Lesson 13: Conservative vector fields

In this lesson, we will focus developing a more refined and nuanced understanding of vector fields

$$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

In particular, we will work to develop theory and techniques to answer the following two questions:

QUESTION 1: EXISTENCE

- [ ] When can we express a "given" vector field

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

as the gradient of a related potential function w

gradient operator

$$\text{Backward Problem:} \quad \vec{\nabla}\Big[\underbrace{\phi(x,y)}_{\text{unknown \& desired potential function}}\Big] = \underbrace{\vec{F}(x,y)}_{\text{"given" vector field}}$$

Remark: Notice that this question is really an existence question. Anoth way to phrase this question is if we have a description of vector field $\vec{F}$, does there exist a potential function ϕ such that $\vec{\nabla}[\phi] = \vec{F}$.

QUESTION 1: EXISTENCE ... (continued)

Notice, there are direct analogies between the existence question we pose for vector fields and the work we did in single-variable, integral calculus (Known as Math 1B at Foothill college). Let's spin this out a little bit:

| Single-Variable Integral Calculus (Foothill's Math 1B) |

ordinary derivative operator

$$\frac{d}{dx}\Big[\underbrace{F(x)}\Big] = \underbrace{f(x)}$$

unknown and desired "antiderivative" function

$F: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

"Known" or "given" derivative function (called the integrand) with $f: \underbrace{D \subseteq \mathbb{R}} \longrightarrow \mathbb{R}$

here is the source of our label "single variable"

| Multivariable Integral Calculus (Foothill's Math 1D |

Gradient operator

$$\vec{\nabla}\Big[\underbrace{\phi(x,y)}\Big] = \underbrace{\vec{F}(x,y)}$$

unknown and desired potential function

$\phi: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

"Known" or "given" vector field with

$\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

QUESTION 1: EXISTENCE ... (continued)

We are wise when we realize that not all vector fields can be expressed as the gradient of a potential function. This is the multivariable analog to the realization that not all functions have antiderivatives that can be expressed as elementary functions (recall: $\frac{d}{dx}[F(x)] = e^{-x^2}$ is not solvable if our hope is to write $F(X)$ as an elementary function).

However, vector fields $\vec{F}(x,y)$ that can be expressed as the gradient of a potential function $\phi(x,y)$ are very special! These vector fields are so special, in fact, that we give them there own name: such vector fields are known as

conservative vector fields: $\vec{\nabla}[\phi(x,y)] = \vec{F}(x,y)$

exists :)

Let's begin our journey in this lesson by cogitating on question 1. Recall:

$\boxed{\text{QUESTION 1: EXISTENCE}}$

- Given a vector field $\vec{F} = \vec{F}(x,y)$ with $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, when can we write this vector field as the gradient of a related potential function $\phi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$\vec{\nabla}\left[\phi(x,y)\right] = \langle \phi_x(x,y), \phi_y(x,y) \rangle = \vec{F}(x,y) = \langle f, g \rangle$$

- To encode this question in more concise language, we introduced the adjective "conservative" into our lexicon!

In addition to our existence question, we will ask ourselves a related, follow-up question:

QUESTION 2: CLASSIFICATIONS

- ☐ What kind of properties do conservative vector fields have?

One of the exciting and powerful conclusions we will discover in this lesson is that, in fact, conservative vector fields share a number of properties that distinguish them from other types of vector fields. With this idea in mind, we will develop several equivalent properties shared by ALL conservative vector fields.

Definition p. 1079) $\underbrace{\text{Conservative}}_{\text{adjective}}$ $\underbrace{\text{vector field}}_{\text{noun phrase}}$

A vector field $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is said to be conservative on a region $D \subseteq \mathbb{R}^2$ iff there exists a scalar function $\phi: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\vec{\nabla}\left[\phi(x,y)\right] = \vec{F}(x,y)$$

Notes on notation: symbolic description

- Sometimes we compress information and write $\vec{\nabla}\phi = \vec{F}$ to make notation more compact.

Notes on Notation: Intuitive and modeling descriptions

☐ The term "conservative" refers to the conservation laws of physics such as:

- conservation of energy
- conservation of mass-energy
- conservation of linear momentum
- conservation of angular momentum
- conservation of electric charge

for more, see wikipedia article on "conservation law"

☐ In some modeling contexts such as heat flow, the interpretation of the vector field and potential function may make us feel more inspired to define our relationship as

$$-\vec{\nabla}[\,\phi\,] = \vec{F}$$

In order to explore question 1 in more detail, let's begin with some assumptions.

☐ Suppose we are "given" a vector field $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

☐ Suppose our vector field $\vec{F}$ is conservative so that we know there exists a potential function $\phi: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\vec{\nabla}[\phi(x,y)] = \vec{F}(x,y)$$

$$\Leftrightarrow \langle \phi_x(x,y), \phi_y(x,y) \rangle = \langle f(x,y), g(x,y) \rangle$$

$$\Leftrightarrow \phi_x(x,y) = f(x,y) \quad \text{and} \quad \phi_y(x,y) = g(x,y>$$

Recall: By Clairaut's Theorem of the Equality of Mixed Partial Derivatives (Theorem 12.4 p. 900), we know the following

$$\phi_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[\phi(x,y)\right]\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\phi(x,y)\right]\right]$$

$$= \phi_{yx}$$

$$\Longleftrightarrow \quad \phi_{xy} = \frac{\partial}{\partial y}\left[\phi_x\right]$$

$$= \frac{\partial}{\partial y}\left[f(x,y)\right] \qquad \text{since} \quad \phi_x = f$$

$$= \frac{\partial}{\partial x}\left[\phi_y\right]$$

$$= \frac{\partial}{\partial x}\left[g(x,y)\right]$$

$$\Rightarrow \quad \frac{\partial}{\partial y}\left[f(x,y)\right] = \frac{\partial}{\partial x}\left[g(x,y)\right]$$

$$\Rightarrow \quad f_y(x,y) = g_x(x,y)$$

These observations give us the first (and most accessible) half of the proof to a theorem that we will call a Test for conservative vector fields.

However, in order to state this theorem, we need to introduce some special notation to describe the domain $D \subseteq \mathbb{R}^2$.

Types of Curves in $\mathbb{R}^n$

Jeff's Thought Bubble

- Here we will introduce important notation that we will come back to throughout the rest of the class
- This notation helps to compress information and will show up in the antecedents of many theorems!

Definition: (p. 1078) $\underbrace{\text{Simple}}_{\text{adjective}}$ $\underbrace{\text{curves}}_{\text{noun}}$ in $\mathbb{R}^n$ where $n \in \mathbb{N}$

Suppose $C \subseteq \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ described parametrically as

$$C = \{ \vec{r}(t) : a \leq t \leq b \}$$

where $\vec{r} : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$. We say that C is a simple curve if and only if

$$\vec{r}(t_1) \neq \vec{r}(t_2)$$

for all $t_1, t_2 \in [a,b]$ with $a < t_1 < t_2 < b$

Jeff's Thought Bubble: Notice that in this definition we do NOT say that $\vec{r}(a) \neq \vec{r}(b)$ so that simple curves may have identical endpoints. Instead, we only require that no points on the "interior" of the curve touch each other

Definition: $\underbrace{\text{Closed}}_{\text{adjective}}$ $\underbrace{\text{curves}}_{\text{noun}}$ in $\mathbb{R}^n$ where $n \in \mathbb{N}$

p. 1078

Suppose $C \subseteq \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ that is described parametrically as

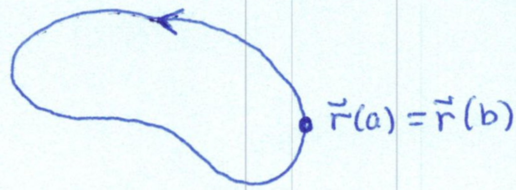
$$\vec{r}(t) = \{ \vec{r}(t) : a \leq t \leq b \}$$

with $\vec{r} : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$. We say that C is a closed curve if and only if

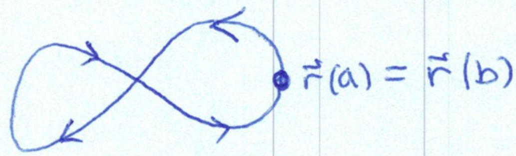
$$\vec{r}(a) = \vec{r}(b).$$

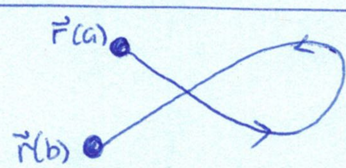
In other words, C is closed iff the initial and terminal points of C are equal.

Let's visualize some cases of curves to give some context to these definitions:

<table>
<tr><th></th><th>Closed</th><th>NOT closed</th></tr>
<tr><td>Simple</td><td>

Closed and simple:
☐ no "internal" intersections
☐ initial point = end point</td><td>

Simple but NOT closed
☐ no "internal" intersections
☐ initial point ≠ terminal point</td></tr>
<tr><td>NOT Simple</td><td>

Closed but NOT simple:
☐ initial point = end point
☐ has at least one "internal" intersection point.</td><td>

NOT simple AND NOT closed
☐ initial point ≠ terminal point
☐ has at least one "internal" intersection point.</td></tr>
</table>

Types of Regions in $\mathbb{R}^n$

In developing classifications of regions $D \subseteq \mathbb{R}^n$,

we will assume all of the regions we consider

are $\underbrace{\text{open}}_{\text{adjective}}$ $\underbrace{\text{regions}}_{\text{noun.}}$.

Note on notation: verbal descriptions

- Recall that the formal definition of an open set means that <u>all</u> points in our set are interior points.

- This formal definition of open sets immediately implies that open sets do NOT contain any boundary points.

Definition: $\underbrace{\text{Connected}}_{\text{adjective}}$ $\underbrace{\text{Regions}}_{\text{noun}}$ $D \subseteq \mathbb{R}^n$ where $n = 2, 3$
(p. 1078)

Suppose $D \subseteq \mathbb{R}^n$ is an open region with $n = 2, 3$.

We say that D is a connected region

if and only if it is possible to connect

any two points in D be a continuous

curve that lies completely inside D.

Jeff's Thought Bubble: Building Intuition via abuela language

- In "abuela" language, we might talk about connected regions as regions $D \subseteq \mathbb{R}^n$ that are "all in one piece".

- The word "abuela" means grandma in spanish. The scholar Dr. Rochelle Gutierrez uses the phrase "abuela language" to describe intuitive language that would make sense to our abuela. I adopt this idea to describe the idea of translating technical terminology into intuitive ideas.

Definition: $\underbrace{\text{Simply Connected}}_{\text{adjective phrase}}$ $\underbrace{\text{Region}}_{\text{noun}}$ in $\mathbb{R}^n$ with $n=2,3$

(p. 1078)

Suppose $D \subseteq \mathbb{R}^n$ is an open region with $n=2,3$.

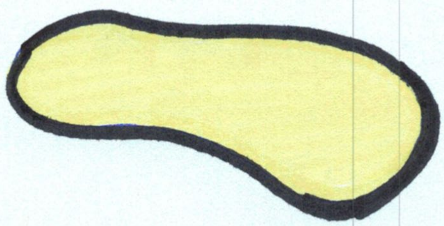
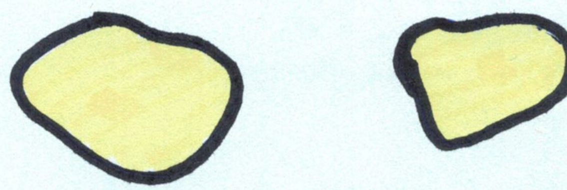
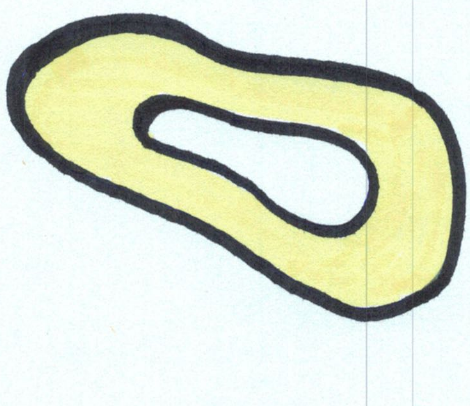
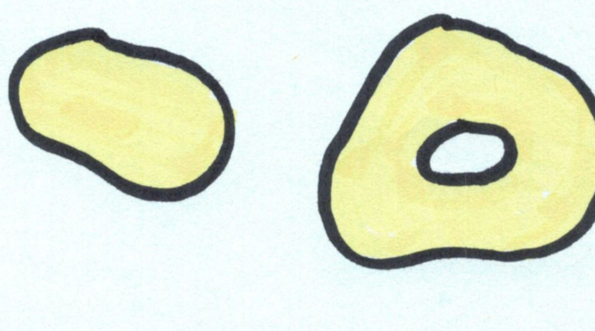
We say that D is a simply connected region iff every closed, simple curve $C_1 \subseteq D$ can be deformed and contracted to a single point inside D.

Jeff's thought bubble: Building intuition via abuela language

- ☐ In "abuela" language, we might say that a simply connected region does not have any holes inside the region.

Types of Regions in $\mathbb{R}^n$ ...

Let's visualize some cases of regions $D \subseteq \mathbb{R}^2$ to get a better idea about the definitions of connected and simply connected regions

	Connected	NOT Connected
simply connected	 Connected and simply connected □ all in one piece □ no holes inside region	
NOT Simply connected		

Theorem 14.3 p. 1079) Test for Conservative Vector fields in $\mathbb{R}^2$

□ Let $\vec{F} = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \langle f, g \rangle$

be a vector field with $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

□ Suppose the domain region $D \subseteq \mathbb{R}^2$ is BOTH:

i. Connected in $\mathbb{R}^2$

ii. simply connected in $\mathbb{R}^2$

□ Suppose the functions

$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$g : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

have continuous first partial derivatives on region D.

Then, the vector field $\vec{F} = \vec{F}(x,y)$ is said to be a conservative vector field on D if and only if

$\boxed{\text{Condition 1:}}$ $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$

Example 14.3.1a pg. 1079) Testing for a Conservative Field

Determine if the vector field

$$\vec{F}(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle$$

is conservative on $\mathbb{R}^2$.

Solution: Recall that since $\vec{F}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, we can use our Thm 14.3: Test for conservative vector fields in $\mathbb{R}^2$. This test has only one condition to check with results from Clairaut's t

Condition 1: Let $\vec{F} = \langle f, g \rangle$. Then $\vec{F}$ is conservative iff $f_y = g_x$.

To this end, notice that

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

$$= \langle e^x \cdot \cos(y), -e^x \sin(y) \rangle$$

$$\Rightarrow \quad f(x,y) = e^x \cdot \cos(y) \quad \text{and} \quad g(x,y) = -e^x \sin(y)$$

$$\Rightarrow \quad f_y(x,y) = \frac{\partial}{\partial y}\left[e^x \cos(y)\right] \quad \text{and} \quad g_x(x,y) = \frac{\partial}{\partial x}\left[-e^x \sin(y)\right]$$

$$\Rightarrow \quad f_y(x,y) = -e^x \sin(y) = g_x(x,y) = -e^x \sin(y)$$

$$\Rightarrow \quad f_y = g_x$$

$\Rightarrow$ $\vec{F}$ is a conservative vector field on $D = \mathbb{R}^2$

$\Rightarrow$ $\vec{F} = \vec{\nabla}\phi$ for some scalar field

$$\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$\Rightarrow \quad \vec{F}(x,y) = \langle e^x \cos(y), -e^x \sin(y) \rangle = \underbrace{\langle \phi_x , \phi_y \rangle}_{} = \vec{\nabla}\phi$$

we want to find $\phi(x,y)$

Finding Potential Functions

We have just generated a mechanism to answer. QUESTION 1: Existence.

Recall:

- Remember that our first queston about vector fields was stated as follows:
  - When can we express a "given" vector field $\vec{F}(x,y)$ as the gradient of a related potential function ϕ with

$$\vec{\nabla}[\phi(x,y)] = \vec{F}(x,y)$$

- Thm 14.3: Testing for conservative vector fields in $\mathbb{R}^2$ gives a single condition that answers this problem for $\vec{F}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ stated as

$$\boxed{\text{Condition 1:}}\ f_y = g_x \iff \vec{\nabla}[\phi] = \vec{F} = \langle f, g \rangle$$

- Thm 14.3: Testing for conservative vector fields in $\mathbb{R}^3$ gave 3 conditions that answer this question for $\vec{F}: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.

Then, with this in mind we see that every conservative vector field

$$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

has some underlying scalar-valued potential function

$$\phi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

such that

$$\vec{\nabla}[\phi(x,y)] = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

Notice, this is a "backward problem" directly analogous to our math 1B study:

Single-variable Backward Problem

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\Rightarrow \underbrace{F(x)}_{\text{this antiderivative was unique up to a constant}} = \int_a^x f(t)\,dt$$

Multivariable Backward Problem

$$\vec{\nabla}[\phi(x,y)] = \vec{F}(x,y)$$

Example 14.3.2a, p. 1080) Find the potential function of our conservative vector field in $\mathbb{R}^3$

Let $\vec{F}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be the conservative vector field

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \langle e^x \cdot \cos(y), -e^x \sin(y) \rangle$$

Find the potential function $\phi(x,y)$ s.t. $\vec{\nabla}\phi = \vec{F}$.

Solution) We know by example 14.3.1a that $\vec{F}$ is conservative

$$\Rightarrow \vec{\nabla}\phi(x,y) = \langle \phi_x , \phi_y \rangle = \langle e^x \cdot \cos(y), -e^x \sin(y) \rangle$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\phi(x,y)\right] = e^x \cdot \cos(y) \quad \& \quad \frac{\partial}{\partial y}\left[\phi(x,y)\right] = -e^x \sin(y)$$

$$\Rightarrow \int \phi_x \, dx = \int e^x \cdot \cos(y) \, dx$$

$$= \cos(y) \int e^x \, dx$$

$$= \cos(y) \, e^x + c(y)$$

$$= e^x \cdot \cos(y) + c(y)$$

some function strictly in terms of variable

Example 14.3.2a continued ...

$$\Rightarrow \quad \phi(x,y) = \int \phi_x \, dx = e^x \cdot \cos(y) + c(y)$$

$$\Rightarrow \quad \phi_y(x,y) = \frac{d}{dy}\left[e^x \cdot \cos(y) + c'(y)\right]$$

$$= -e^x \sin(y) + c'(y)$$

$$= g(x,y) + c'(y)$$

$$\Rightarrow \quad c'(y) = 0$$

$\Rightarrow$ $c(y) \in \mathbb{R}$ is a constant function

$$\Rightarrow \quad \phi(x,y) = e^x \cdot \cos(y)$$

Note on Constants:

- Notice that this chosen potential function is unique up to an additive constant. This is true in general. However, when using the potential function $\phi(x,y)$ to model a real-world phenomena, we expect that the additive constant has some physical meaning.

- For the "toy examples" we work with, we are not modeling reality and thus we "omit" the additive constant since no meaning exists.

Procedure 14.3 p. 1081) Finding Potential Functions in $\mathbb{R}^2$

Let $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ be a conservative vector field. To find $\phi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t. $\vec{\nabla}\phi = \vec{F}$, we might use the following steps:

Step 1: Integrate $\phi_x = f$ with respect to x to obtain $\phi(x,y)$:

$$\phi(x,y) = \int \phi_x(x,y)\,dx$$

$$= \int f(x,y)\,dx$$

$$= \hat{\phi}(x,y) + c(y)$$

Notice that in our first pass, we will include the arbitrary function $c(y)$

Step 2: compute $\phi_y(x,y) = \frac{\partial}{\partial y}[\phi(x,y)]$

$$= \frac{\partial}{\partial y}\left[\hat{\phi}(x,y) + c(y)\right]$$

$$= \hat{\phi}_y + c'(y)$$

Equate this to $g(x,y)$ to obtain an expression for $c'(y) = \frac{\partial}{\partial y}[c(y)]$.

Step 3: Get an expression for $c(y)$ using ordinary integration techniques

$$c(y) = \int c'(y)\, dy$$

Fundamental Theorem for Line Integrals

Now that we have relatively robust mechanisms to answere Question 1: Existence, let us shift our focus to question 2. Recall:

> **QUESTION 2: CLASSIFICATON**
>
> - What Kind of properties do conservative vector fields have?

We will focus on Three equivalent principles

$\vec{F}$ is conservative $\Longleftrightarrow$ line integrals on $\vec{F}$ are path independent

(Thm 14.4: Fundamental Theorem for line integrals)

$\Longleftrightarrow \oint_{C} \vec{F} \cdot \vec{T}\, ds = 0$ for simple, closed curves C

(Thm 14.5: Line integrals of conservative fields on closed curves)

Thm 14.4 : Fundamental Theorem for Line Integrals of Conservative Vector fields (a.k.a. Path Independence
p. 108)

Let $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ be a continuous vector field on an open, connected region $D \subseteq \mathbb{R}^2$ with $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

The vector field $\vec{F}(x,y)$ is conservative on D if and only if

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{\nabla}\phi \cdot \vec{T} \, ds$$

$$= \phi(B) - \phi(A)$$

for all points $A, B \in D \subseteq \mathbb{R}^2$ and for all piecewise-smooth oriented curves $C \subseteq D$ from A to B

Remarks about the Fundamental Thm for Line Integrals

- We can translate this theorem, written in formal, technical jargon into a more intuitive version: "If $\vec{F}$ is a conservative vector field, then the value of a line integral of $\vec{F}$ depends only on the endpoints of the path."

- An even more concise way to say this is:

  "For conservative vector fields, line integrals are independent of paths."

- This theorem implies that a parametrization of a path

$$C = \{ \vec{r}(t) : a \leq t \leq b \}$$

is not needed to evaluate a line integral $\int_C \vec{F} \cdot d\vec{r}$ of a conservative vector field.

- Let's compare the Fundamental Theorem of Calculus from Math 1B to the Fundamental Theorem of Line Integrals from Math 1D:

Fundamental Theorem of (single-variable) Calculus

$$\int_a^b f(x)\,dx = \int_a^b \underbrace{\frac{d}{dx}\left[F(x)\right]}_{f(x)} dx$$

$$= F(b) - F(a)$$

Fundamental Theorem of Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \vec{\nabla}\phi(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \phi(x(b), y(b)) - \phi(x(a), y(a)) = \phi(B) - \phi(A)$$

Proof of thm 14.4:

We will prove only one of the two directions of this biconditional proposition. In particular, we will prove:

If $\underbrace{\vec{F} \text{ is conservative}}_{\text{Antecedent}}$,

then $\underbrace{\text{the line integral is path independent.}}_{\text{Consequent}}$

This is from Math 22

To this end, let's assume the antecedents are true. In other words, suppose $\vec{F} = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ is a continuous vector field on an open connected region $D \subseteq \mathbb{R}^2$. Suppose $\vec{F}$ is conservative so there exists some $\phi(x,y)$ with $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ with

$$\vec{\nabla}\phi(x,y) = \vec{F}(x,y).$$

Let $C \subseteq D$ be a piecewise smooth oriented curve with

$$C = \{ \vec{r}(t) : a \le t \le b \}$$

where $\vec{r}(t) = \langle x(t), y(t) \rangle$ with $B = \vec{r}(b) = \langle x(b), y(b) \rangle$ and $A = \vec{r}(a) = \langle x(a), y(a) \rangle$

Proof of thm 14.4 ...)

Then let's consider

$$\phi(t) = \phi(\vec{r}(t)) = \phi(x(t), y(t))$$

$$\Rightarrow \quad \phi'(t) = \frac{d}{dt}\left[\phi(\vec{r}(t))\right]$$

$$= \frac{d}{dt}\left[\phi(x(t), y(t))\right]$$

$$= \frac{\partial \phi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dt}$$

$$= \left< \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right> \cdot \left< x'(t), y'(t) \right>$$

$$= \vec{\nabla} \phi(x(t), y(t)) \circ \vec{r}\,'(t)$$

$$= \vec{F}(x(t), y(t)) \circ \vec{r}\,'(t)$$

$$= \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)$$

Proof of thm 14.4 ...)

Now, let's consider the line integral

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_1} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \cdot \|\vec{r}\,'(t)\|_2 \, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_a^b \frac{d}{dt}\left[\phi(\vec{r}(t))\right] dt$$

$$= \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

$$= \phi(B) - \phi(A).$$

This is exactly what we wanted to show. ∎

Example 14.3.3 p. 1082) Verify path independence for line integrals of conservative vector fields

Consider the potential function

$$\phi(x,y) = \frac{(x^2 - y^2)}{2}$$

Notice that the gradient field of ϕ is given as

$$\vec{F}(x,y) = \vec{\nabla}\phi(x,y) = \langle x, -y \rangle$$

A. Let C_1 be the "quarter" circle $C_1 = \{\vec{r}_1(t) : 0 \le t \le \pi/2\}$

where $\vec{r}_1(t) = \langle \cos(t), \sin(t) \rangle$ whose initial point is

$A(1,0) = \vec{r}(0) = \langle 1, 0 \rangle$ and whose terminal point is

$B(0,1) = \vec{r}(\pi/2) = \langle 0, 1 \rangle$

B. Let C_2 be the line segment $C_2 = \{\vec{r}_2(t) : 0 \le t \le 1\}$

where $\vec{r}_2(t) = \langle 1, 0 \rangle + t \cdot \langle -1, 1 \rangle = \langle 1-t, t \rangle$ connecting initial point $A(1,0)$ to terminal point $B(0,1)$.

Show $\displaystyle\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \phi(B) - \phi(A)$.

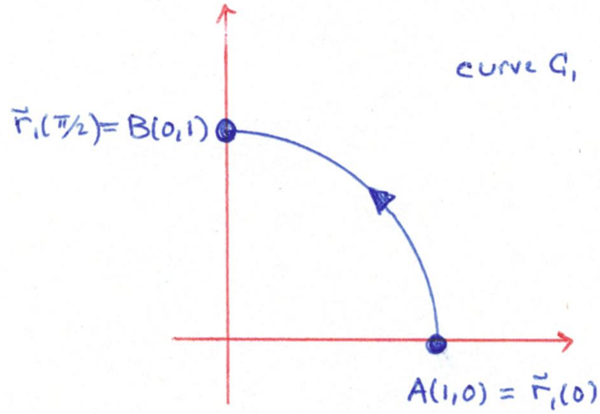
Example 14.3.3 p. 1082, solution to part A)

Lets begin by calculating $\int_{C_1} \vec{F}\cdot\vec{T}\,ds$. First, we visualize

$$C_1 = \{\vec{r}_1(t) : 0 \le t \le \tfrac{\pi}{2}\}$$

$$= \{\langle \cos(t), \sin(t)\rangle : 0 \le t \le \pi/2\}$$

$$= \{\langle x_1(t), x_2(t)\rangle : 0 \le t \le \pi/2\}$$



Now, we recall our desired line integral

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}(\vec{r}_1(t)) \overset{\text{dot product}}{\cdot} \underbrace{\frac{\vec{r}_1{}'(t)}{\|\vec{r}_1{}'(t)\|_2}}_{\vec{T}(t)} \, ds$$

$$= \int_{C_1} \vec{F}(x_1(t), y_1(t)) \cdot \frac{\vec{r}_1{}'(t)}{\|\vec{r}_1{}'(t)\|_2} \underbrace{\|\vec{r}_1{}'(t)\|_2 \, dt}_{ds}$$

$$= \int_{C_1} \vec{F}(x_1(t), y_1(t)) \cdot \vec{r}_1{}'(t)\, dt$$

$$= \int_{C_1} \langle x_1(t), -y_1(t)\rangle \cdot \langle x_1'(t), y_1'(t)\rangle \, dt$$

$$= \int_{C_1} \langle \cos(t), -\sin(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle \, dt$$

Example 14.3.3 p. 1082, solution to part A ...)

$$\Rightarrow \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} -2\sin(t)\cos(t)\,dt$$

$$= \int_0^{\pi/2} -\sin(2t)\,dt$$

Side note: Double Angle Formula

$$\sin(2t) = \sin(t+t)$$

$$= \sin(t)\cos(t) + \cos(t)\sin(t)$$

$$= 2\sin(t)\cos(t)$$

$$= \left.\frac{\cos(2t)}{2}\right|_0^{\pi/2}$$

$$= \frac{1}{2}\left(\cos(2\cdot\pi/2) - \cos(2\cdot 0)\right)$$

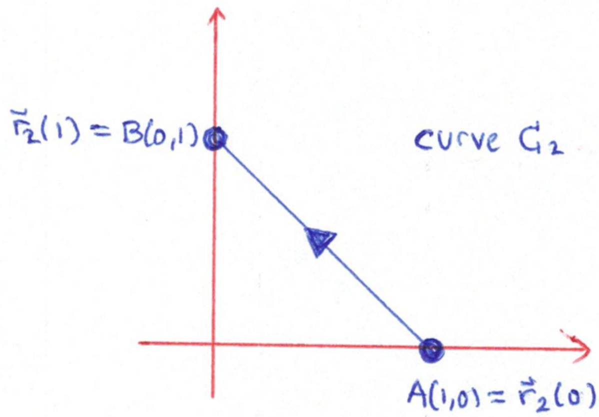
$$= \frac{1}{2}\left(\cos(\pi) - \cos(0)\right)$$

$$= \frac{1}{2}(-1-1) = -2/2 = -1.$$

$$\Rightarrow \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_{C_1} \vec{F}\cdot d\vec{r} \boxed{= -1.}$$

Example 14.3.3 p. 1082, solution to part B)

We continue our work on this example by calculating our line integral $\int_{C_2} \vec{F}\cdot\vec{T}\,ds$. Once again, we start our work by visualizing our curve



$$C_2 = \{\vec{r}_2(t) : 0 \le t \le 1\}$$

$$= \{\langle 1-t, t\rangle : 0 \le t \le 1\}$$

$$= \{\langle x_2(t), y_2(t)\rangle : 0 \le t \le 1\}$$

Now we calculate our desired line integral

$$\int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}(\vec{r}_2(t)) \overset{\text{dot product}}{\cdot} \underbrace{\frac{\vec{r}_2{}'(t)}{\|\vec{r}_2{}'(t)\|_2}}_{\vec{T}(t)}\, ds$$

$$= \int_{C_2} \vec{F}(x_2(t), y_2(t)) \cdot \frac{\vec{r}_2(t)}{\|\vec{r}_2{}'(t)\|_2} \cdot \underbrace{\|\vec{r}_2{}'(t)\|_2\, dt}_{ds}$$

$$= \int_{C_2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2(t)\, dt$$

$$= \int_{C_2} \langle x_2(t), -y_2(t)\rangle \cdot \langle x_2'(t), y_2'(t)\rangle\, dt$$

Example 14.3.3 p. 1082, solution to part B ...)

$$\Rightarrow \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \langle 1-t, -t\rangle \cdot \langle -1, 1\rangle\, dt$$

$$= \int_0^1 t-1-t\, dt$$

$$= \int_0^1 -1\, dt$$

$$= -t \Big|_0^1$$

$$= -(1-0) = -1$$

$$\Rightarrow \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{r}'(t)\,dt = \int_{C_2} \vec{F}\cdot d\vec{r} \boxed{= -1}.$$

Example 14.3.3 p. 1082, Thm 14.4 analysis ...)

Finally, let's analyze this example using our ideas from the fact that $\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$. For any smooth, oriented curve $C_1 \subseteq D$ from initial point $A(1,0)$ to terminal point $B(0,1)$ with $C_1 = \{\vec{r}(t) : a \le t \le b\}$ where $\vec{r}(a) = A$ and $\vec{r}(b) = B$, we have

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_{C_1} \vec{\nabla}\phi(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[\phi(t)\right]dt$$

$$= \int_a^b \frac{d\phi}{dt}\,dt$$

$$= \phi(\vec{r}(t))\Big|_a^b$$

$$= \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

$$= \phi(B) - \phi(A).$$

Example 14.3.3 p. 1082, Thm 14.4 analysis...)

$$\Rightarrow \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \phi(B) - \phi(A)$$

$$= \phi(0,1) - \phi(1,0)$$

$$= \frac{(0^2-1^2)}{2} - \frac{(1^2-0^2)}{2}$$

$$= -\frac{1}{2} - \frac{1}{2}$$

$$\Rightarrow \phi(B) - \phi(A) = -1 \ \checkmark$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$$ and we have confirmed

path independence for this example.

Line Integrals of Conservative Vector Fields on Closed Curves

In our previous analysis leading up to Thm 14.4, we assumed only that C was a simple, smooth, oriented curve from point A to point B in D.

However, we can also consider line integrals of conservative vector fields on simple, closed piecewise-smooth oriented curves $C \subseteq D$, by attempting to calculate

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}\,ds$$

$$= \oint_C \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}\,\|\vec{r}\,'(t)\|_2\,dt$$

$$= \oint_C \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$$= \oint_C \vec{F}\cdot d\vec{r}$$

By thm 14.4, we Know

$$\oint_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

since $\vec{F}$ is conservative, we Know $\vec{F} = \vec{\nabla}\phi$ and we apply Thm 14.4

$$= \phi(A) - \phi(A)$$

since C is closed, we Know $B = A$

$$= 0 \checkmark$$

This is a stunning conclusion:

We Know A can be an arbitrary point on C.

We have just shown, via a "rigorous" mathematical argument, that the line integral of any conservative vector field on a closed curve is zero.

Jeff's thought bubble: logical description of the concept definition above

- The above derivation actually encodes a (one-directional) conditional statement in the form $P \Rightarrow Q$ given as follows:

If $\underbrace{\vec{F} \text{ is conservative}}_{\text{proposition } P}$, then $\underbrace{\oint_C \vec{F} \cdot d\vec{r} = 0 \text{ on all simple, closed curves } C \subseteq}_{\text{proposition } Q}$

Jeff's thought bubble: logical description for $\oint_C \vec{F} \cdot d\vec{r}$ of conservative $\vec{F}$

- While (one-directional) conditional statements are a good start, mathematicians often pray for logical equivalence of two propositions. Logical equivalence means that we can encode a theorem as a (two-directional) biconditional proposition in the form $P \Leftrightarrow Q$ which is actually two (one-directional) conditional statements

$$P \Leftrightarrow Q \quad \Leftrightarrow \quad \begin{array}{ll} \text{i.} & P \Rightarrow Q \quad \text{AND} \\ \text{ii.} & Q \Rightarrow P \end{array}$$

We might ask ourselves if we can make an argument "in the opposide direction". In particular, we might ask ourselves what we can say if $\oint_C \vec{F} \cdot d\vec{r} = 0$ on all close, simple, piecewise-smooth oriented curves in region D?

To answer this question, let's make some assumptions.

Let $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous vector field on $D \subseteq \mathbb{R}^2$.

Let $A, B \in D$ be distinct points on the interior of the open, connected region D, where $A \neq B$. Let C be a closed, simple, oriented curve consisting of curves C_1 and C_2 where

- C_1 is a simple smooth, oriented curve from A to B
- C_2 is a simple, smooth, oriented curve from B to A
- $C_1 \neq C_2$
- C_1 does not intersect any points on C_2 (except at the end points)

Finally, suppose $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Then $$0 = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_{C_1 \cup C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{-C_2} \vec{F} \cdot d\vec{r}$$

where $-C_2$ is the curve C_2 with opposite orientation from point A to point B

$\Rightarrow$ the line integral of $\vec{F}$ has the same value on any two arbitrary simple paths from point A to point B

$\Rightarrow$ the line integral is path independent

$\Rightarrow \vec{F}$ is conservative by thm 14.4

$\Rightarrow$ there exists a $\phi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$$

This gives us a biconditional statement:

$\boxed{\text{Thm 14.5: Line Integrals on Closed Curves}}$
p. 1083

Let $D \subseteq \mathbb{R}^2$ be an open connected region.

Let $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous vector field on D.

The vector field $\vec{F}$ is conservative on D

if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ on all

simple, closed, piecewise-smooth oriented curves

$C \subseteq D$.

Summary of Properties of Conservative Vector Fields

In this lesson, we have established "equivalencies" between three properties of conservative vector fields $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ on open, connected regions $D \subseteq \mathbb{R}^2$.

Property 1: There exists a (scalar-valued) potential function $\phi: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

(Definition of conservative vector field)

$$\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$$

Property 2: For all points $A, B \in D$ and all piecewise, smooth, oriented curves $C_1 \subseteq D$ from A to B we have

(path independence)

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \underbrace{\phi(B) - \phi(A)}$$

this is any arbitrary curve

value of line integral stays constant independent of the path we travel

Property 3: For all simple, closed, piecewise-smooth, oriented curves $C_1 \subseteq D$, we have

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = 0$$

Then, we have created multiple mechanisms to classify conservative fields:

(Path independence) (F is conservative)

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A) \underset{\text{(Thm 14.4)}}{\Longleftrightarrow} \vec{F}(x,y) = \vec{\nabla}\phi(x,y)$$

$$\underset{\text{(Thm 14.5)}}{\Longleftrightarrow} \oint_{C_1} \vec{F} \cdot d\vec{r} = 0$$

Exercise 14.3.33 p. 1085) Line Integral of a vector field on a closed curve

Evaluate the line integral $\oint_{C} \vec{F} \cdot d\vec{r}$ for the vector field

$\vec{F}(x,y) = \langle x, y \rangle$ on the curve

$$C = \{ \vec{r}(t) : 0 \leq t \leq 2\pi \}$$

where $\vec{r}(t) = \langle 4 \cdot \cos(t), 4 \sin(t) \rangle = \langle x(t), y(t) \rangle$

Solution: Let's consider

$$\oint_{C} \vec{F} \cdot d\vec{r} = \oint_{C} \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \, ds$$

$$= \oint_{C} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \cdot \underbrace{\|\vec{r}\,'(t)\|_2 \, dt}_{ds}$$

$$= \oint_{C} \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) \, dt$$

Exercise 14.3.33 p. 1085 solution ...)

$$\Rightarrow \oint_{C} \vec{F} \cdot d\vec{r} = \oint_{0}^{2\pi} \langle x(t), y(t) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$= \oint_{0}^{2\pi} \langle 4 \cdot \cos(t), 4\sin(t) \rangle \cdot \langle -4\sin(t), 4\cos(t) \rangle \, dt$$

$$= \oint_{0}^{2\pi} -16 \sin(t) \cos(t) + 16 \sin(t) \cos(t) \, dt$$

$$= \oint_{0}^{2\pi} 0 \, dt$$

$$= 0$$

$$\Rightarrow \oint_{C} \vec{F} \cdot d\vec{r} = 0$$

$\Rightarrow$ $\vec{F}$ is conservative and $\vec{F}(x,y) = \vec{\nabla} \phi(x,y)$.

Exercise 14.3.33 p. 1085 solution ...)

We can go further. Since we know

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = 0 \iff \vec{F} = \vec{\nabla} \phi$$

we can search of $\phi(x,y)$ using our procedure 14.3 p. 1081.

To this end, we know

$$\vec{\nabla}\phi = \langle \phi_x , \phi_y \rangle = \langle x , y \rangle = \vec{F}$$

$$\Rightarrow \quad \phi_x = x$$

$$\Rightarrow \quad \phi(x,y) = \int x \, dx$$

$$\Rightarrow \quad \phi(x,y) = \frac{x^2}{2} + c(y)$$

$$\Rightarrow \quad \phi_y = c'(y) = y$$

$$\Rightarrow \quad c(y) = \int c'(y) \, dy = \int y \, dy = \frac{y^2}{2}$$

$$\Rightarrow \quad \boxed{\phi(x,y) = \frac{x^2 + y^2}{2}}$$