Lesson 12, part b) "Line" Integrals of Vector Fields

The general form of an integral in this class looks like

$$\int_D f \, dw$$

So far, our line integrals have taken the form

integrand f: has scalar-valued output

$$\int_C f \, ds = \int_C f \cdot \underbrace{\|\vec{r}\,'(t)\| \, dt}_{ds}$$

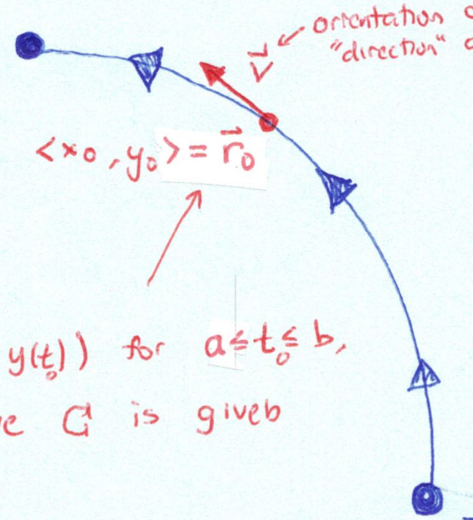
However, we can introduce a "line" integral that involves an integrand $\vec{F} : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is a vector field. The major idea is to properly define the significance of the integral in this context.

To define a "line" integral $\int_C f \, ds$ of a vector field $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, we need to start with the following:

- a curve $C \subseteq D$:

Let $C = \{ \vec{r}(t) : a \leq t \leq b \}$ be an oriented curve contained within the domain D of vector field $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$\vec{r}(b) = \langle x(b), y(b) \rangle$



positive orientation is in the "direction" in which the parameter increases.

At point $P(x_0, y_0) = (x(t_0), y(t_0))$ for $a \leq t_0 \leq b$, the tangent line to curve C is giveb by equation

$\vec{r}(a) = \langle x(a), y(a) \rangle$

$$\vec{r}_0 + t\vec{v} = \langle x_0, y_0 \rangle + t \cdot \langle x'(t_0), y'(t_0) \rangle$$

where the "direction" of the tangent line is vector $\vec{v} = \vec{r}\,'(t_0) = \langle x'(t_0), y'(t_0) \rangle$

The "line" integral of a vector field "along" the oriented curve C is an infinite sum of the components of $\vec{F}$ "in the same direction" as the tangent vectors to the curve C.

To begin, suppose

$$C = \{ \vec{r}(s) : a \leq s \leq b \}$$

we begin with the assumption that curve C is parameterized w/ respect to arc length s

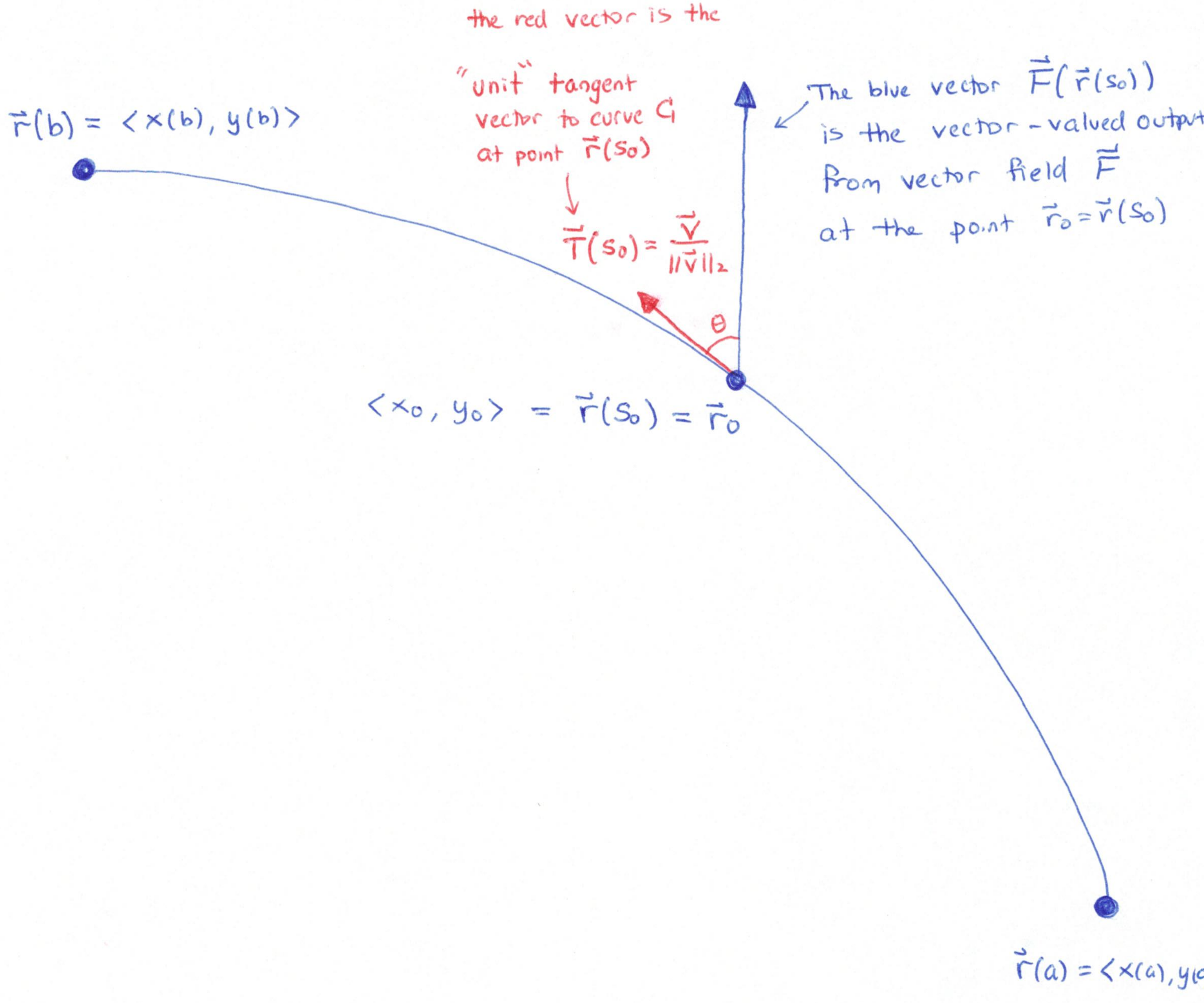
$$= \{ \langle x(s), y(s) \rangle : a \leq s \leq b \}$$

Let $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a vector field

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

continuous on region D that contains C.

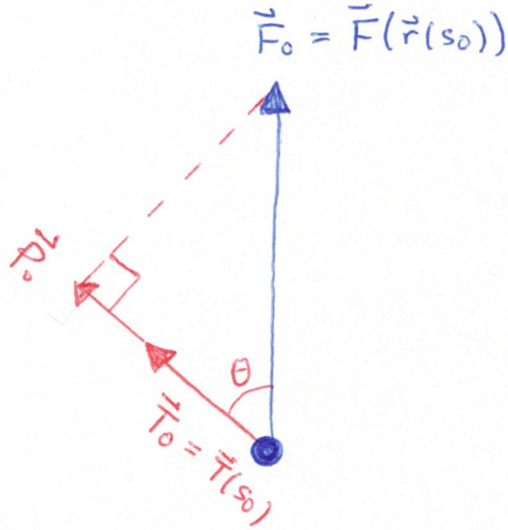
Then, consider the diagram:



At the point $\vec{r}_0 = \vec{r}(s_0) = P(x_0, y_0) = \langle x(s_0), y(s_0) \rangle$ for some value of the arc length parameter $a \le s_0 \le b$, we have

- the (not-necessarily-unit-lengthed) tangent vector to the curve C_1 at point $\vec{r}_0$ given by $\vec{v} = \vec{r}'(s_0) = \langle x'(s_0), y'($
- the unit tangent vector $\vec{T}_0 = \frac{\vec{v}}{\|\vec{v}\|_2} = \vec{T}(s_0) = \frac{\vec{r}'(s_0)}{\|\vec{r}'(s_0)\|_2}$
- the output vector $\vec{F}_0 = \vec{F}(s_0)$ from the vector field.

Recall from Math 1C (and Math 2B) that we can use the idea of an $\boxed{\text{orthogonal projection}}$ to measure the "component" of $\vec{F}$ in the direction of the tangent vector to C at the point $\vec{r}_0 = \vec{r}(s_0)$.



the vector $\vec{P}_0$ is the orthogonal projection of $\vec{F}_0$ onto the "direction" of the unit tangent vector $\vec{T}_0$

Remark: If you don't remember the ideas behind orthogonal projections, I encoura you to go back and rederive all the appropriate formula for yourself

We write this in symbols as

$$\vec{P}_0 = \mathrm{Proj}_{\vec{T}_0}(\vec{F}_0)$$

$$= \frac{\vec{F}_0 \cdot \vec{T}_0}{\|\vec{T}_0\|_2^2} \cdot \vec{T}_0$$

$$= \underbrace{\frac{\vec{F}_0 \cdot \vec{T}_0}{\|\vec{T}_0\|_2}}_{\substack{\text{the signed length} \\ \text{of the vector } \vec{p} \\ \text{Known as the scalar} \\ \text{projection of } \vec{F}_0 \text{ onto } \vec{T}_0}} \cdot \underbrace{\left[\frac{\vec{T}_0}{\|\vec{T}_0\|_2}\right]}_{\text{unit vector in the direction of } \vec{T}_0}$$

However, since $\vec{T}_0$ is a unit vector, we know $\|\vec{T}_0\|_2 = 1$

and we can conclude:

$$\vec{p} = \underbrace{(\vec{F}_0 \cdot \vec{T}_0)}_{\text{signed length of projection}} \cdot \vec{T}_0$$

The "Line" Integral of a Vector Field when Arc Length is the parameter

Now that we have a mechanism to measure the component of vector $\vec{F}_0$ in the direction of the unit tangent $\vec{T}_0$ to curve C_1 at the point $\vec{r}_0 = \vec{r}(s_0)$, we can use this intuition to form an infinite sum and create a "line" integral of the vector field $\vec{F}$ along the curve C_1:

$$\int_{C_1} f \, ds = \int_{C_1} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_1} \vec{F}(\vec{r}(s)) \cdot \vec{T}(s) \, ds$$

$$= \int_a^b \vec{F}(\vec{r}(s)) \cdot \frac{\vec{r}\,'(s)}{\|\vec{r}\,'(s)\|_2} \, ds$$

Recall, though, that since s is the arc length parameter we Know that $\|\vec{r}\,'(s)\|_2 = 1$ for every $a \le s \le b$ and thus our "line" integral becomes

Recall: the vector field $\vec{F} = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$

$$\int_{C_1} f\, ds = \int_{C_1} \vec{F}(\vec{r}(s)) \cdot \vec{r}\,'(s)\, ds$$

$$= \int_{C_1} \vec{F}(x(s), y(s)) \cdot \langle x'(s), y'(s) \rangle\, ds$$

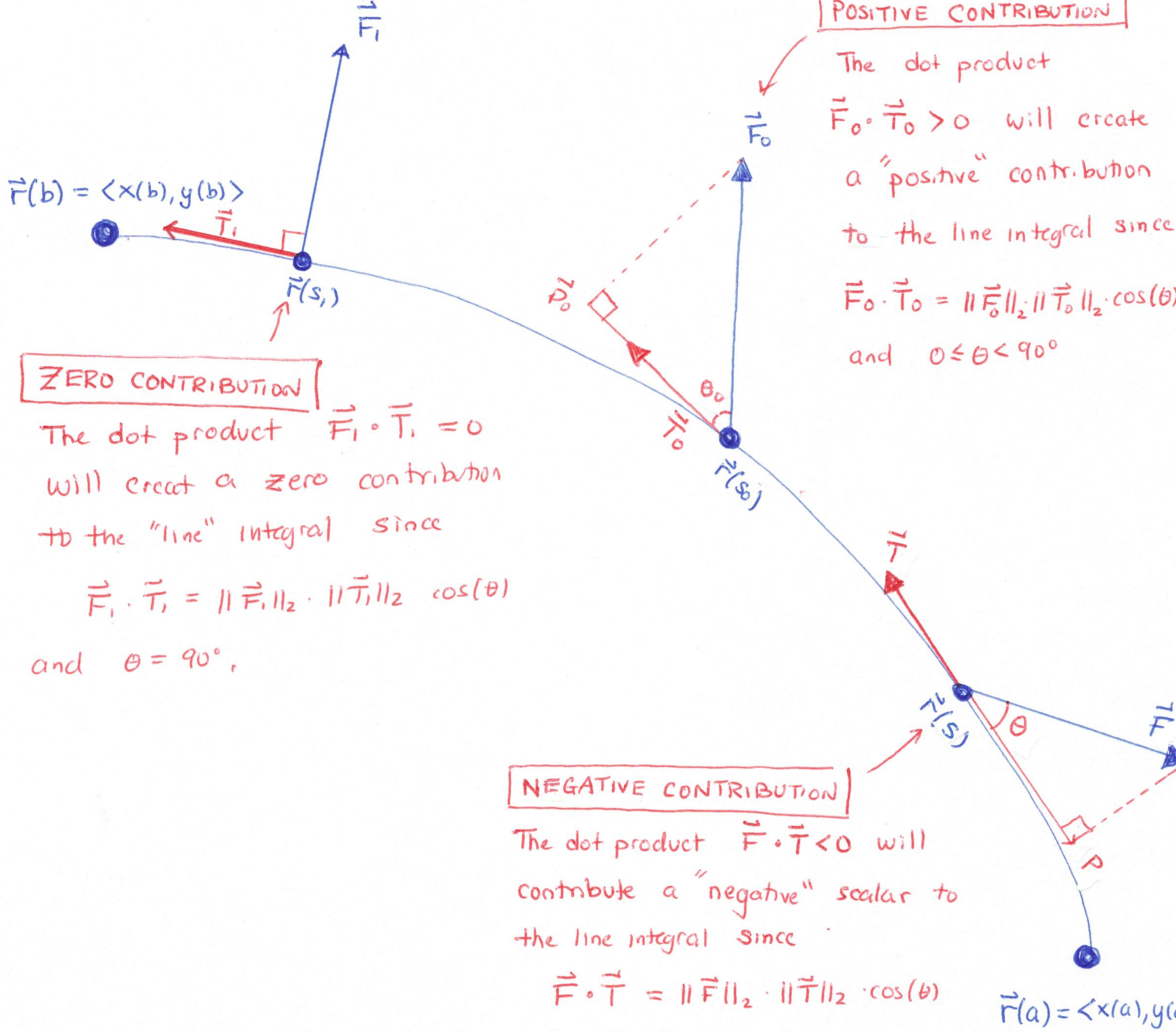
$$= \int_{C_1} \langle f(x(s), y(s)), g(x(s), y(s)) \rangle \cdot \langle x'(s), y'(s) \rangle\, ds$$

$$= \int_a^b f(x(s), y(s)) \cdot x'(s) + g(x(s), y(s)) \cdot y'(s)\, ds$$

$$= \int_a^b f \cdot x' + g \cdot y'\, ds$$

Now let's take a look at some geometry.

Notice that when we take the "line" integral of a vector field $\vec{F}$ over a curve C parameterized via function $\vec{r} = \vec{r}(s) = \langle x(s), y(s) \rangle$ where s is the arc length parameter, we consider the diagram



POSITIVE CONTRIBUTION

The dot product $\vec{F}_0 \cdot \vec{T}_0 > 0$ will create a "positive" contribution to the line integral since $\vec{F}_0 \cdot \vec{T}_0 = \|\vec{F}_0\|_2 \cdot \|\vec{T}_0\|_2 \cdot \cos(\theta)$ and $0 \leq \theta < 90°$

ZERO CONTRIBUTION

The dot product $\vec{F}_1 \cdot \vec{T}_1 = 0$ will creat a zero contribution to the "line" integral since

$$\vec{F}_1 \cdot \vec{T}_1 = \|\vec{F}_1\|_2 \cdot \|\vec{T}_1\|_2 \cos(\theta)$$

and $\theta = 90°$.

NEGATIVE CONTRIBUTION

The dot product $\vec{F} \cdot \vec{T} < 0$ will contribute a "negative" scalar to the line integral since

$$\vec{F} \cdot \vec{T} = \|\vec{F}\|_2 \cdot \|\vec{T}\|_2 \cdot \cos(\theta)$$

and $90 < \theta \leq 180$

The "Line" Integral of a Vector Field when Arc Length s

in NOT the parameter.

Now let's develop a method for evaluating vector "line"

integrals of a vector field $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$$

along a parameterized curve C_1 where

$$C_1 = \{ \vec{r}(t) : a \leq t \leq b \}$$

and the parameter t is not necessarily the arc length

parameter. To this end, recall that the unit tangent

vector to curve C_1 at any point $\vec{r}(t)$ w/ $a \leq t \leq b$

is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}.$$

Since we know that arc length function

$$s(t) = \int_a^t \|\vec{r}\,'(u)\|_2 \, du$$

has derivative

$$s'(t) = \frac{d}{dt}\left[s(t) \right]$$

$$= \frac{d}{dt}\left[\int_a^t \|\vec{r}\,'(u)\|_2 \, du \right]$$

$$= \|\vec{r}\,'(t)\|_2$$

we know that $ds = \|\vec{r}\,'(t)\|_2 \, dt$. We can use this description of the differential form ds to create the "line" integral of $\vec{F}$ over C where $\vec{r}(t)$ is not parametrized using the arc length parameter.

In this case, our line integral becomes

$$\int_{C} f \, ds = \int_{C} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \, ds$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|_2} \, ds$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|_2} \cdot \|\vec{r}'(t)\|_2 \, dt$$

$$= \int_{C} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

This integral may be expressed in several equivalent forms depending on how we encode these ideas using symbols. We note

$$\int_{C} f \, ds = \int_{C} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_{a}^{b} \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_{a}^{b} \langle f(x(t),y(t)), g(x(t),y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$= \int_{a}^{b} f(x(t),y(t)) \cdot x'(t) + g(x(t),y(t)) \cdot y'(t) \, dt$$

$$= \int_{a}^{b} \left(f(t) \cdot x'(t) + g(t) \cdot y'(t) \right) dt$$

where $f(t) = f(x(t), y(t$

and $g(t) = g(x(t), y(t$

Another useful variation on the notation we use to write line integrals comes from the creative use of differential form notation. Recall from our discussion that

$$\vec{r}(t) = \langle x, y \rangle = \langle x(t), y(t) \rangle$$

$$\Rightarrow \quad dx = d[x(t)] = x'(t)\,dt$$

$$dy = d[y(t)] = y'(t)\,dt$$

$$\Rightarrow \int_C f\,ds = \int_C \vec{F} \cdot \vec{T}\,ds$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}\,ds$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}\,\|\vec{r}\,'(t)\|_2\,dt$$

$$\Rightarrow \int_{C_1} f\, ds = \int_a^b \left(f(t)\cdot x'(t) + g(t)\cdot y'(t)\right) dt$$

$$= \int_a^b f(t)\cdot x'(t)\, dt + \int_a^b g(t)\cdot y'(t)\, dt$$

$$\text{"="} \int_a^b f\, dx + \int_0^b g\, dy$$

this is "fuzzy" math (not exact !!)

$$= \int_a^b f\, dx + g\, dy$$

$$= \int_a^b \langle f, g\rangle \cdot \langle dx, dy\rangle$$

$$= \int_a^b \langle f, g\rangle \cdot d[\langle x, y\rangle]$$

This notation is designed for experts.

$$= \int_a^b \vec{F} \cdot d\vec{r}$$ ← cute but highly suspect! I will avoid this notation in early acquisition phase

Example 14.2.5 p. 1067

Evaluate $\int_{C_1} \vec{F} \cdot \vec{T} \, ds$ where $\vec{F}(x,y) = \langle y-x, x \rangle$

on each of the following oriented paths in $\mathbb{R}^2$

Path a: The oriented quarter circle C_1 from $P(0,1)$ to $Q(1,0)$

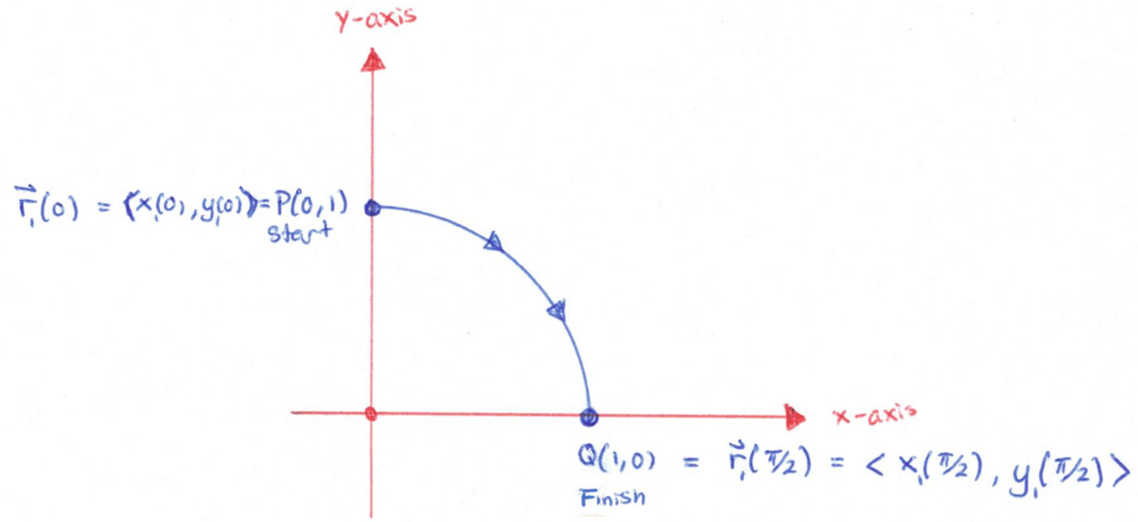
Path b: The reverse oriented quarter circle $C_3 = -C_1$ from $Q(1,0)$ to $P(0,1)$.

Path c: The path C_2 from $P(0,1)$ to $Q(1,0)$ via two line segments connected at origin $O(0,0)$.

Solution: Let's begin with part a of this problem.

We will parameterize C_1 from point $P(0,1)$ to $Q(1,0)$ using the arc length parameter s:

Example 14.2.5a continued ...



Let's parametrize C_1 as follows:

$$C_1 = \{ \vec{r}_1(s) : 0 \le s \le \pi/2 \}$$

where $\vec{r}_1(s) = \langle x_1(s), y_1(s)\rangle = \langle \sin(s), \cos(s)\rangle$.

Then, we saw that the "line" integral of $\vec{F}$ over C_1 was

$$\int_{C_1} f \, ds = \int_{C_1} \vec{F} \cdot \vec{T} \, ds = \int_0^{\pi/2} F(\vec{r}(s)) \cdot \vec{T}(s) \, ds$$

Example 14.2.5a continued ...

But, we see that for our vector field

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = \langle y - x, x \rangle$$

when we make the substitution $x = x_1(s) = \sin(s)$ and $y = y_1(s) = \cos(s)$

we have

$$\vec{F}(\vec{r}_1(s)) = \vec{F}(x_1(s), y_1(s))$$

$$= \langle y_1(s) - x_1(s), x_1(s) \rangle$$

$$= \langle \cos(s) - \sin(s), \sin(s) \rangle$$

We also note that our unit tangent vector is given as

$$\vec{T}(s) = \frac{\vec{r}_1{}'(s)}{\|\vec{r}_1{}'(s)\|_2}$$

Side note:

$\vec{r}_1{}'(s) = \frac{d}{ds}\left[\vec{r}_1(s)\right] = \frac{d}{ds}\left[\langle \sin(s), \cos\right.$

$\Rightarrow r_1'(s) = \langle \cos(s), -\sin($

$\Rightarrow \|\vec{r}_1{}'(s)\|_2 = 1$

$$\Rightarrow \vec{T}(s) = \langle \cos(s), -\sin(s) \rangle$$

Example 14.2.5a continued ...

We can use these calculations to find the value of the line integral

$$\int_{C_1} f \, ds = \int_{C_1} \vec{F}(\vec{r}_1(s)) \cdot \vec{T}(s) \, ds$$

$$= \int_0^{\pi/2} \langle \cos(s) - \sin(s), \sin(s) \rangle \cdot \langle \cos(s), -\sin(s) \rangle \, ds$$

$$= \int_0^{\pi/2} \cos^2(s) - \sin(s)\cos(s) - \sin^2(s) \, ds$$

$$= \int_0^{\pi/2} \underbrace{\cos^2(s) - \sin^2(s)}_{\cos(2s)} - \underbrace{\sin(s) \cdot \cos(s)}_{\frac{1}{2} \cdot \sin(2s)} \, ds$$

Side note: Trig Identities w/ Double Angles

- $\cos(2s) = \cos(s+s)$
  $= \cos(s) \cdot \cos(s) - \sin(s) \cdot \sin(s)$
  $= \cos^2(s) - \sin^2(s)$

- $\sin(2s) = \sin(s+s)$
  $= \sin(s) \cdot \cos(s) + \cos(s) \cdot \sin(s)$
  $= 2 \sin(s) \cdot \cos(s)$

Example 14.2.5a continued...

Using our trigonometric identities, we have

$$\int_{C_1} \vec{F}\cdot\vec{T}\, ds = \int_0^{\pi/2} \cos(2s) - \frac{1}{2}\sin(2s)\, ds$$

$$= \frac{1}{2}\sin(2s) + \frac{1}{4}\cos(2s) \Big|_0^{\pi/2}$$

$$= \left(\frac{1}{2}\sin(\pi) + \frac{1}{4}\cos(\pi)\right) - \left(\frac{1}{2}\sin(0) + \frac{1}{4}\cos(0)\right)$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\boxed{= -\frac{1}{2}}$$

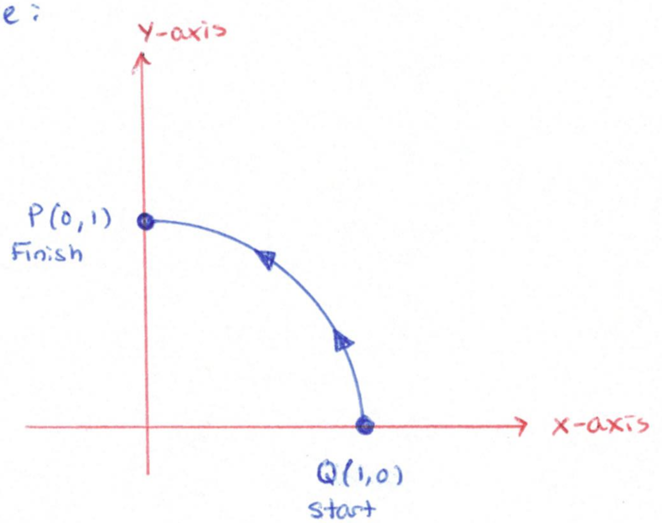
The fact that $\int_{C_1} f\, ds = \int_{C_1} \vec{F}\cdot\vec{T}\, ds = -\frac{1}{2} < 0$ indicates that the "line" integral "mostly" points in the opposite direction of the unit-tangent vectors to oriented curve C_1.

Example 14.2.5b ...

Now let's try the same line integral with a new path

$C_3 = -C_1$ from $Q(1,0)$ to $P(0,1)$ via a

quarter circle:



Now, we will parameterize C_3 as follows

$$C_3 = \{ \vec{r}_3(s) : 0 \leq s \leq \pi/2 \}$$

where $\vec{r}_3(s) = \langle x_3(s), y_3(s) \rangle = \langle \cos(s), \sin(s) \rangle$.

By similar reasoning as before, we see:

- $\vec{F}(\vec{r}_3(s)) = \langle y_3(s) - x_3(s), x_3(s) \rangle = \langle \sin(s) - \cos(s), \cos(s) \rangle$

- $\vec{T}(s) = \vec{r}\,'(s) = \frac{d}{ds}\left[\langle \cos(s), \sin(s) \rangle \right] = \langle -\sin(s), \cos(s) \rangle$

• we notice $\|\vec{r}\,'(s)\|_2 = 1$ since the parameter s is, indeed, the arc length parameter!

Example 14.2.5b, contnued ...

Now, we can calculate the "line" integral of the vector field

$\vec{F}$ over curve C_3 :

$$\int_{C_3} f \, ds = \int_{C_3} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_3} \vec{F}(\vec{r}_3(s)) \cdot \vec{r}_3\,'(s) \, ds$$

$$= \int_0^{\pi/2} \langle \sin(s) - \cos(s), \cos(s) \rangle \cdot \langle -\sin(s), \cos(s) \rangle \, ds$$

$$= \int_0^{\pi/2} -\sin^2(s) + \sin(s)\cos(s) + \cos^2(s) \, ds$$

$$= \int_0^{\pi/2} \underbrace{\cos^2(s) - \sin^2(s)}_{\cos(2s)} + \underbrace{\sin(s)\cos(s)}_{\frac{1}{2}\sin(2s)} \, ds$$

$$= \int_0^{\pi/2} \cos(2s) + \frac{1}{2}\sin(2s) \, ds$$

Then, we can take this integral using our Knowledge of Math 1B techniques to find

$$\int_{C_3} f \, ds = \int_{C_3} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_0^{\pi/2} \cos(2s) + \frac{1}{2} \sin(2s) \, ds$$

$$= \left(\frac{1}{2} \sin(2s) - \frac{1}{4} \cos(2s) \right) \Big|_0^{\pi/2}$$

$$= \left(\frac{1}{2} \sin(\pi) - \frac{1}{4} \cos(\pi) \right) - \left(\frac{1}{2} \sin(0) - \frac{1}{4} \cos(0) \right)$$

$$= 0 + \frac{1}{4} - \left(-\frac{1}{4} \right)$$

$$= \boxed{+\frac{1}{2}}$$

Notice : $\int_{C_1} \vec{F} \cdot \vec{T} \, ds = - \int_{-C_1} \vec{F} \cdot \vec{T} \, ds$

□ Reversing the orientation of our path flips the sign of our "line" integral over that path.

Example 14.2.5, part c)

Finally, let's compute the "line" integral of our vector field

$$\vec{F}(x,y) = \langle y-x,\ x \rangle$$

along a path C_2 that is made up of two line segments:

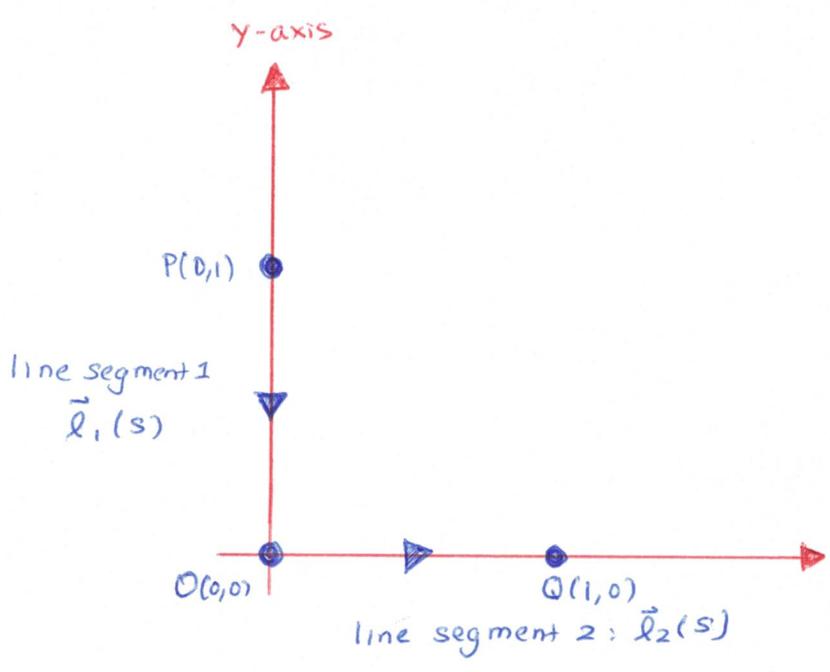
Line segment 1: Connects point $P(0,1)$ to origin $O(0,0)$

Line segment 2: connects point $O(0,0)$ to origin $Q(1,0)$

We can parameterize both these segments using a parametric function and the arc length parameter:

Line segment 1: $\vec{\ell}_1(s) = \langle 0,1 \rangle + s \cdot \langle 0,-1 \rangle = \langle 0, 1-s \rangle$

Line segment 2: $\vec{\ell}_2(s) = \langle 0,0 \rangle + s \langle 1,0 \rangle = \langle s, 0 \rangle$



Then, we can take the "line" integral over this path

$$\int_{C_2} f \, ds = \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{\vec{l}_1} \vec{F} \cdot \vec{T} \, ds + \int_{\vec{l}_2} \vec{F} \cdot \vec{T} \, ds$$

Side note:

On line segment 1 with $\vec{l}_1(s) = \langle 0, 1-s \rangle$:

- $\vec{F}(x,y) = \langle y-x, x \rangle = \langle 1-s, 0 \rangle$
- $\vec{T}(s) = \langle 0, -1 \rangle$

On line segment 2 with $\vec{l}_2(s) = \langle s, 0 \rangle$:

- $\vec{F}(x,y) = \langle y-x, x \rangle = \langle -s, s \rangle$
- $\vec{T}(s) = \langle 1, 0 \rangle$

$$= \int_0^1 \langle 1-s, 0 \rangle \cdot \langle 0, -1 \rangle \, ds + \int_0^1 \langle -s, s \rangle \cdot \langle 1, 0 \rangle \, ds$$

$$= \int_0^1 0 \, ds + \int_0^1 -s \, ds$$

$$= -\frac{s^2}{2} \Big|_0^1 = \boxed{\frac{-1}{2}}$$

Remark:

Notice that our line integrals from parts a and c are equal w/

$$\frac{-1}{2} = \int_{C_1} \vec{F} \cdot \vec{T} ds = \int_{C_2} \vec{F} \cdot \vec{T} ds$$

This is a property of a special type of vector field, seen in sec 14.3.

Circulation of a Vector Field (think line integrals on "closed" curves)

Let $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ be a continuous vector field

with $\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Let $C \subseteq D$ be a closed, smooth, oriented curve where

$$C = \{ \vec{r}(t) : a \le t \le b \}$$

$$= \{ \langle x(t), y(t) \rangle : a \le t \le b \}$$

Note on Notation: see Section 14.3 p. 1078

□ The curve $C \subseteq \mathbb{R}^2$ given by parameterization

$$C = \{ \vec{r}(t) : a \le t \le b \}$$

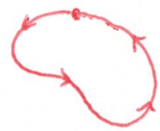
$$= \{ \langle x(t), y(t) \rangle : a \le t \le b \}$$

is closed iff $\vec{r}(a) = \vec{r}(b)$, that

is iff the initial and terminal points

of the curve C are the same.

$\boxed{\text{closed}}$



$\boxed{\text{Not closed}}$

$\boxed{\text{closed}}$

$\boxed{\text{Not closed}}$

The circulation of the vector field $\vec{F}$ on the closed oriented curve C is the "line" Integral

$$\int_{C} \vec{F}\cdot\vec{T}\,ds = \oint_{C} \vec{F}\cdot\vec{T}\,ds$$

where $\vec{T} = \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2}$ is the unit tangent vector to C

at any point $\vec{r}(t)$ as indicated by the chosen orientation.

Remark: How to interpret the circulation

- Notice that circulation is a fancy name for ordinary line Integrals of vector fields over parameterized curves C. However, in this special case, the curve must be closed. To emphasize that the curve must be closed, we can write the circulation as $\oint_{C} \vec{F}\cdot\vec{T}\,ds$ with a small (closed) circle on the integral sign.

- Circulation is a quantitative measurement of the net tendancy of a vector field to "point" in the "same direction" as an oriented closed curve C. In other words, the circulation of $\vec{F}$ along closed oriented curve C is a measure of how much the vector field is at your back (+) and how much is in your face (-) as you travel along C in posi[tive] direction

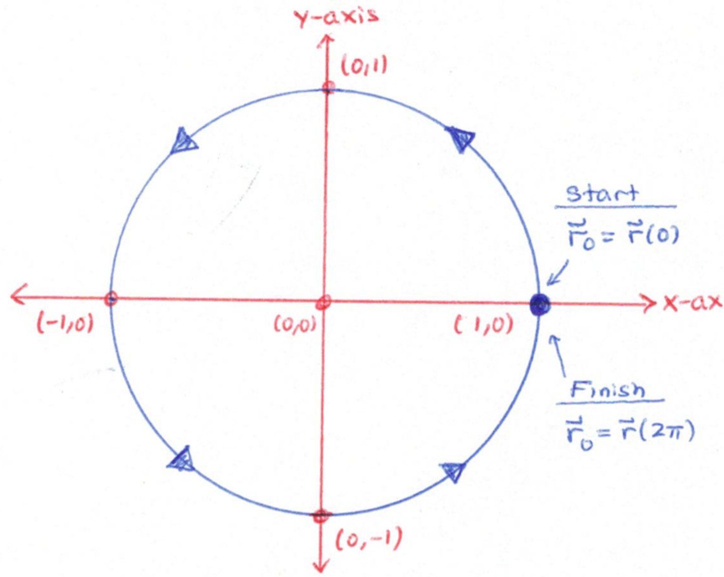
Example 14.2.7 p. 1070 - 1071) Circulation of two-dimensional flows

Let $C \subseteq \mathbb{R}^2$ be the unit circle with a counter clockwise orientation where

$$C = \{ \vec{r}(s) : \quad 0 \leq s \leq 2\pi \}$$

$$= \{ \langle x(s), y(s) \rangle : 0 \leq s \leq 2\pi \}$$

$$= \{ \langle \cos(s), \sin(s) \rangle : 0 \leq s \leq 2\pi \}$$

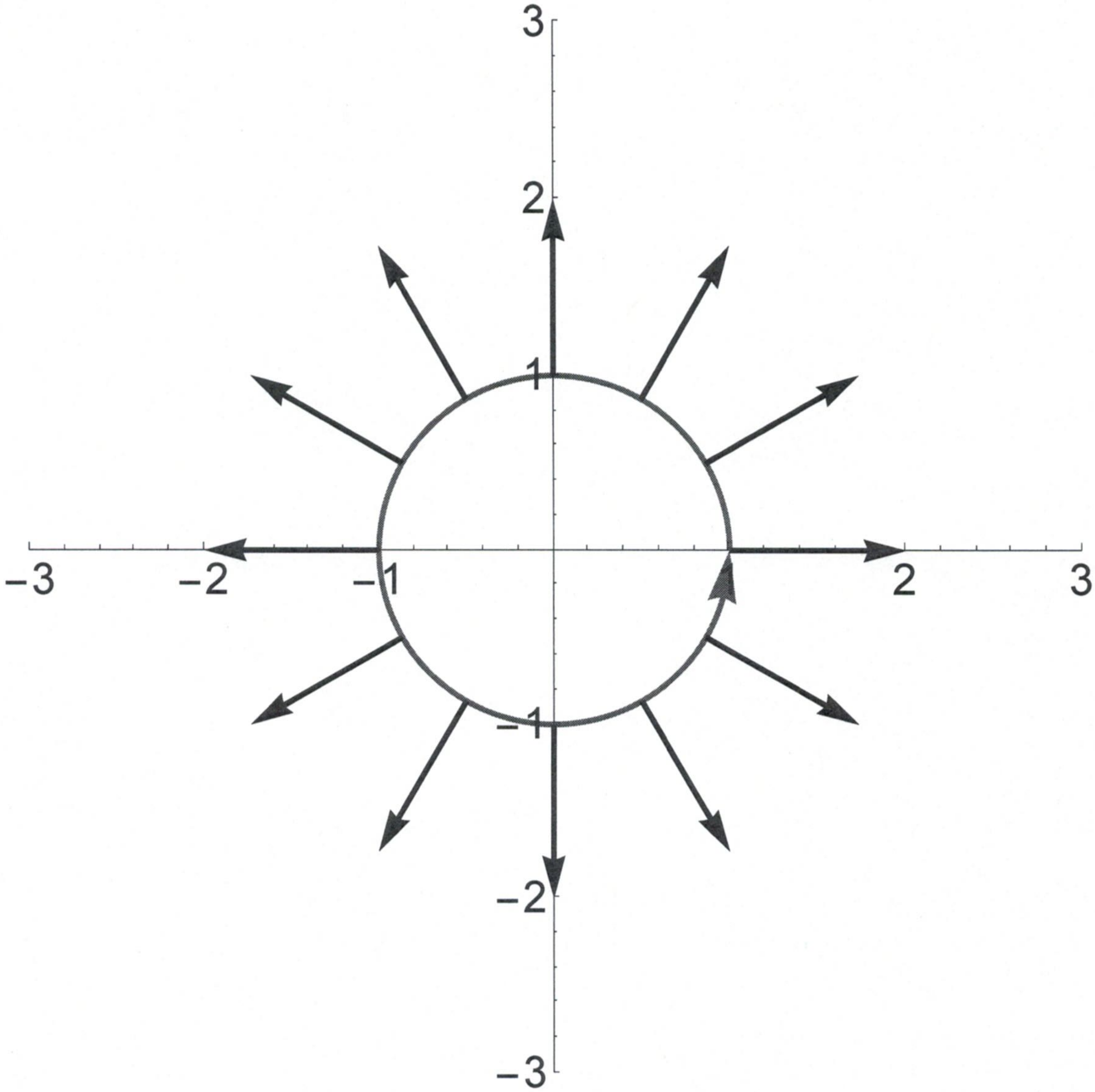


Find the circulation on C of each of the following vector field

Part a) The radial Field: $\vec{F}(x,y) = \langle x, y \rangle$

Part b) The rotational field: $\vec{F}(x,y) = \langle -y, x \rangle$

Let's begin with part a! We visualize a subset of vectors along the curve C_1 below:



We notice that with our given parameterization, we have

☐ $\vec{F}(x,y) = \vec{F}(x(s), y(s)) = \langle x(s), y(s) \rangle = \langle \cos(s), \sin(s) \rangle$

☐ $\vec{T}(s) = \dfrac{\vec{r}\,'(s)}{\|\vec{r}\,'(s)\|_2} = \dfrac{d}{ds}\left[\langle \cos(s), \sin(s) \rangle\right] = \langle -\sin(s), \cos(s) \rangle$

Then, our desired circulation of the radial field $\vec{F}$ on unit circle C_1 i

$$\int_{C_1} f\, ds = \oint_{C_1} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_0^{2\pi} \langle \cos(s), \sin(s) \rangle \cdot \langle -\sin(s), \cos(s) \rangle\, ds$$

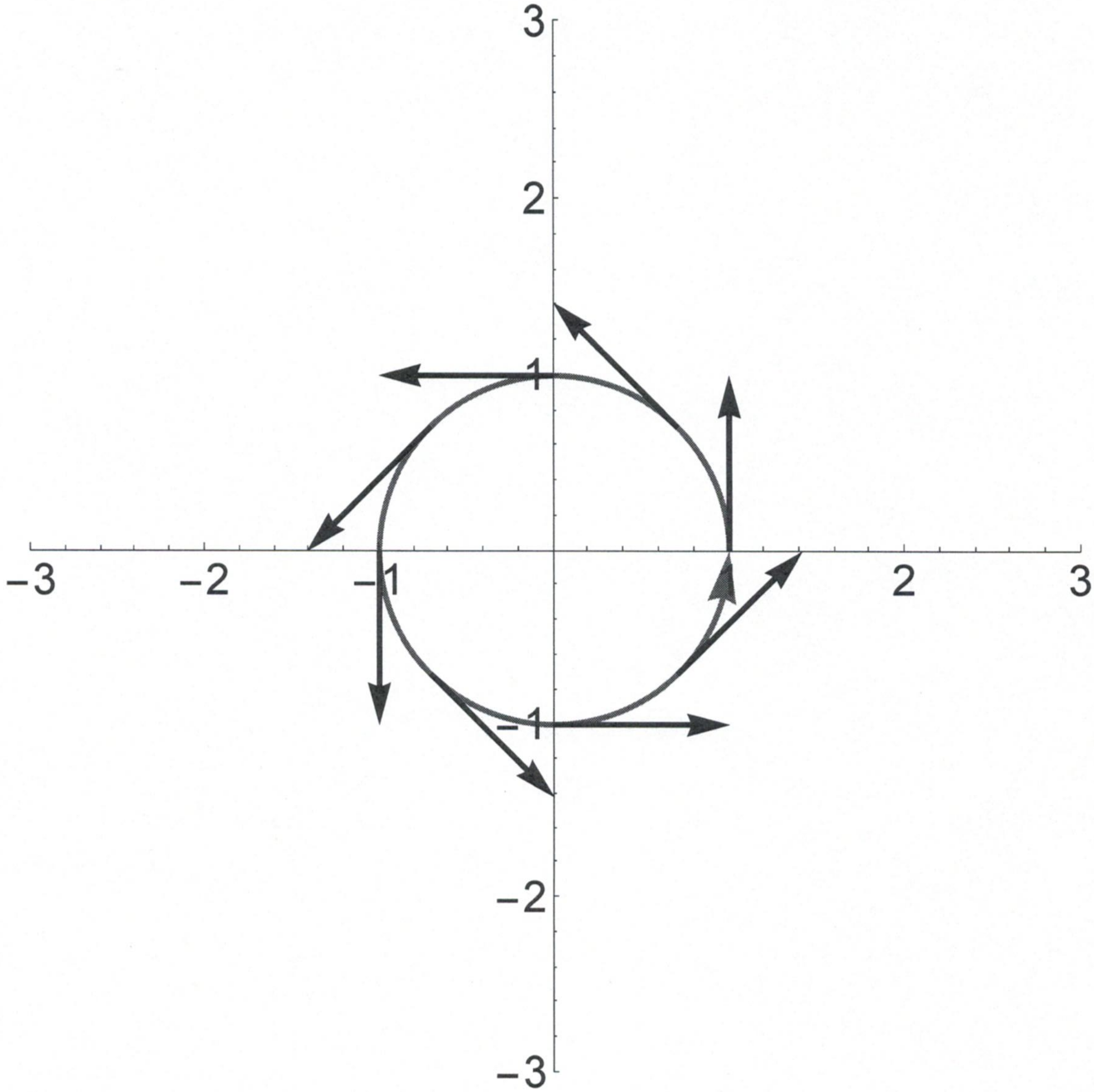
$$= \int_0^{2\pi} -\cos(s)\sin(s) + \sin(s) \cdot \cos(s)\, ds$$

$$= \int_0^{2\pi} 0\, ds$$

$$= \boxed{0}.$$

Indeed we see that since the vectors from radial vector field $\vec{F}$ are orthogonal to the unit tangent vectors of C_1 everywhere along the close oriented curve C_1, the circulation of $\vec{F}$ on C_1 is zero.

Example 14.2.4, part B (p. 1071) Now let's move on to calculate the circulation of the rotational field $\vec{F} = \langle -y, x \rangle$ seen below:



Given our parameterization of C, we notice that

- $\vec{F}(x,y) = \vec{F}(x(s), y(s)) = \langle -y(s), x(s) \rangle = \langle -\sin(s), \cos(s) \rangle$
- $\vec{T}(s) = \vec{r}\,'(s) = \langle -\sin(s), \cos(s) \rangle$

Example 14.2.7, part b p. 1071)

Then our desired circulation of rotational field $\vec{F}$ on the unit circle C_1 is given as

$$\int_{C_1} f \, ds = \oint_{C_1} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_0^{2\pi} \langle -\sin(s), \cos(s) \rangle \cdot \langle -\sin(s), \cos(s) \rangle \, ds$$

$$= \int_0^{2\pi} \sin^2(s) + \cos^2(s) \, ds$$

$$= \int_0^{2\pi} 1 \, ds$$

$$= \boxed{2\pi}$$

In the case of our rotational field $\vec{F} = \langle -y, x \rangle$ on our closed, oriented unit circle, at every point on C_1 the vector-valued output from the rotational field is in the exact direction of the unit tangent vector $\vec{T}$ at that point. Since the vector field is always "at our back", we get positive circulati

Flux of a vector field

Let $\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ be a continuous vector field with

$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Let $C \subseteq D$ be a smooth, oriented, simple curve that does not intersect itself where

$$C = \{ \vec{r}(t) : a \le t \le b \}$$

$$= \{ \langle x(t), y(t) \rangle : a \le t \le b \}$$

Note on Notation: see section 14.3 p. 1078

- The curve $C \subseteq \mathbb{R}^2$ given by parameterization $C = \{ \vec{r}(t) : a \le t \le b \} = \{ \langle x(t), y(t) \rangle : a \le t \le b \}$ is simple iff $\vec{r}(t_1) \neq \vec{r}(t_2)$ for all t_1 and t_2 with $a < t_1 < t_2 < b$

- The terminology that a curve C is simple means that C never intersects itself between its end points

* Notice that a simple curve C may or may not be closed

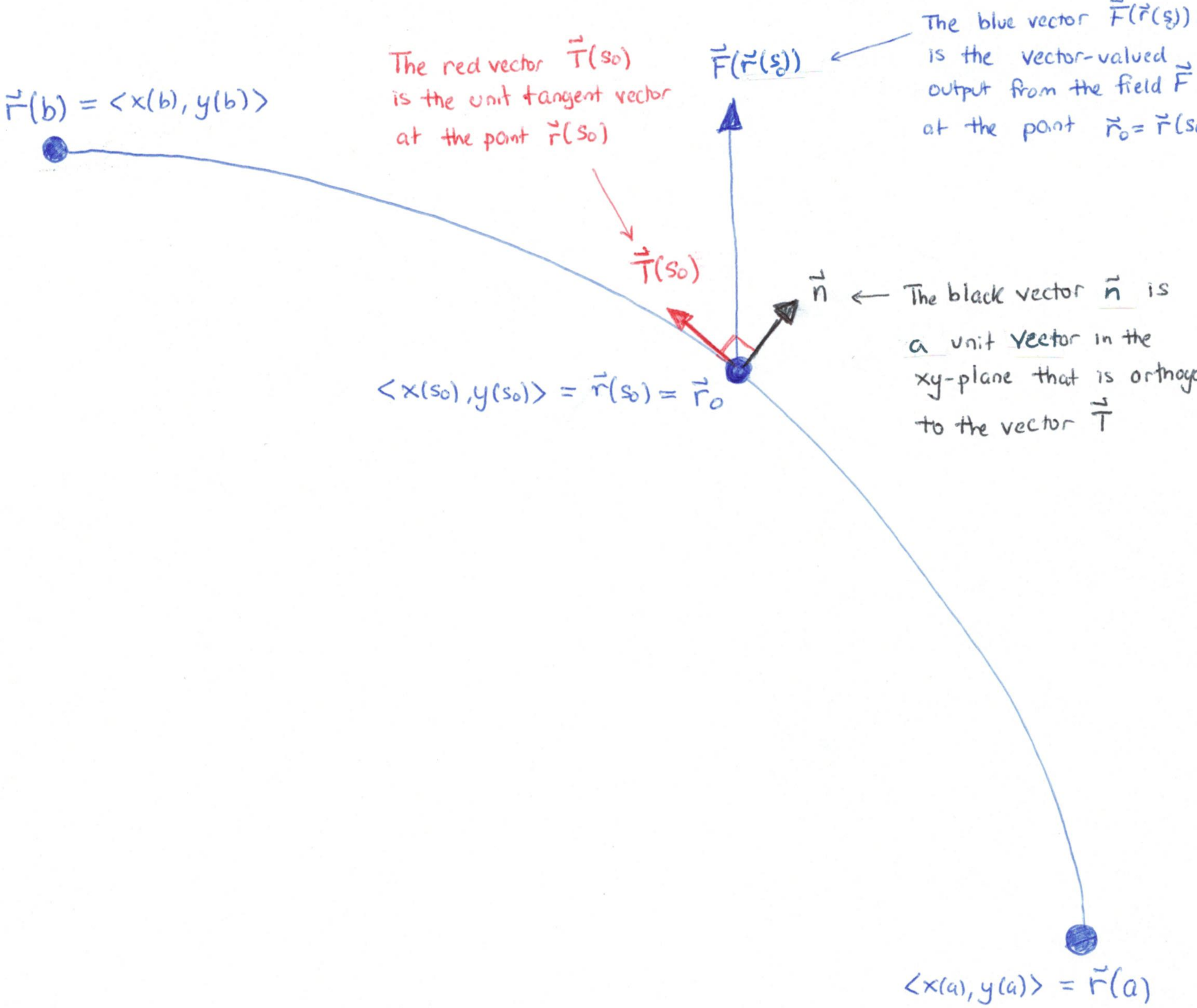
* $\boxed{\text{SIMPLE}}$

$\boxed{\text{NOT SIMPLE}}$

* $\boxed{\text{SIMPLE}}$

$\boxed{\text{NOT SIMPLE}}$

Now, let's reconsider our diagram:

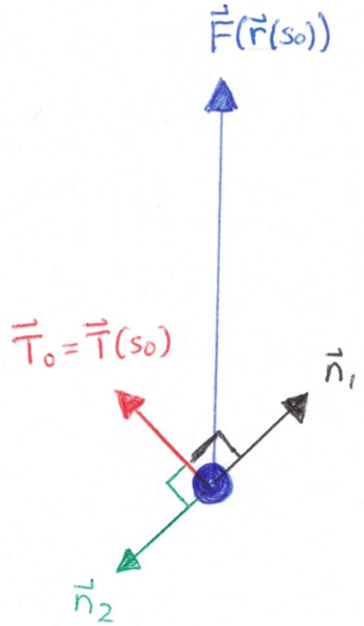


At the point $\vec{r}_0 = \vec{r}(s_0) = P(x_0, y_0) = \langle x(s_0), y(s_0) \rangle$ with a specific value of the arc length parameter $a \leq s_0 \leq b$, we have the following:

- ☐ the (not-necessarily-unit-length) tangent vector to the curve C at the point $\vec{r}_0$ given by $\vec{v} = \vec{r}'(s_0) = \langle x'(s_0), y'(s_0) \rangle$
- ☐ the unit tangent vector $\vec{T}_0 = \dfrac{\vec{v}}{\|\vec{v}\|_2} = \vec{T}(s_0) = \dfrac{\vec{r}'(s_0)}{\|\vec{r}'(s_0)\|_2}$
- ☐ the output vector $\vec{F}_0 = \vec{F}(\vec{r}(s_0)) = \vec{F}(x(s_0), y(s_0)) = \langle f(x(s_0), y(s_0)), g(x(s_0), y(s_0)) \rangle$
- ☐ a unit vector $\vec{n}$ in the xy-plane such that $\vec{n} \perp \vec{T}_0$ (where $\vec{n} \cdot \vec{T}_0 = 0$)

Let's focus in on the point $\vec{r}(s_0) = \langle x(s_0), y(s_0) \rangle$

and visualize all three of these vectors



Recall that we claimed that the black vector $\vec{n}$ was a unit vector in the xy-plane that is orthogonal to $\vec{T}_0 = \vec{T}(s_0)$.

However, as we see above, at every point $\vec{r}(s_0)$, there are two possible candidates for the unitvector $\vec{n}$ orthogonal or normal to the curve C: we could choose $\vec{n}$ as $\vec{n}_1$ or $\vec{n}_2$ from the diagram above. However, we choose we want to make sure that we are all making the "same" choice so that any calculations we do individually produce the same results.

With this in mind, we go back to Math 1C and introduce some clever tools to guarantee that we pick a unique $\vec{n}$. Specifically, in order to define our desired unit normal vector $\vec{n}$, we do the following:

Step 1: Embed the entire problem into $\mathbb{R}^3$ to enable the use of the cross product!

$\boxed{\text{Vectors on } C \subseteq \mathbb{R}^2}$	$\boxed{\text{vectors embedded into } \mathbb{R}^3}$
◦ $\vec{r}(s_0) = \langle x(s_0), y(s_0) \rangle$	$\vec{r}(s_0) = \langle x(s_0), y(s_0), 0 \rangle$
◦ $\vec{r}'(s_0) = \langle x'(s_0), y'(s_0) \rangle$	$\vec{r}'(s_0) = \langle x'(s_0), y'(s_0), 0 \rangle$
◦ $\vec{T}(s_0) = \dfrac{\vec{r}'(s_0)}{\lVert \vec{r}'(s_0) \rVert_2}$	$\vec{T}'(s_0) = \dfrac{\vec{r}'(s_0)}{\lVert \vec{r}'(s_0) \rVert_2}$
◦ $\vec{F}(\vec{r}(s_0)) = \langle f(\vec{r}(s_0)), g(\vec{r}(s_0)) \rangle$ w/ $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$	$\vec{F}(\vec{r}(s_0)) = \langle f(\vec{r}(s_0)), g(\vec{r}(s_0)), 0 \rangle$ w/ $\vec{F}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$\vec{i} = \langle 1, 0 \rangle$ $j = \langle 0, 1 \rangle$ NONE	$\vec{i} = \langle 1, 0, 0 \rangle$ $\vec{j} = \langle 0, 1, 0 \rangle$ $\vec{k} = \langle 0, 0, 1 \rangle$

Notice: in the table above, we are blatently abusing notation by overloading the symbols that we use to represent ideas. For example, in this table we see:

$$\langle x(s_0), y(s_0) \rangle = \vec{r}(s_0) \neq \vec{r}(s_0) = \langle x(s_0), y(s_0), 0 \rangle$$

Step 2: Remeber everything we learned abot cross products and apply this operation creatively to calculate the vector $\vec{n}$ using the appropriat embedded versions of our vectors $\vec{T}(s_0), \vec{F}(\vec{r}(s_0)) \in \mathbb{R}^3$.

- Recall that the cross product operation takes two vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ as input and outputs a third vector in $\mathbb{R}^3$. We can summarize these ideas as follows

domain of cross product (cross prodct takes n two vectors)

$$\underbrace{\times}_{\text{symbol to represelt cross product of two vectors in } \mathbb{R}^3} : \overbrace{\mathbb{R}^3 \times \mathbb{R}^3} \longrightarrow \mathbb{R}^3$$

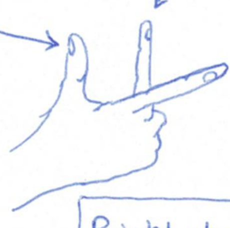
codomain of cross product (output of the cross product is a vector in $\mathbb{R}^3$)

- If we state that $\vec{n} = \vec{x} \times \vec{y}$, then the cross product output $\vec{n}$ is orthogonal to both $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$ with

  - $\vec{n} \perp \vec{x} \iff \vec{n} \cdot \vec{x} = 0$
  - $\vec{n} \perp \vec{y} \iff \vec{n} \cdot \vec{y} = 0$
  - $\vec{n}$ is chosen w/r to the right-hand rule

Point middle finger of our right hand in direction of vector $\vec{y} \in \mathbb{R}^3$

take our thumb and point it orthogonal to both our index and middle fingers. The direction of our thumb is the direction of the vector $\vec{n}$



← point index finger of our right hand "in direction" of vector $\vec{x} \in \mathbb{R}^3$

Right-hand rule

Now, since we wanted to define the unit normal vector $\vec{n}$ in a unique and easily reproducible method to the curve $C \subseteq \mathbb{R}^2$ at any point $\vec{r}(s_0)$, we state:

Note on Notation:
In order to make calculations easi…

Let $\vec{n} = \underbrace{\vec{T}_0}_{\text{embedding of } \vec{T}_0 \text{ into } \mathbb{R}^3} \times \vec{k}$

$= \langle T_x, T_y, 0 \rangle \times \langle 0, 0, 1 \rangle$

component form

$= \langle T_y \cdot 1 - 0 \cdot 0,\ 0 \cdot 0 - T_x \cdot 1,\ T_x \cdot 0 - T_y \cdot 0 \rangle$

$= \langle T_y, -T_x, 0 \rangle$

$= T_y \cdot \vec{i} - T_x \cdot \vec{j}$

Side note: Determinent method for Cross

$$\vec{T} \times \vec{k} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ T_x & T_y \\ 0 & 0 \end{matrix}$$

$= T_y \cdot \vec{i} + 0 \cdot \vec{j} + 0 \cdot \vec{k}$
$- 0 \cdot \vec{i} - \vec{T}_x \vec{j} - 0 \cdot \vec{k}$

Step 3: Take embedded vector $\vec{n} = \vec{T_0} \times \vec{k} \in \mathbb{R}^3$ and deflate it so that it lives in $\mathbb{R}^2$

vector embedded in $\mathbb{R}^3$	vector deflated into $\mathbb{R}^2$

$$\vec{n} = \vec{T_0} \times \vec{k}$$

$$\Rightarrow \quad \vec{n} = \langle T_x, T_y, 0\rangle \times \langle 0,0,1\rangle$$

$$\Rightarrow \quad \vec{n} = T_y \cdot \vec{j} - T_x \cdot \vec{i}$$

$$\Rightarrow \quad \vec{n} = \langle T_y, -T_x, 0\rangle \xrightarrow[\text{third component}]{\text{deflate the}} \vec{n} = \langle T_y, -T_x\rangle$$

Remark: Checking $\vec{n} \perp \vec{T_0}$

- We can very quickly check that $\vec{n} \in \mathbb{R}^2$ is orthogonal to $\vec{T_0} \in \mathbb{R}^2$ since

$$\vec{n} \cdot \vec{T_0} = \langle T_y, -T_x\rangle \cdot \langle T_x, T_y\rangle$$

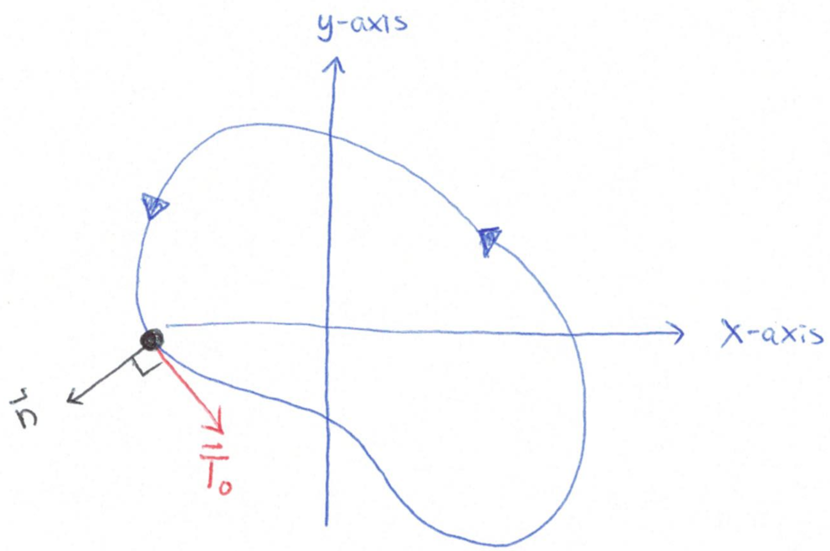
$$= T_y \cdot T_x - T_x \cdot T_y$$

$$= 0 \checkmark$$

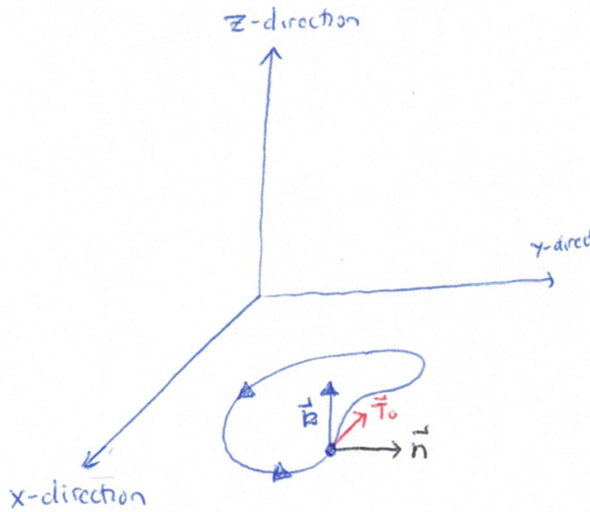
Step 4: Interpret this choice for $\vec{n}$ geometrically for the curve C_1.

This choice for the vector $\vec{n}$ has some important implications with respect to the curve C_1 including the following:

- If the curve C_1 is closed and oriented counterclockwise (when viewed from above), the unit normal vector $\vec{n}$ points "outward" along the curve



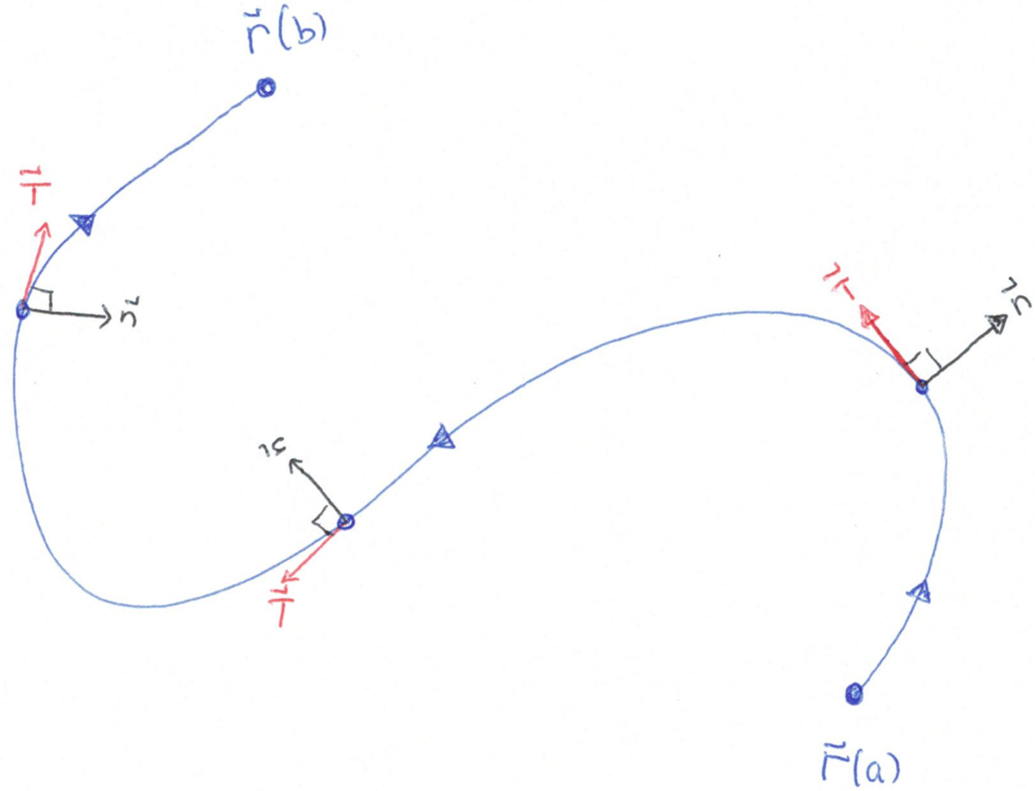
curve $C_1 \subseteq \mathbb{R}^2$



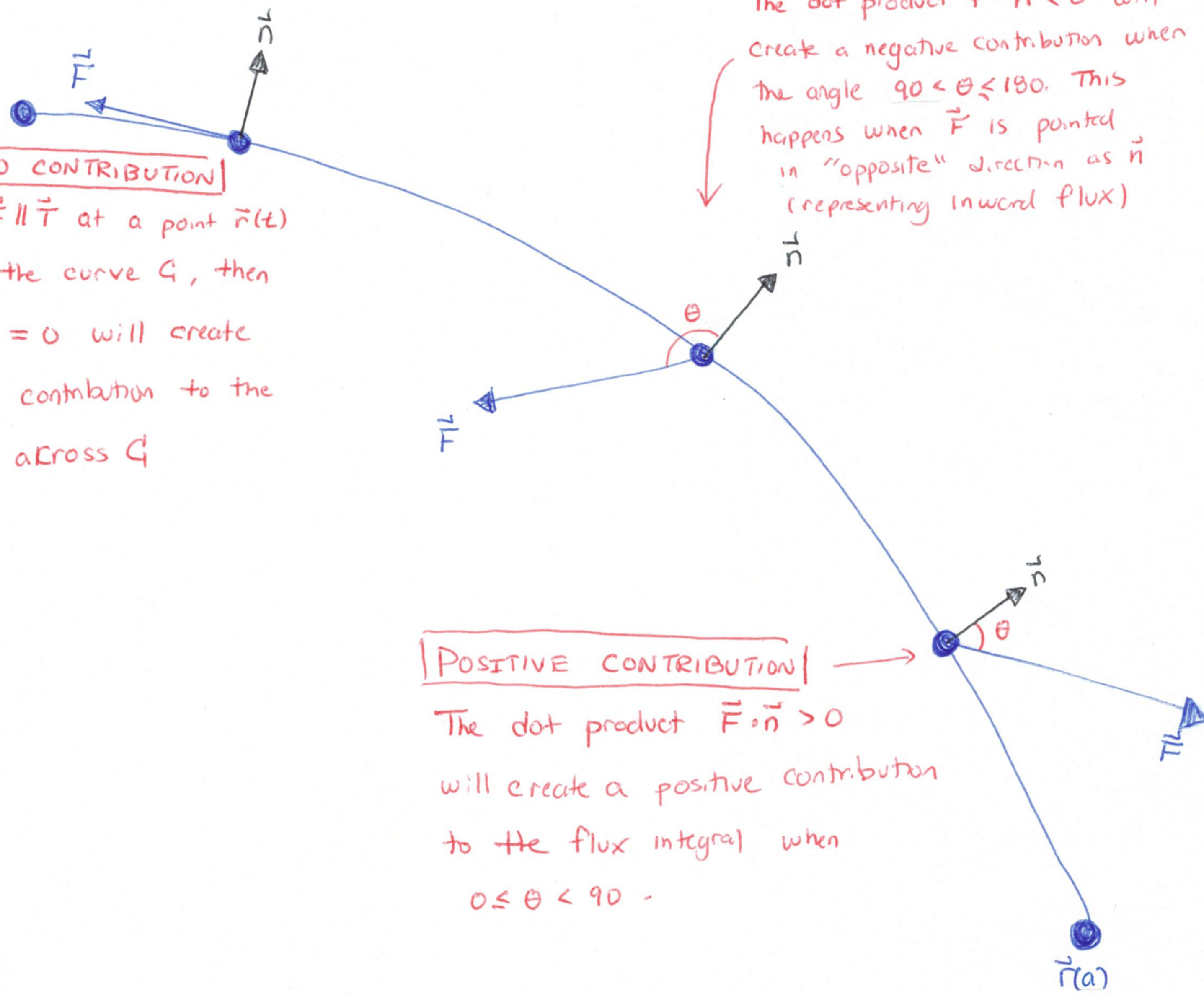
curve C_1 embedded in $\mathbb{R}^3$

☐ If C is not a closed curve, the unit normal vector points to the right of the curve as we travel along the curve in the "positive" direction

(assuming we are looking down on the curve from above).



Notice, that if we now visualize the vectors of the vector field $\vec{F}$ at each point on $C \subseteq \mathbb{R}^2$ and we project these vectors onto the unit normal vector $\vec{n}$ we can interpret the geometry of the scalar components of these projections as before:



NEGATIVE CONTRIBUTION

The dot product $\vec{F} \cdot \vec{n} < 0$ will create a negative contribution when the angle $90 < \theta \leq 180$. This happens when $\vec{F}$ is pointed in "opposite" direction as $\vec{n}$ (representing inward flux)

ZERO CONTRIBUTION

If $\vec{F} \parallel \vec{T}$ at a point $\vec{r}(t)$ along the curve C, then $\vec{F} \cdot \vec{n} = 0$ will create zero contribution to the flux across C

POSITIVE CONTRIBUTION

The dot product $\vec{F} \cdot \vec{n} > 0$ will create a positive contribution to the flux integral when $0 \leq \theta < 90$.

☐ In terms of actually calculating the components, the definition $\vec{n} = \vec{T} \times \vec{k}$ results in the equation

$$\vec{n} = \vec{T} \times \vec{k}$$

$$= \begin{vmatrix} i & j & k \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle T_y, -T_x, 0 \rangle$$

If C is parameterized w/r to general parameter $C = \{\vec{r}(t) : a \le t \le b\}$, then we have $\vec{T} = \langle T_x, T_y, 0 \rangle = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|_2} = \dfrac{\langle x'(t), y'(t), 0 \rangle}{\|\vec{r}'(t)\|_2}$

$$\Rightarrow \quad \langle T_x, T_y, 0 \rangle = \frac{\langle x'(t), y'(t), 0 \rangle}{\|\vec{r}'(t)\|_2}$$

$$\Rightarrow \quad \vec{n} = \langle T_y, -T_x \rangle = \frac{\langle y'(t), -x'(t) \rangle}{\|\vec{r}'(t)\|_2}$$

Just as we defined the circulation as a line integral of the net, "projection" of $\vec{F}$ onto $\vec{T}$, we will define the flux of our field along our curve as the net projection of $\vec{F}$ onto $\vec{n}$ along our curve.

$$\int_C f \, ds = \int_C \vec{F} \cdot \vec{n} \, ds$$

$$= \int_C \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, ds$$

$$= \int_C \vec{F}(x(t), y(t)) \cdot \frac{\langle y'(t), -x'(t) \rangle}{\| \vec{r}\,'(t) \|_2} \, ds$$

$$= \int_C \langle f(x(t), y(t)), g(x(t), y(t)) \rangle \cdot \frac{\langle y'(t), -x'(t) \rangle}{\| \vec{r}\,'(t) \|_2} \, ds$$

$$= \int_C \langle f(x(t), y(t)), g(x(t), y(t)) \rangle \cdot \frac{\langle y'(t), -x'(t) \rangle}{\| \vec{r}\,'(t) \|_2} \cdot \overbrace{\| \vec{r}\,'(t) \|_2 \, dt}$$

$$= \int_{C_1} \langle f(x(t), y(t)), g(x(t), y(t)) \rangle \cdot \langle y'(t), -x'(t) \rangle \, dt$$

$$= \int_a^b f(x(t), y(t)) \cdot y'(t) - g(x(t), y(t)) \cdot x'(t) \, dt$$

$$= \int_a^b f(t) \cdot y'(t) - g(t) \cdot x'(t) \, dt$$

Side note: Expert notation

$x = x(t) \Rightarrow dx = x'(t)\,dt$

$y = y(t) \Rightarrow dy = y'(t)\,dt$

"fuzzy" math: symbols used to compress information and make this easy to write!

$$= \int_a^b f(t)\,dy + g(t)\,dx$$

$$= \int_a^b f\,dy + g\,dx$$

$$\Rightarrow \int_{C} f \, ds = \int_{C} \vec{F} \cdot \vec{n} \, ds \quad w/ \quad \vec{n} \text{ "=" } \vec{T} \times \vec{k} \in \mathbb{R}^3$$

(deflated into $\mathbb{R}^2$)

$$= \int_{C} \langle f(t), g(t) \rangle \cdot \langle y'(t), -x'(t) \rangle \, dt$$

$$= \int_{C} (f(t) \cdot y'(t) - g(t) \, x'(t)) \, dt$$

$$= \int_{C} f \, dy - g \, dx$$

← this is called the flux integral.

Definition : Flux Integrals (think line integrals on "simple" curves in $\mathbb{R}^2$)

Let $\vec{F} = \vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$ be a continuous vector field on a region $D \subseteq \mathbb{R}^2$ with

$$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

Let $C = \{ \vec{r}(t) : a \leq t \leq b \}$ be a smooth, oriented, simple curve (recall: C is simple if it does not intersect itself).

The flux of the vector field $\vec{F}$ across C is calculated as the line integral

$$\int_C f \, ds = \int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \left(f(t) \cdot y'(t) - g(t) \, x'(t) \right) dt$$

where $\vec{n}$ "=" $\underset{\uparrow \atop \vec{T} \text{ embedded into } \mathbb{R}^3}{\vec{T}} \times \vec{k}$ deflated into $\mathbb{R}^2$ and $\vec{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|_2} \in \mathbb{R}^2$.

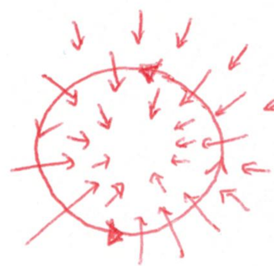
If C is also a closed curve with counterclockwise orientation, then the unit normal vector $\vec{n}$ is the "outward" (pointing) unit normal.

In this case the flux integral gives the outward flux across C.

Remark : Sources and sinks

$$\oint_C \vec{F} \cdot \vec{n} \, ds < 0$$

(net in flux)

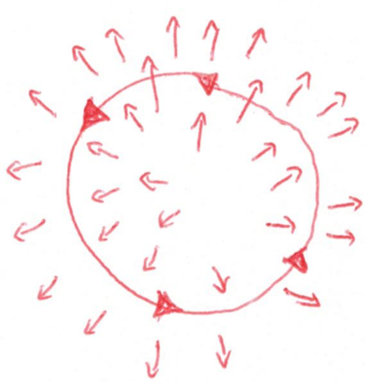


$\leftarrow$ sink

Think about this as a sink

$$\oint_C \vec{F} \cdot \vec{n} \, ds > 0$$

(net out flux)



$\rightarrow$ source

Think about this as a source

Example 14.2.9 p. 1073 - 1074) Flux of two-dimensional flows

Find the outward flux across the unit circle w/ counterclockwise orientation for the following vector fields:

Part a) The radial field $\vec{F}(x,y) = \langle x, y \rangle$

Part b) The rotational field $\vec{F}(x,y) = \langle -y, x \rangle$

Solution: Let's begin with part a. We know from Example 14.2.7 what the geometry of this problem looks like. We also know we can parameterize

$$C_1 = \{ \vec{r}(s) : 0 \le s \le 2\pi \}$$

$$= \{ \langle \cos(s), \sin(s) \rangle : 0 \le s \le 2\pi \}$$

$$\Rightarrow \vec{r}\,'(s) = \langle -\sin(s), \cos(s) \rangle = \langle x'(s), y'(s) \rangle$$

and $$\vec{F}(\vec{r}(s)) = \langle x(s), y(s) \rangle$$

$$= \langle \cos(s), \sin(s) \rangle.$$

Example 14.2.4 p 1073–1074 continued...)

We calculate the flux

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C \vec{F}(\vec{r}(s)) \cdot \vec{n}(s)\, ds$$

$\vec{n} \text{ "=" } \vec{T} \times k$ with

$\vec{n} = \langle T_y, -T_x \rangle$

$= \langle y'(s), -x'(s) \rangle$

$= \langle \cos(s), \sin(s) \rangle$

$$= \oint_C \vec{F}(x(s), y(s)) \cdot \vec{n}(s)\, ds$$

$$= \oint_C \langle \cos(s), \sin(s) \rangle \cdot \langle \cos(s), \sin(s) \rangle\, ds$$

$$= \oint_C \cos^2(s) + \sin^2(s)\, ds$$

$$= \int_0^{2\pi} 1\, ds = \boxed{2\pi}$$

This makes sense: The radial vector field has only flux motion and no circulatory motion!!

Example 14.2.9 p. 1073 - 1074, continued ...

Solution to part b) For the rotational field

$$F(x,y) = \langle -y, x \rangle$$

we can calculate the outward flux of $\vec{F}$ across C as the line integral $\oint_C \vec{F} \cdot \vec{n} \, ds$.

Since $\vec{r}(s) = \langle \cos(s), \sin(s) \rangle$ for $0 \leq s \leq 2\pi$,

we see

$$\vec{F}(\vec{r}(s)) = \vec{F}(x(s), y(s))$$

$$= \langle -y(s), x(s) \rangle$$

$$= \langle -\sin(s), \cos(s) \rangle$$

Moreover, we have $\vec{n} \overset{\text{"}}{=} \vec{T} \times K$

$$= \langle T_y, -T_x \rangle$$

$$= \langle y'(s), -x'(s) \rangle = \langle \cos(s), \sin(s) \rangle$$

Then, the outward flux is

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \oint_C \vec{F}(\vec{r}(s)) \cdot \vec{n}(s)\, ds$$

$$= \oint_C \vec{F}(x(s), y(s)) \cdot \vec{n}(s)\, ds$$

$$= \oint_C \langle -\sin(s), \cos(s)\rangle \cdot \langle \cos(s), \sin(s)\rangle\, ds$$

$$= \oint_C -\sin(s)\cdot\cos(s) + \cos(s)\sin(s)\, ds$$

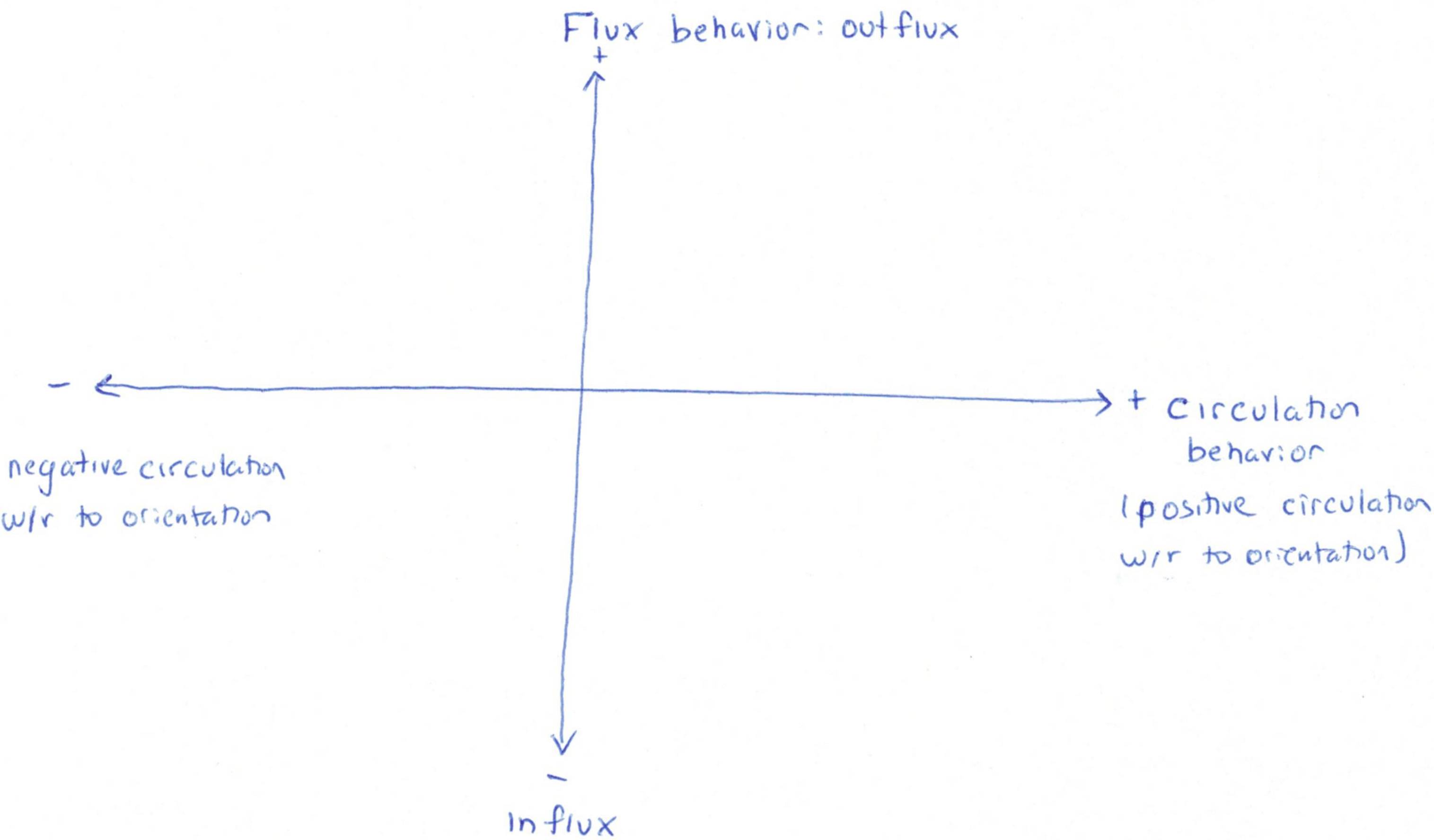
$$= \int_0^{2\pi} 0\, ds$$

$$= 0$$

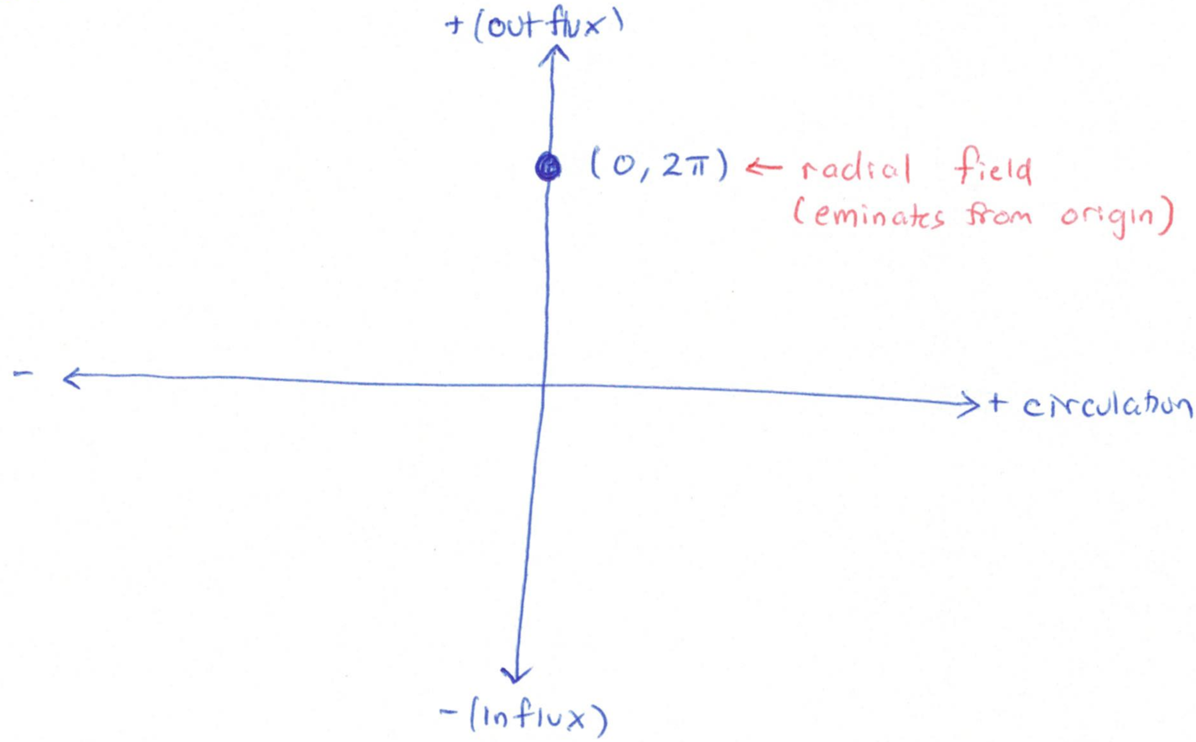
This makes sense: a rotational vector field that is purly rotating (and not at all fluxing) should have no outward or inward flux !!

Recapitulation & Review of Examples 14.2.7 & 14.2.9

These examples 14.2.7 and 14.2.9 are worth remembering (or getting tattooed). They represent ideal versions of physical behavior that Math 1D is designed to model. We can think of these on a "spectrum"



The radial field $\vec{F}(x,y) = \langle x, y \rangle$ has outward flux of 2π across the unit circle w/ counter clockwise orientation AND ZERO circulation



The rotational field $\vec{F}(x,y) = \langle -y, x \rangle$ has zero outward flux across the unit circle w/ counter clockwise orientation and positive circulation of 2π.