled as a geometric region and each "point" in this space can be assigned a position.

- For a two-dimensional position model, we assign position values using two coordinates. In this case, we assume the space we study has "only two degrees of freedom"

- For a three-dimensional position model, we assign position data using three coordinates.

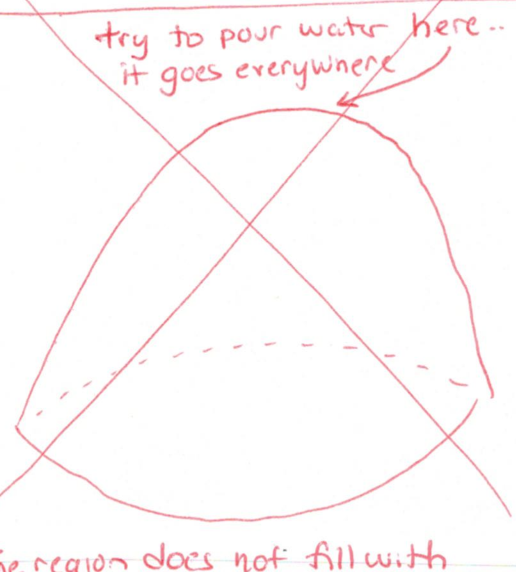
Concept 1: Space, continued...

- As described by concept 1, we imagine observing a physical space that has objects we can touch & feel. In our case, since we are trying to model fluid, this space should probably ...... vaguely resemble a bath tub, sink, lake bed, etc... In other words, if we pour a bunch of water into this space, the water drops should tend to collect, build up, and stay together:

Yes: water fits inside

⇒ The region fills with water and then we can observe

NO: water doesn't build up and collect



The region does not fill with water and we can't observe the macro behavior of all the water drops we poured easily

Concept 2: Mass

- ☐ Once we have chosen a physical space to observe that can hold a huge number of water droplets together, we now actually have to fill that space with water (fluid)

- ☐ As we fill the space up with water, we must imagine pouring enough water to be able to study the properties of moving water molecules as they interact with each other. Thus, we assume that we observe a "non-trivial" amount of water. In other words, we need a body of water that has a relatively large mass,

- ☐ Formally speaking, mass is a measurement of the inertia of a body (the resistence of the body to changes in velocity). However, to build more useful intuition, we might think about mass as a measurement of the quantity of matter inside our body. The point here is that we need many, many, many drops of water grouped together in space so we can observe what happens as the water flows & moves.

Concept 2: Mass, continued...

- In our models of fluid statics (and the more complex analog models of fluid dynamics), we will study the macroscopic motion of the fluid. To do so, we must assume that any "small" area (in $\mathbb{R}^2$) or volume (in $\mathbb{R}^3$) of fluid still contains a very large number of molecules of the fluid (water).

- We also assume that the fluid (water) is a continuous medium so that the observations we make and the models we contruct are based on an assumption of continuity in the macroscopic sense. Accordingly, when we introduce limits and infinitely small elements of area (in $\mathbb{R}^2$) or volume (in $\mathbb{R}^3$), we think of elements that are "physically" infinitely small compared with the total area / volume of the entire mass of fluid (water).

Concept 3: Particles

- Now that we have a physical space that we are observing filled with a (relatively) large, continuous mass of water, we also need to be able to distinguish and identify individual water "particles"

- For our purposes, we might think of being able to "see" individual drops of water or individual water molecules. Such an assumption is as if we wear a pair of magical glasses that have a super human "zoom" feature.
- If the "zoom" feature is off, we can look at the water from above and see the macroscopic behavior of the water: how it flows, moves, swirls, and mixes.

- If we turn the "zoom" feature on, we can now continue to see the water move throughout the entire region in space. However, we can also pick out and focus on any water molecule(s) at any "point" in our body of water and track its motion.

Concept 3, continued ...

- ☐ While this "zoom" feature is incredible because it allows us to see microscopic water particles within the macroscopic mass of water, the "zoom" feature also has other magical properties, as described below:

- ☐ The zoom feature allows us to focus on a single water molecule in the region or to see a whole collection of many molecules or to see every single molecule in the entire mass of water. WOW!
(maybe we should figure out how to manufacture these and become filthy **rich**? But only if we are then taxed at $\geq 50\%$ to combat income inequality !!).

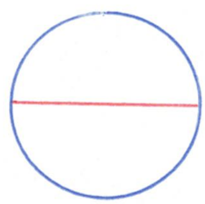
- ☐ The ability to see individual water particles is very important to us as applied mathematicians. Specifically, these individual water molecules do occupy space and do have geometric and physical properties that can be measured including "length", "width", "height", "mass", "density", "area", "volume" (for 3D objects), "surface area" (for 3D objects), "pressure", etc.

Concept 3: Particles, continued...

- In our case, we choose to observe the top layer of our mass of water and we restrict our observations to a two-dimensional space.

  Thus, when we turn on the zoom feature of our magical glasses, we see each water particle as a circle on the top of our mass of water.

diameter = h units
(we might think that h is measured in $1 \mu m = 10^{-6} m$)



← individual water molecule represented as a "circle" in space on the top layer of our mass of water (fluid)

- This individual water molecule occupies an infinitely small area and has infinitely small dimensions (radius, diameter, area, circumference mass) compared to the dimensions of the surface area of water in the region we are observing.

- Experimentally, we might say that the macroscopic region we observe has an area measured in meters while the size of each water molecule is measured in micrometers so that the dimensions of each water particle are effectively "infinitely" small compared to our macroscopic region.

Concept 3: Particles, continued ...

- For the purposes of our mathematical models, the diameter h of each particle of fluid (water) being infinitely small compared to the area of the region of our observation of the macroscopic motion of the continuous fluid implies that h is our differential element of the continuous macroscopic mass of fluid.

- Thus, each particle of water has a mass concentrated at a single "point".

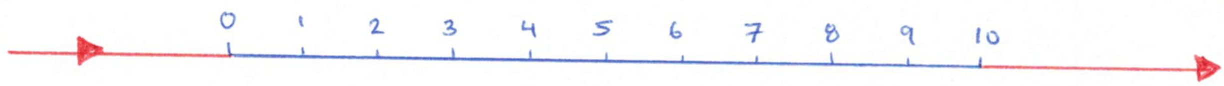
- With this in mind, the phrases "fluid particle" and a "point in a fluid" are understood to be similar for our purposes.

Concept 3: Particles, continued...

- A useful analogy is in order:
  - Imagine we consider the interval

$$[0, 10] = \{ x \in \mathbb{R} : 0 \leq x \leq 10 \}$$

we can visualize this interval on the real number line:



- Although we know there are infinitely many real numbers $x \in [0, 10]$, we might say that when studying the macroscopic geometry of this interval, we can break up the intervals into "particles" (subintervals) of length $10^{-5} = \frac{1}{10^5} = \frac{1}{100000}$

$$\Rightarrow [0, 10] = \underbrace{\left[0, \tfrac{1}{10^5}\right]}_{\text{1st particle}} \cup \underbrace{\left[\tfrac{1}{10^5}, \tfrac{2}{10^5}\right]}_{\text{2nd "particle"}} \cup \underbrace{\left[\tfrac{2}{10^5}, \tfrac{3}{10^5}\right]}_{\text{3rd "particle"}} \cup \cdots \cup \underbrace{\left[10 - \tfrac{1}{10^5}, 10\right]}_{\text{last "particle"}}$$

Concept 3: Particles, continued ...

☐ If we were then determined to study the macroscopic behavior of a "continuous" function

$$F : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

where the domain $D = [0, 10] \subseteq \mathbb{R}$, we might conclude that any time we constructed a limit as $h \to 0$ involving F, that we consider the step sizes of our limit (the h-values) to be on the same order as the width of each particle.

Concept 4: Time

- Time is a measurement of the succession of events, one after the nexts.

- Time shows up as one of the basic independent variables in fluid dynamics. For example, if we are modeling the motion of a fluid in $\mathbb{R}^3$ over time, we could introduce a multivariable, vector-valued function

$$\vec{v} : D \subseteq \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

given by $\vec{v} = \vec{v}(x, y, z, t) = \langle f, g, h \rangle$

$$= \langle f(x,y,z,t), g(x,y,z,t), h(x,y,z,t) \rangle$$

to assign a variable velocity vector $\vec{v}$ to every particle of fluid at each point in time.

Concept 4: Time, continued...

- One of the fascinating features of our model of a snapshot of the top layer of water at a moment in time is that our modeled velocity function

$$\vec{F} = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \langle f, g \rangle$$

does not include the basic time variable t as an input. This is quite subtle. Instead, we have to imagine that time does exist implicitly underneath our model but that we "ignore" and "supress" any mention of time's effect on motion for the sake of simplicity.

- To do this, we can imagine our fluid exhibits beautiful and complex macroscopic motion over time, described by a velocity function $\vec{v} = \vec{v}(x,y,z,t)$. Then, we take our snapshot of the surface of the water and set the time variable $t = t_0$ constant and the z-variable $z = z_0$ constant so that

$$F(x,y) = v(x, y, z_0, t_0)$$

Concept 5: Distance between particles is "zero"

- Finally, the last basic concept we will agree on is that when studying the macroscopic motion of our fluid, we agree that the mass of the fluid is "continuous" and that the differential element of the fluid is the size of each water particle. We also agree that the fluid is made up of a very great number of individual particles, packed tightly together...

← each particle has diameter h which is the differential element of the mass (i.e. h << Area of the region we observe...) we might say, h is an infinitely small distance compared to the area of the mass of fluid

- While the diameter h of each fluid particle is infinitely small compared to the size measurements of the region we observe, these diameters are much larger compared to the distances between each fluid molecule.

☐ When assigning our "velocity" vector, we should be very careful! Normally (in Single-variable Calculus, Physics, Dynamics, etc.), the term velocity refers to the rate of change of position over time:

$$\text{Velocity} = v(t) = \dot{u}(t)$$

$$= u'(t)$$

$$= \frac{d}{dt}\left[u(t)\right]$$

$$= \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

where the function $u(t)$ meausers the position (or displacement) of a single object at a point in time. This model resides in the study of <u>dynamics</u> since the model assumes that we study the motion of an object <u>over</u> a period of time (time is NOT static/constant).

☐ In particular, the difference quotient

$$\frac{u(t+h) - u(t)}{h}$$

depends on our ability to measure the position of our object at two different points in time t and $t+h$ where h is a step size in time.

☐ While this classic interpretation of velocity is quite helpful in Math 1AB to develop application-based interpretations of derivatives and integrals, we cannot use this interpretation in our development of our model to describe the velocity of water particles in our snapshot of our fluid.

☐ Specifically, in our snapshot of fluid motion in our region, we are not measuring the position of each drop of water at multiple points in time. Instead, each drop of water gets a single position measurement that is static / constant in our picture. Thus, the assignment of a "velocity" vector $\vec{F} = \langle f, g \rangle$ cannot depend on a difference quotient that requires position measurements at different points in time!

☐ Instead, we will do something more sublte that will depend on the position measurement (x, y) and an implicit reference to the macroscopic motion of the fluid over time that "existed" before and after our photo was taken and will be designed into our velocity vector.

□ First, we introduce a coordinate system onto $D \subseteq \mathbb{R}^2$ so that to every "molecule" of water contained in D, we can assign a "location". This location will be written as an ordered pair of real numbers

$$(x, y) \in D$$

where $x, y \in \mathbb{R}$.

□ Notice, since $x, y \in \mathbb{R}$, we know that each of these numbers must have a sign:

Choices for x: $x > 0$ (x positive)

$x = 0$ (x zero)

$x < 0$ (x negative)

Choices for y: $y > 0$ (y positive)

$y = 0$ (y zero)

$y < 0$ (y negative).

☐ The choices for $x, y \in \mathbb{R}$ depend on our imposed coordinate system (where we place our axes and how we encode positions as numbers). Indeed, for our circular region, the placement of the

cartesian-based x-axis and y-axis has vastly different effects on each (x, y) assignment than if we use a polar-base (r, θ) system.

☐ So, we can imagine that before we've ever constructed a model $\vec{F}(x, y) = \langle f, g \rangle$ for the "velocity" of each molecule of our fluid, we have to construct a "position function"

$$P_c\left(\begin{array}{c}\text{individual}\\ \text{molecule}\\ \text{in region } D\end{array}\right) = \underbrace{(x, y)}$$

this notation assumes that we encode positions based on cartesian coordinates

$$P_p\left(\begin{array}{c}\text{individual}\\ \text{molecule}\\ \text{in region } D\end{array}\right) = \underbrace{(\vec{r}, \theta)}$$

this notation suggests a polar coordinate system to track positions

☐ Then, before we've ever constructed a vector field, we've introduced a "position function"

$$\underset{\substack{\uparrow \\ \text{input is an individual} \\ \text{water molecule from} \\ \text{region } D \subseteq \mathbb{R}^2}}{P(\quad)} = \underbrace{(x, y)}_{\substack{\uparrow \\ \text{output is an} \\ \text{ordered pair of} \\ \text{real-numbers}}}$$

(requires a chosen coordinate system and $\therefore$ measurement techniques)

Jeff's Thought Bubble: "All models are wrong but some models are useful"

☐ Strictly speaking, the assumption that I, as a modeler, can identify each individual molecule of water in $D \subseteq \mathbb{R}^2$ is preposterous. As a human being, my senses are not designed to "see", "touch", or "hear" the location of individual molecules.

☐ However, the assumption that I can impose position measurements to different water molecules is not completely absurd as long as I am clever, intentional, and thoughtful with my assumptions.

☐ For example, if I want to assign an individual molecule a position (x,y), I might require myself not to think on an atomic scale when measuring position*... Thus, my view is

NOT focused on protons, electrons, and orbitals (which require length measurements in the range of anstroms with 1 angstrom $= 10^{-10}$ meters $=$ 0.1 nanometers $=$ 100 picometers)

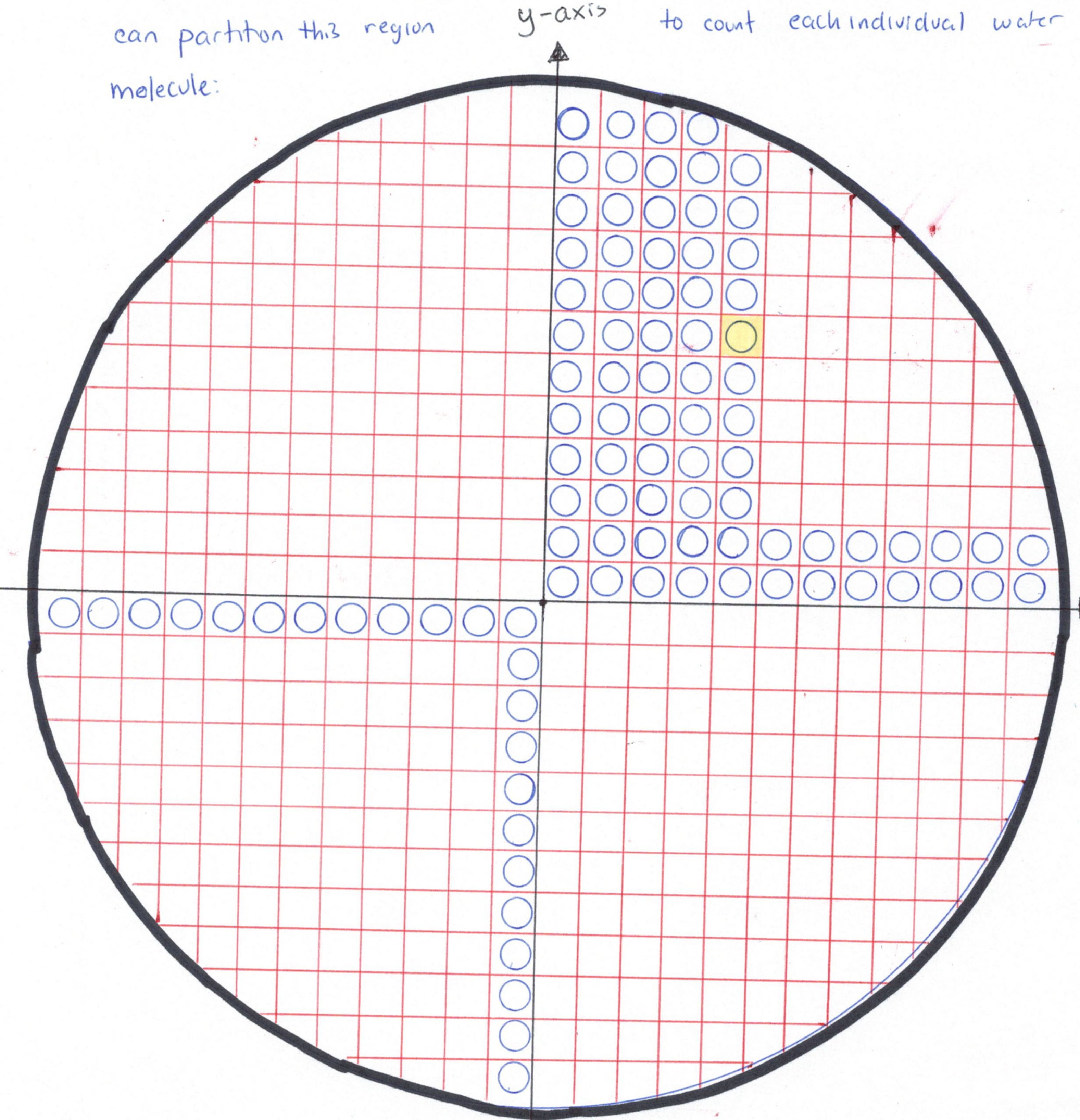
* see wikipedia article on "orders of magnitude"

☐ Instead, to assign positions to individual molecules in my fluid, I might focus on assigning ordered pairs (x,y) where $10^{-6}\,m \leq x,y \leq 10^{-3}\,m$. Recall

$$10^{-6}\,m = 1\,\mu m = 1 \text{ micrometre}$$

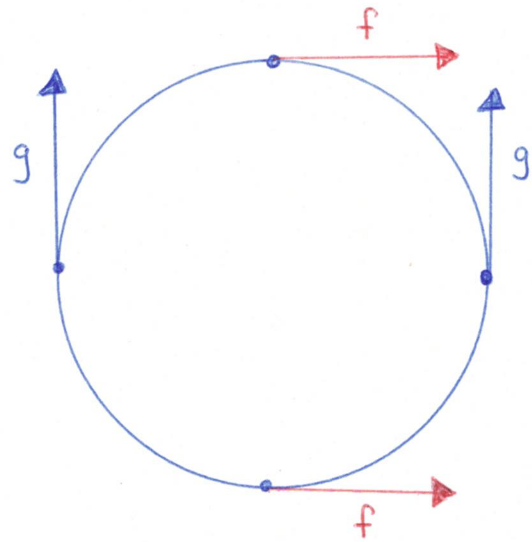
$$10^{-3}\,m = 1\,mm = 1 \text{ millimeter}.$$

☐ Now, let's visualize this scenario on our simple, representative region $D \subseteq \mathbb{R}^2$ with our water: we can partition this region to count each individual water molecule:



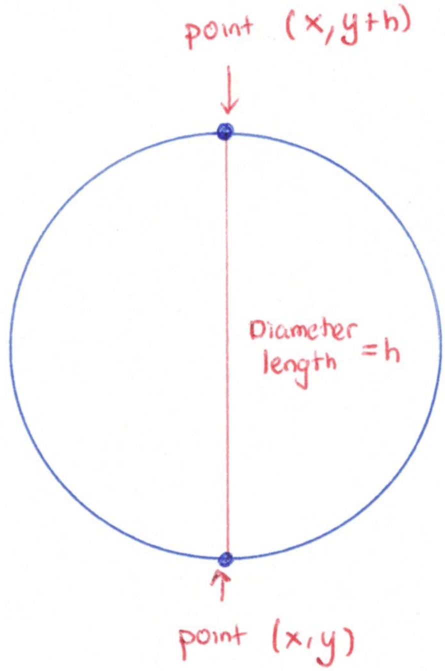
Let's zoom in a focus on a specific "drop" of our fluid (water) highlighted in yellow above.

For this drop of fluid, we can visualize the velocity vectors from the vector field. To this end we will focus on velocity vectors at 4 "points" on drop of water:

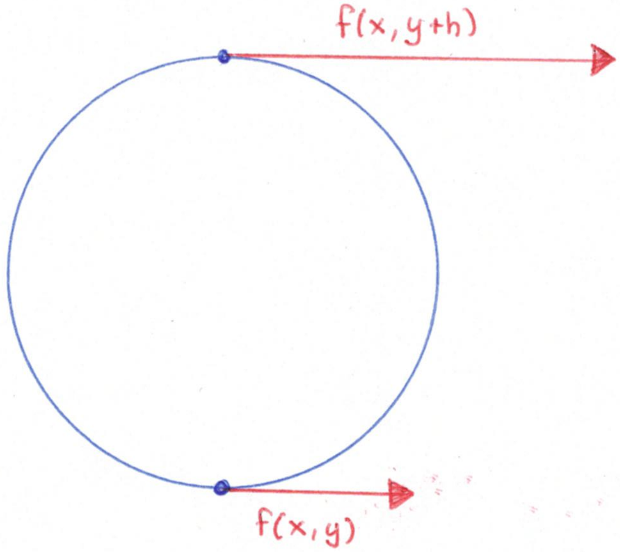


- ☐ Let's analyze each of the pairs of vectors separately to get a sense of how the velocity vector field might cause microrotations of each water drop. Remember that microrotations occur around the "center point" of the drop in an axis orthogonal to the plane.

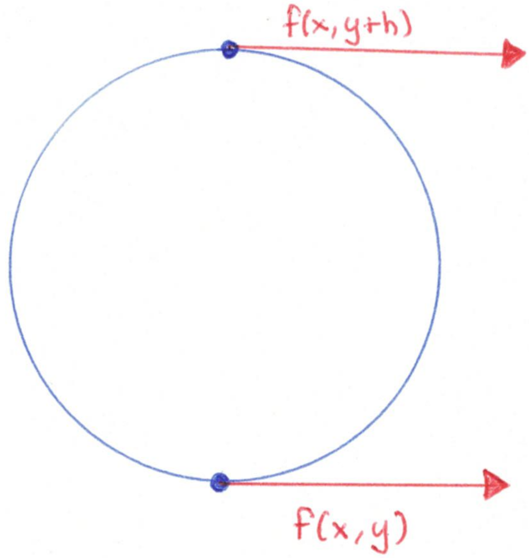
- We begin by considering the velocity vectors in the direction of $\vec{i}$. Below we introduce notation to help us describe this situation



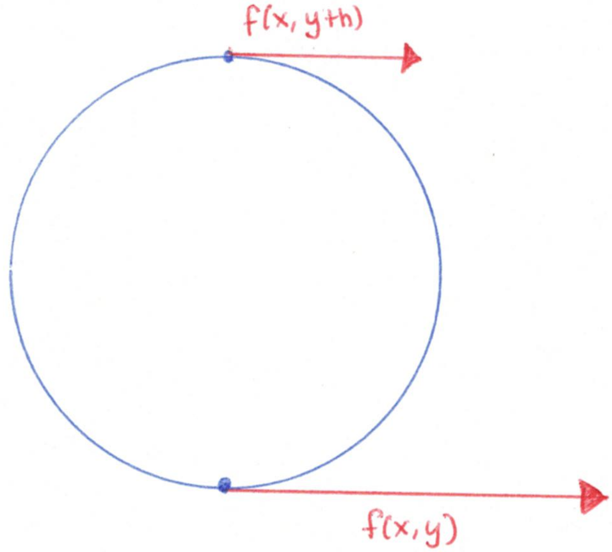
- Each of these points has a velocity vector $\vec{F}(x,y) = \langle f, g \rangle$ from the vector field. In our analysis here, we will focus on the 1st component $f(x,y)$ and $f(x,y+h)$ which point in the $\vec{i}$ direction. When considering the effects of these velocity vectors on micro rotations, we have three scenarios to consider, Scenario 1: $f(x, y+h) > f(x,y)$



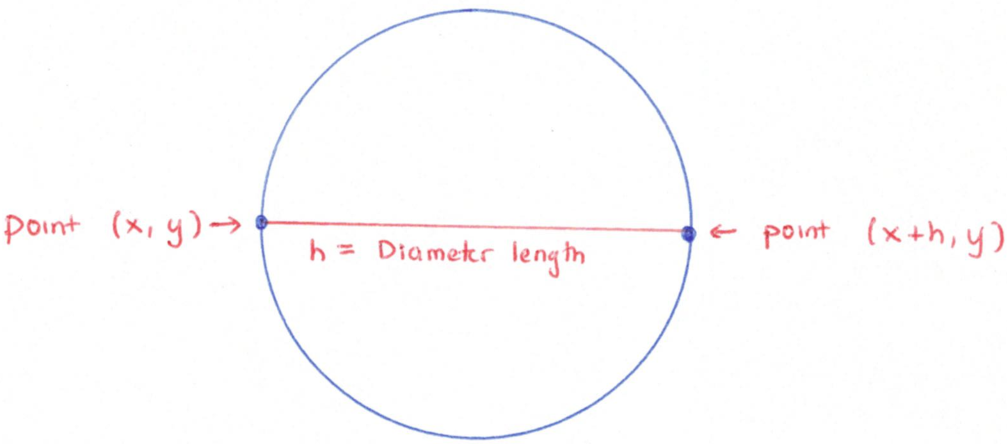
$\boxed{\text{Scenario 2 :} \quad f(x, y+h) = f(x, y)}$



$\boxed{\text{Scenario 3:} \quad f(x, y+h) < f(x,y)}$



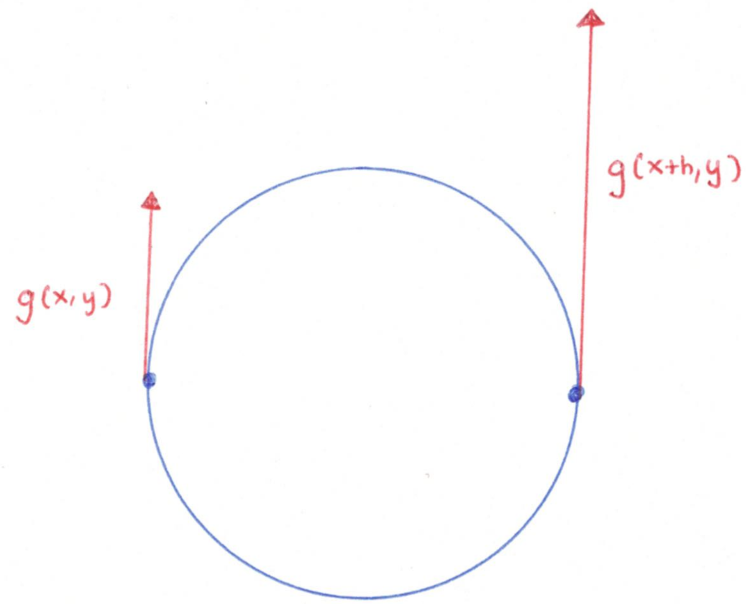
☐ Next, we can consider the velocity vectors in the direction of $\vec{j}$. Again we introduce notation to help us visualize and describe this situation. Notice we abuse notation a bit by reusing



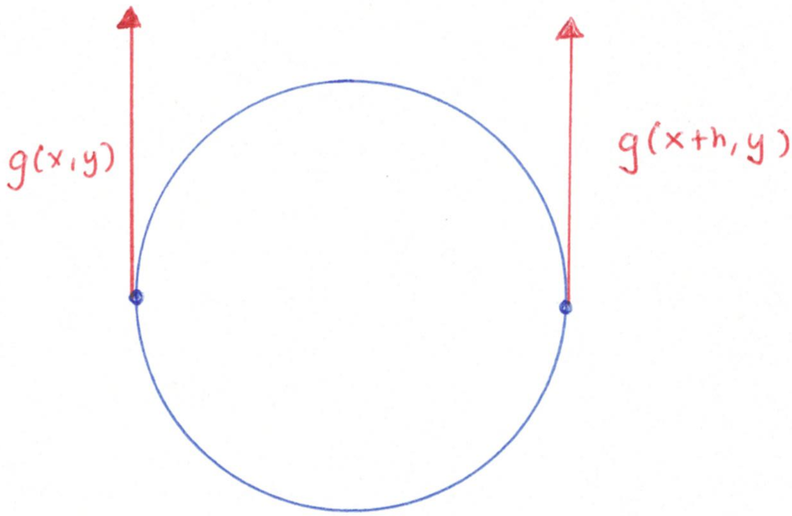
☐ Once again, we know that each of these points has a velocity vector $\vec{F} = \langle f, g \rangle$ from the vector field. In our analysis, we now focus on the 2nd components $g(x,y)$ & $g(x+h,y)$ which point in the $\vec{j}$ direction.

☐ We want to study and understand the effects of these velocity vectors on producing microscopic rotations. To do this, we focus on three scenarios, analogus to the case of the f values in the $\vec{i}$ direction.

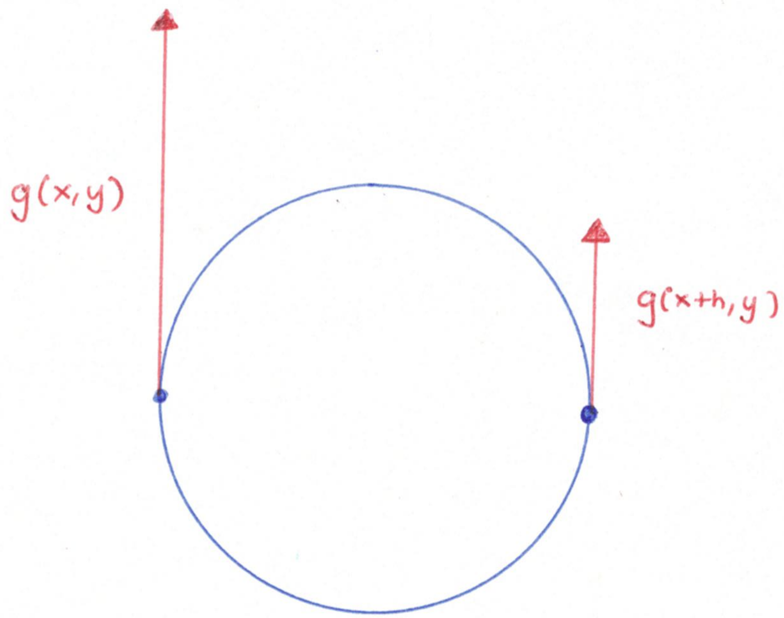
$$\boxed{\text{Scenario 1:} \quad g(x+h, y) > g(x, y)}$$



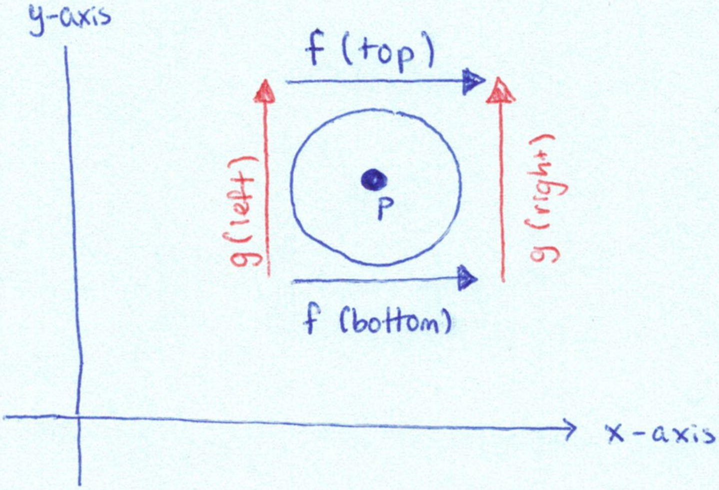
$$\boxed{\text{Scenario 2:} \quad g(x+h, y) = g(x,y)}$$

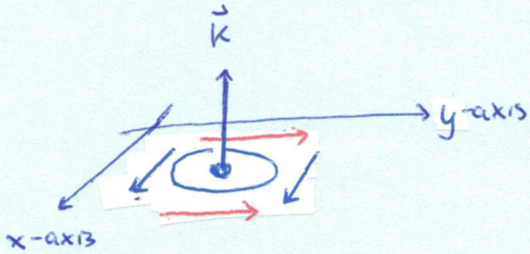


Scenario 3: $g(x+h, y) < g(x, y)$



Now, imagine that at each point on the interior $D \subseteq \mathbb{R}^2$ we place a small disk with a rough edge and we consider how the vector field might effect this disk:





☐ What type of "flow" via the four vectors above would produce

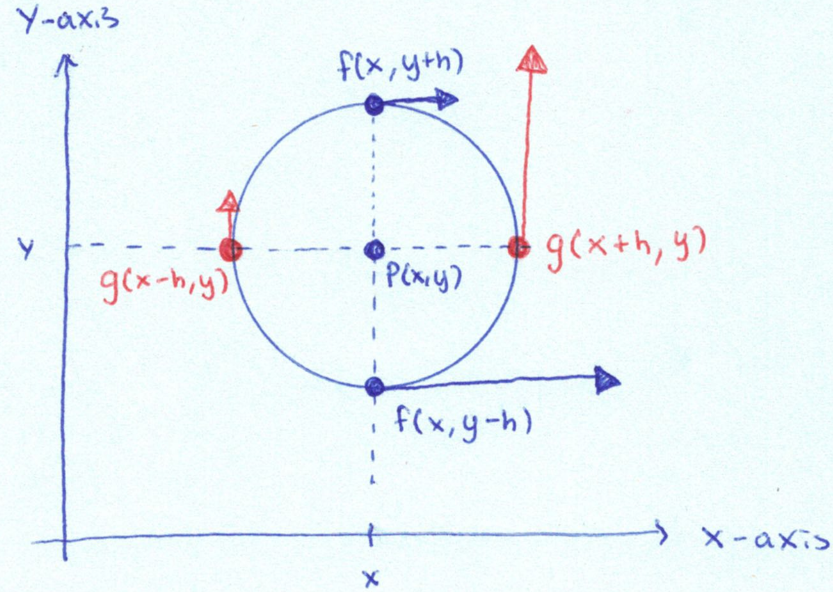
A. No rotation of the disk : $g_x = f_y \Leftrightarrow g_x - f_y = 0$

B. counter clockwise rotation : $g_x > f_y \Leftrightarrow g_x - f_y > 0$

C. Clockwise rotation: $g_x < f_y \Leftrightarrow g_x - f_y < 0$

Counterclockwise rotation

Consider the following diagram



Example: Macroscopic rotation with no microscopic rotation

Let $\vec{F}(x,y) = \dfrac{\langle -y, x \rangle}{x^2 + y^2}$. Lets calculate

the circulation (macro-rotation) on the unit circle

centered at $(0,0)$ oriented counter clockwise. To this end,

we set

$$C_1 = \{ \vec{r}(s) : 0 \leq s \leq 2\pi \}$$

where $\vec{r}(s) = \langle x(s), y(s) \rangle = \langle \cos(s), \sin(s) \rangle$.

Now we consider

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T} \, ds$$

$$= \oint_C \vec{F}(\vec{r}(s)) \cdot \vec{r}'(s) \, ds$$

$$= \oint_C \vec{F}(x(s), y(s)) \cdot \langle x'(s), y'(s) \rangle \, ds$$

$$= \oint_C \frac{\langle -y(s), x(s) \rangle}{(x(s))^2 + (y(s))^2} \cdot \langle x'(s), y'(s) \rangle \, ds$$

$$= \oint_C \frac{\langle -\sin(s), \cos(s) \rangle}{\cos^2(s) + \sin^2(s)} \cdot \langle -\sin(s), \cos(s) \rangle \, ds$$

$$= \oint_C \sin^2(s) + \cos^2(s) \, ds$$

$$\Rightarrow \oint_{C} \vec{F} \cdot d\vec{r} = \int_{C} \sin^2(s) + \cos^2(s) \, ds$$

$$= \int_{0}^{2\pi} 1 \, ds$$

$$= 2\pi .$$

Indeed, we see that this vector field has non zero circulation on curve C, leading to a macro rotation phenomena along the curve.